Fourier - 1

We now have an exact, complete sol'n to the damped, driven oscillator if the driving force is $f(t) = f_0 \cos\omega t$ (or, $f_0 e^{i\omega t}$)

So if the source (driver) is sinusoidal, we know the system's response.

Sinusoidal $f(t)$ is common in both mechanical & electrical settings, but the real importance of this (special) sol'n is the following:

— If we have a "driver" ($f(t)$) that's periodic (with any shape or functional dependence at all), we can "build it up" out of a sum of sinusoids (with different ω's).

Thus we can solve the general case of any periodic "driver"

This is the "method of Fourier", or Fourier Series

Notation: $f(t)$ is periodic, with period τ, ("τ-periodic") if

$f(t+\tau) = f(t)$ for any/all times t.

Fourier's claim: Any τ-periodic function $f(t)$ can be uniquely written as

$$f(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin\omega t + b_2 \sin 2\omega t + \ldots$$

$$= \sum_{n=0}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \quad (\text{with } \omega = 2\pi/\tau)$$

(Any periodic fn looks like a superposition of pure sin's + cos's)

Fourier - 2.

<u>Claim</u>: If $y_1(t)$ solves $\ddot{y}_1 + 2\beta\dot{y}_1 + \omega_0^2 y_1 = \cos(\omega_1 t)$

and $y_2(t)$ solves $\ddot{y}_2 + 2\beta\dot{y}_2 + \omega_0^2 y_2 = \cos(\omega_2 t)$

then $y = c_1 y_1 + c_2 y_2$ solves $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = c_1\cos\omega_1 t + c_2\cos\omega_2 t$

(Just <u>plug it in</u>, the linearity ensures it, it's a 1-step proof!)

So since Fourier says <u>any</u> periodic $f(t) = \sum_n a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$

then apparently we know, by <u>inspection</u>, how to solve the ODE

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f(t);$$

it will simply be

$$y(t) = \sum_{n=0}^{\infty} a_n\, y_{cn}(t) + \sum_{n=1}^{\infty} b_n\, y_{sn}(t)$$

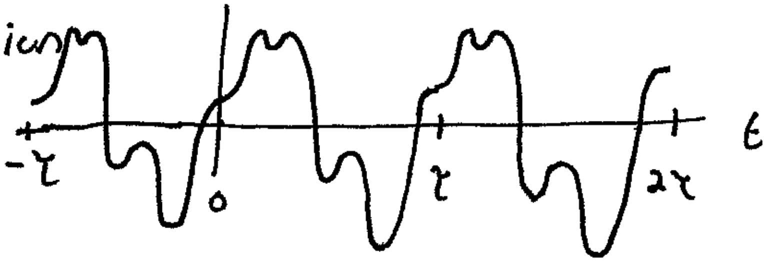
[where y_{cn} is the sol'n of our ODE driven by $\cos n\omega t$, and y_{sn} is the sol'n of our ODE driven by $\sin n\omega t$]

we just need to know the (constants!) a_n's + b_n's up here.

<u>(And we need to remember, from the last section, what the sol'ns to the ODE are.)</u>

Some examples of "τ-periodic" functions, <u>anything</u> that repeats!

- Hitting a nail once every τ seconds
- Singing a "pitch" of nominal frequency f (with overtones)
- Noisy electronic signals built on a base frequency like 60 Hz
- The function



Air pressure from An oboe??

Fourier -3

Given $f(t)$, we need a method to find those a_n's + b_n's.
(Once we know them, we immediately know the response of a damped oscillator to this driver $f(t)$, i.e. $y(t)$.)
This method is straightforward, it's "Fourier's Trick".

Let me 1st give the result
then motivate it,
+ lastly "derive" it.

Result: If $f(t)$ is "τ-periodic", then define $\omega = 2\pi/\tau$,

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \cos n\omega t,$$ with coefficients:

$$\boxed{a_n = \frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} f(t)\cos n\omega t\, dt}$$

$$\boxed{\left(a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, dt\right) \text{ is a special case}}$$

$$\boxed{b_n = \frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} f(t)\sin n\omega t\, dt}$$

(Note: See notes p. 11a if you want to know the math behind these "magic" formulas!)

That's it. We have formulas to compute a_n's + b_n's. These are definite integrals, a_n's + b_n's are all numbers, constants.

$f(t)$ is a "superposition" of pure sinusoids, (always!)
+ then the system's response, $y(t)$ is the same superposition (same coefficients) of "pure responses" (to pure frequency drivers)

Fourier - 4

Motivation: Any <u>vector</u> $\vec{V}$ can be uniquely expanded as

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

(for some "basis set" of orthogonal unit vectors, the $\hat{e}_i$'s.)

V_1, V_2: a constant, the "component"

$\hat{e}_2, \hat{e}_3$: a unit vector

If you know $\vec{V}$, you can find the components, by

$$V_i = \vec{V} \cdot \hat{e}_i \qquad (\text{so e.g } V_x = \vec{V} \cdot \hat{i} \text{ in Cartesian})$$

• Our unit vectors are orthogonal. In general, <u>any</u> 2 vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a} \cdot \vec{b} = 0$. In components

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i \, b_i = 0 \iff \text{orthogonality}$$

Now, imagine a vector in N-dimensional space. Can you see that

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{N} a_i b_i = 0 \iff \text{orthogonality}.$$

Now, let $N \to \infty$, + replace this <u>sum</u> with an integral! (This is a leap, we're making an analogy, not an equality) But the "dot product of two functions" (really called the <u>inner</u> product) $a(t)$ and $b(t)$ will be

$$\frac{\tau}{2} \int_{-\tau/2}^{\tau/2} a(t) \, b(t) \, dt .$$

(If this integral is 0, we say the fn's a and $b(t)$ are orthogonal!)

(this factor is just for later convenience.)

So just as vectors can be "expanded" in a basis set of unit vectors
So too can functions be "expanded" in a basis set of functions!

Here, $\hat{e}_1, \hat{e}_2, \hat{e}_3$ was our 3-D basis set of vectors
and $\cos(\omega t), \cos(2\omega t), \cos(3\omega t), \ldots$ etc forms our "basis set" of functions!

— Just as the coefficient $V_n = \vec{V} \cdot \hat{e}_n$, so too, ← a dot product

the coefficient $a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos n\omega t \, dt$ ← our inner product.

When I see a Fourier series, $f(t) = \sum_n a_n \cos n\omega t$,
I think of this as "expanding f in the basis fn's $\cos n\omega t$"
the coefficients a_n are numbers, just like the "components" of a vector.
Just as $\{V_1, V_2, V_3\}$ completely + uniquely defines a vector in 3D
so $\{a_0, a_1, a_2, a_3, \ldots\}$ " " " " a function

<u>Recap</u>: Any τ-periodic fn $f(t)$ (with period $\tau = 2\pi/\omega$)
can be uniquely written as

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$\hookrightarrow n=0$ is irrelevant, since $\sin 0 = 0$

you need to find those constants a_n and b_n, easy!

$$a_0 = 1/\tau \int_{-\tau/2}^{\tau/2} f(t)\, dt$$

$$a_n = 2/\tau \int_{-\tau/2}^{\tau/2} f(t) \cos n\omega t\, dt \qquad n \geq 1$$

$$b_n = 2/\tau \int_{-\tau/2}^{\tau/2} f(t) \sin n\omega t\, dt \qquad n \geq 1.$$

---

when I write $\vec{V} = \sum_{n=1}^{3} V_n \hat{e}_n$, I think of V_n as telling me "how much of $\vec{V}$ is in the $\hat{e}_n$ direction"

when I write $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$, I think of a_n as telling me "how much of the function $f(t)$ is like a <u>pure</u> sinusoid, $\cos(n\omega t)$".

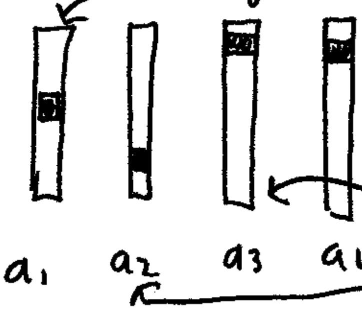
In music, different n's correspond to <u>harmonics</u>. E.g a singer singing concert A = 440 Hz = $\omega_0/2\pi$ produces a complex wave form $f(t)$

then a_1 tells me "How much is pure $\cos(\omega_0 t)$"

a_2 tells me "How much is the 1st overtone, $\cos(2\omega_0 t)$, at $f = 880$ Hz" (Not strong)

etc.

On a stereo equalizer, each knob controls the strength of the a_n's.



I have some a_1 here, so <u>some</u> pure ω_0

I have a <u>little</u> a_2 here, not much $2\omega_0$

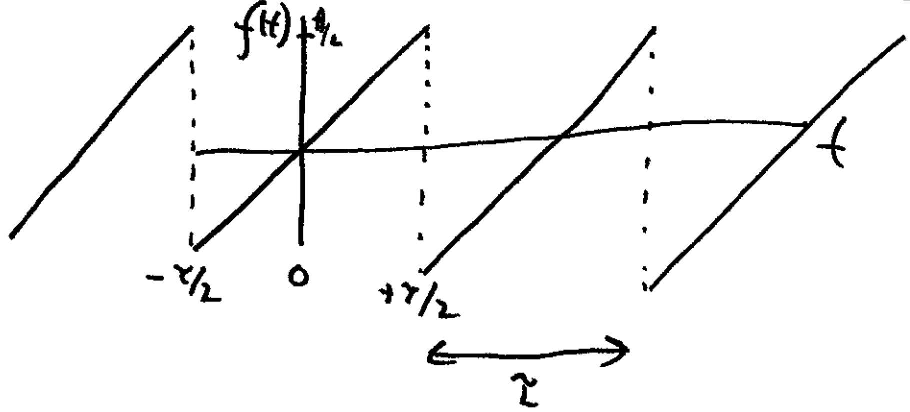
I have a <u>lot</u> of a_3, plenty of $3\omega_0$

Apparently I like "treble" here, + am adjusting the sound to emphasize the $n=3$ and 4 "high harmonics" of the base frequency

In this way, given an $\omega = 2\pi/\tau$, you can build up a complex periodic wave, with the same pitch (it's <u>still</u> periodic in τ seconds!) but with many overtones + a rich functional time dependence.

<u>Example</u>: Consider

$$\left[\begin{array}{ll} f(t) = A\dfrac{t}{\tau} & -\frac{\tau}{2} < t < +\tau/2 \\ \text{\underline{repeating} with period } \tau. & \end{array}\right]$$



- This <u>is</u> periodic, with period τ (not $\tau/2$, look at the graph!)
- This is certainly no sin wave! Not even close!

Fourier insists we can write it <u>uniquely</u> as

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t.$$

(Note: It's an <u>odd</u> fn, so I <u>suspect</u> the a_n's will all vanish! we'll see this shortly!)

Let's compute the b_n's, $b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin n\omega t\, dt$.

Here, we have $b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \frac{A}{\tau}\, t \sin n\omega t\, dt$.

Can integrate by parts! Or just let MMA do it, we get

$$b_n = \frac{2}{\tau}\cdot\frac{A}{\tau}\cdot\left[-\frac{t}{n\omega}\cos n\omega t + \frac{\sin n\omega t}{n^2\omega^2}\right]_{-\tau/2}^{+\tau/2}$$

But note $\omega = \frac{2\pi}{\tau}$, so we get $n\omega\frac{\tau}{2} \Rightarrow n\pi$

$$b_n = \frac{2A}{\tau^2}\left[-\frac{2\cdot\tau/2}{n\omega}\underbrace{\cos n\pi}_{\substack{=+1 \text{ for even } n\\ =-1 \text{ for odd } n}} + \frac{2}{n^2\omega^2}\underbrace{\sin n\pi}_{\text{always zero!}}\right]$$

$$b_n = \frac{2A}{2\pi}\cdot\frac{-1}{n}(-1)^n$$

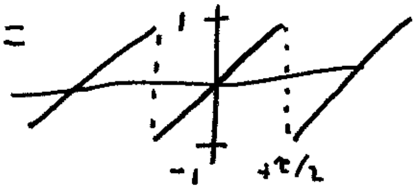
- you can do the a_0 and a_n integrals, in MMA, but no need to bother! Consider $a_n = \frac{2A}{\tau^2}\int_{-\tau/2}^{+\tau/2} \underbrace{t}_{\text{odd function}}\cdot \underbrace{\cos n\omega t}_{\text{even function}}\, dt$

But odd * even = odd,

and $\int_{-\tau/2}^{\tau/2} (\text{odd } fn)\, dt = 0$!

Let's recap + see what we've got. $b_n = \frac{A}{\pi}\frac{(-1)^{n+1}}{n}$

Let $A = 2$, so $f(t) =$



$b_1 = \frac{2}{\pi}$, $b_2 = -\frac{2}{2\pi}$, $b_3 = +\frac{2}{3\pi}$, $b_4 = \frac{-2}{4\pi}$, etc...

- As n grows, b_n shrinks. This is common – perhaps we only need a few terms to get a good approximation to $f(t)$!

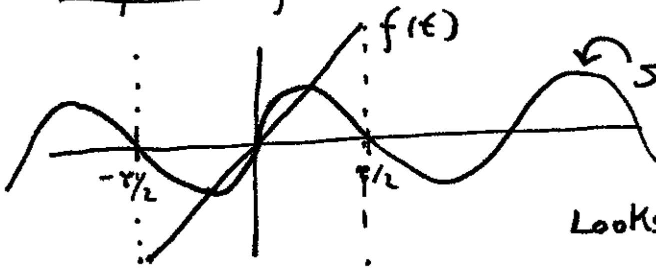
- In general, $a_n \propto \int_{-\tau/2}^{+\tau/2} f(t) \underbrace{\cos n\omega t}_{\text{even}}$

so if $f(t)$ is an odd fn, only get b_n's (sin functions)

if $f(t)$ is an even fn, only get a_n's (cos functions)

- We are sculpting $f(t)$ here out of pure $\sin(n\omega t)$ functions.

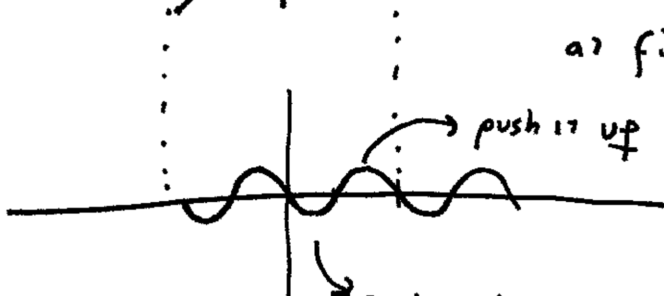
n=1 only:



$\sin \omega t = \sin \frac{2\pi t}{\tau}$

Looks like this ↑ slightly overshoots at first, then undershoots. So, fix it up!

add in n=2

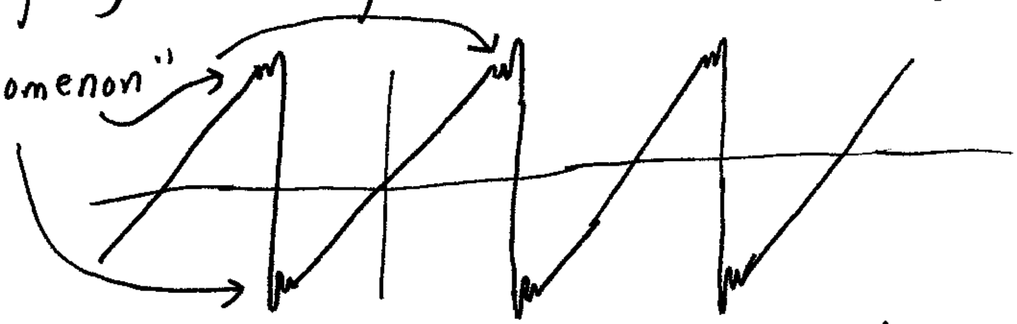


push it up a little here!

Pull it down a little, hence the negative b_2

Check out the PhET sim + try sculpting a bit yourself, you'll quickly see what those a_n's are doing!

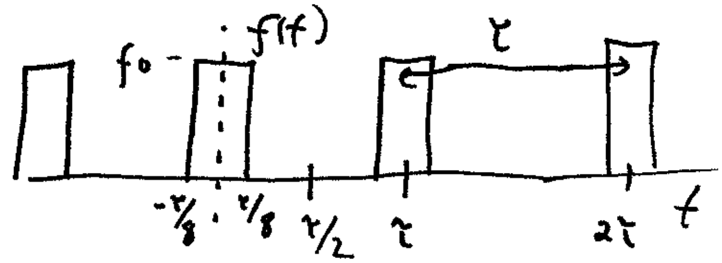
- Functions with discontinuities (like this example) turn out to generate some funny busines right <u>at</u> the discontinuities, the "Gibbs phenomenon"



If you truncate the series, there's a little "ringing" at the discontinuities. Even with $n \to \infty$, you can overshoot by ~9% " " " ! But the effect is localized to <u>just</u> the discontinuities, and of course in real life, nothing has true discontinuities!

Fourier -11-

Taylor works an example:



- It's an even function, he only gets a_n's. (No b_n's)
- It doesn't average to 0 (like $\cos(n\omega t)$ always does), so he must get an a_0 term!
- His a_n's also get smaller as n grows, slowly but surely.
- Even 4 or 5 terms is pretty good! 10 terms is a lovely fit!
- Printing 1.0 of Taylor's book has errors in the coefficients. Work it out, check for yourself! I get $\omega = 2\pi/\tau$ and

$$f(t) \approx f_0 \left(\frac{1}{4} + .45\cos\omega t + .32\cos 2\omega t + .15\cos 3\omega t + 0\cos 4\omega t + -.09\cos 5\omega t + \cdots \right)$$

If this were a sound (pressure) wave, a_0 tells you there's some steady high pressure offset (not usually present in music!!)

This wave has a lot ($a_1 = .45$) of "fundamental", pure ω, and also quite a bit ($a_2 = .32$) of "1st overtone, pure 2ω, but progressively less & less higher harmonics

- The PhET sim can produce this & then play it for you (except $a_0 = 0$) so you can here & see the waveform!

Math digression, Fourier's TRICK. Where did that mystery formula for the a_n's + b_n's come from? Let's just focus on the a_n's: Math Fact

For integers n, m:

$$\frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} \cos n\omega t \, \cos m\omega t \, dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

you can easily prove/derive this yourself! Write $\cos n\omega t = \dfrac{e^{in\omega t} - e^{-in\omega t}}{2}$ and just do the integral!

I write δ_{nm} = "Kronecker delta" $\equiv \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$

Now, assume (a la Fourier) $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$

Fourier's trick: 1st multiply both sides by $\cos m\omega t$

$$f(t) \cos m\omega t = \cos m\omega t \sum_{n=0}^{\infty} a_n \cos n\omega t$$

cos mwt can be pulled into the sum, it's the same in each term of the sum

$$= \sum_{n=0}^{\infty} a_n \cos n\omega t \cos m\omega t$$

Next, integrate both sides: (with same limits + coefficient out front)

$$\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t) \cos m\omega t \, dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \sum_{n=0}^{\infty} a_n \cos n\omega t \cos m\omega t \, dt$$

Integral of sum = sum of integrals

$$= \sum_{n=0}^{\infty} \frac{2 a_n}{\tau} \int_{-\tau/2}^{\tau/2} \cos n\omega t \cos m\omega t \, dt$$

My "math fact" from above

$$= \sum_{n=0}^{\infty} a_n \cdot \delta_{nm}$$

every term vanishes (!!) except one.

$$= a_m$$

Fourier -11- 6

What I just got was

$$\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos m\omega t = a_m.$$

this is the magic formula we've been using (the dummy index is m here, but it's just a dummy!)

What just happened? The idea is, $\cos n\omega t$ and $\cos m\omega t$ are orthogonal functions, meaning the inner product $\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\cos n\omega t\,\cos m\omega t = 0$

So if we expand $f(t)$ in this basis set of orthogonal functions then the coefficient of $f(t)$ with one basis fn is the coefficient!

Just like, if $\vec{V} = \sum_{n=1}^{3} V_n \hat{e}_n$, then $V_n = \vec{V}\cdot\hat{e}_n$

similarly, if $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$, then a_n = inner product of $f(t)$ and $\cos n\omega t$.

Fact: $\cos n\omega t$ is orthogonal to $\sin m\omega t$ $\quad \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\underbrace{\cos n\omega t}_{\text{even}}\;\underbrace{\sin m\omega t}_{\text{odd}} = 0$!

Fact: $\sin n\omega t$ " " " $\sin m\omega t$ if $n\neq m$ $\quad \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\sin n\omega t\,\sin m\omega t = \delta_{nm}$

This gives the b_n formula!

Fact: $\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\cos n\omega t\cdot 1\,dt = 2\,\delta_{n,0}$, (this is why the funny factor of $\frac{1}{2}$ differs for the a_0 formula)

Let's use Fourier series, then, to solve the general driven oscillator.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$ (with $f(t)$ a τ-periodic fn. (assumed even for simplicity!)

- Fourier says $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$ (and we know a formula for each a_n here!)

- But we already solved

$$\ddot{x}_n + 2\beta\dot{x}_n + \omega_0^2 x_n = a_n \cos n\omega t \, .$$

Remember? It has a "homogeneous" part that dies off like $e^{-\beta t}$, and leaves behind the "particular" response (we've just got a driving freq $n\omega$ instead of ω)

$$x_n(t) \underset{\substack{\text{large times,} \\ \text{after transients} \\ \text{die away}}}{=} A_n \cos(n\omega t - \delta_n)$$

where
$$\begin{cases} A_n = a_n \Big/ \sqrt{(\omega_0^2 - (n\omega)^2)^2 + 4\beta^2 (n\omega)^2} \\ \delta_n = \tan^{-1} 2\beta n\omega / (\omega_0^2 - (n\omega)^2) \end{cases}$$

By the linearity of our ODE, $x(t)$ is given very simply

$$x(t) \underset{\text{large } t}{=} \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$$

($n=0$ works just fine - check it for yourself!)

Summary:

- Given a driver $f(t)$, write it as a Fourier series (this means compute all a_n's & b_n's. We have definite integrals to do, but they can always be numerically computed!)
- For each term, calculate A_n and δ_n. (these are a little ugly, but well-defined, formula on previous page) (If you have b_n's, you'll need to go back + take $\text{Im}(z)$ when we first solved the driven SHO.)
- Add the sol'ns back up, to get the sum.

Most realistic (periodic) functions $f(t)$ will need only a few terms. This does seem like a job for a computer, but it gives excellent approximations, often needing only a couple terms

Comments:

(1) Energy of an oscillator (freq ω_n) is given (after transients die off!) by a steady $\frac{1}{2} K A_n^2$. Taylor then shows that if you drive an oscillator with a periodic $f(t)$

$$\frac{1}{2} K \underbrace{\langle x^2 \rangle}_{\text{time average}} = \frac{1}{2} K \left[A_0^2 + \sum_{n=1}^{\infty} \frac{1}{2} A_n^2 \right]$$

So knowing fourier coefficients $\Rightarrow$ A_n's $\Rightarrow$ energy of resulting response

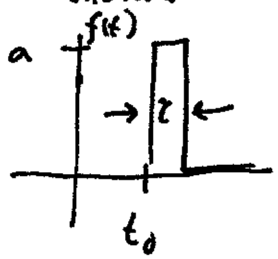
(2) If driver, $f(t) = \sum_n a_n \cos n\omega t$ needs many terms, and your system has a resonant frequency ω_0, then the "response" will be dominated by that one term in $f(t)$ where $n\omega \approx \omega_0$.

So e.g. if your driver has a low freq, $\omega \ll \omega_0$, you might not expect much response. But because of harmonics, one of them is likely to be close, $N\omega \simeq \omega_0$ for some N, so you may get a response after all! (you will respond at $N\omega$, not at ω. Response is $\approx$ resonant freq!)

Ex: If you push a kid on a swing at low f, say once every 3 swings, she can still get going... at the natural resonant frequency. It's not as good (she'll complain) because she's only picking up your "overtones"... but the system will still "resonate".

Finally, what if $f(t)$ is not periodic? It turns out we can solve for the response from any driver! We won't work out the details this term, but here's the basic logic:

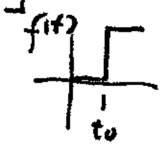
- Consider 1st how an underdamped system responds to this driver:



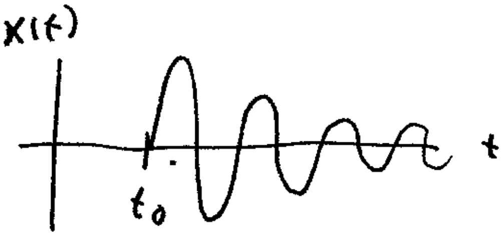
You can solve this. If $x = \dot{x} = 0$ before this "impulse", + if τ is short, then (with $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ as usual)

$$\underset{(t>t_0)}{x(t)} \approx \frac{a\tau}{\omega_1} e^{-\beta(t-t_0)} \sin\omega_1 (t-t_0)$$

Proof is a few short steps. think about solving for a "step up" force first

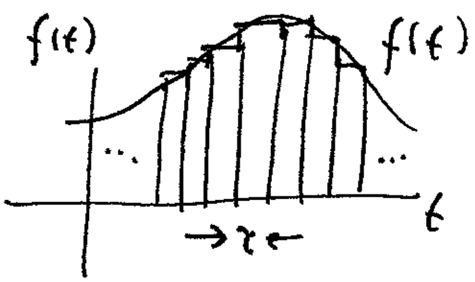


, and superpose with sol'n to "step down" force



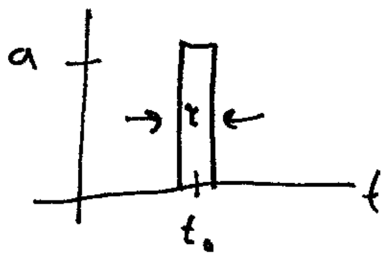
↓ whack an oscillator, it rings + then dies away, nothing unusual here.

But any function $f(t)$ is just a superposition of such "little impulses"



So you can find the response to non-periodic drivers too. This is the method of "Green's functions", and involves Fourier Transforms rather than Fourier series. (our sum of $a_n \cos n\omega t$ turns into a continuous integration over all ω, i.e. $\int_{-\infty}^{\infty} A(\omega) \cos\omega t \, d\omega$)

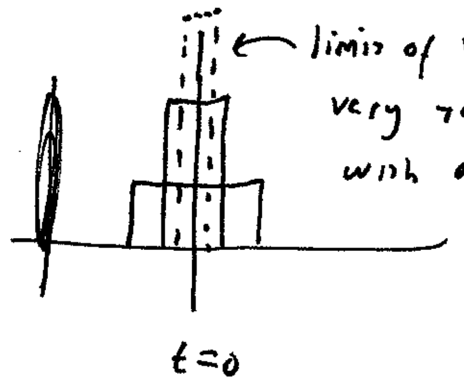
- We won't pursue this (wait a semester!), but I do want to follow up on this "short impulse" function idea!

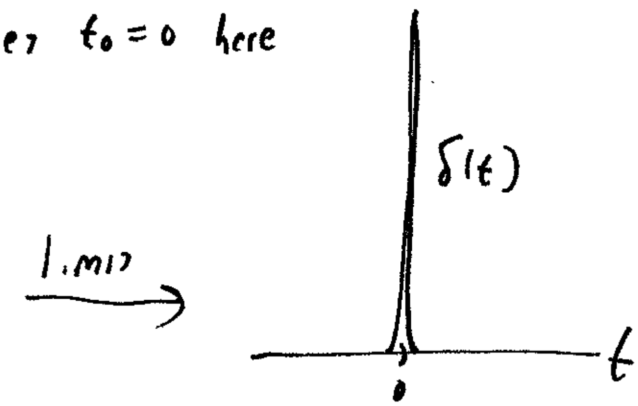


Consider a little quick impulse $f(t)$ as shown,
in the limit that $\tau \to$ small (very quick!)
$a = 1/\tau$ (very strong!)

Note that the impulse $\equiv \int F(t)\,dt = a\cdot\tau = \frac{1}{\tau}\cdot\tau = 1$ is finite.

Now take the limit $\tau \to 0$. Let's let $t_0 = 0$ here





The Dirac Delta function, $\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0, \end{cases}$

such that $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$\delta(t)$ is not a legitimate mathematical fn, but is very useful,
+ integrals involving it are not problematic!

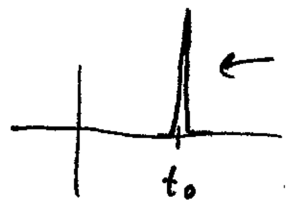
This function was introduced on previous page, but it has many applications. Let's investigate it just a bit more...

I claim $\int_{-a}^{+a} \delta(t)\, dt = 1$ ← For any a!

Limits of integration are irrelevant, since $\delta(t)$ is so narrow! It has area 1, as long as you integrate over the "spike"

Similarly, $\int_{a}^{3a} \delta(t)\, dt = 0$ because these limits don't "catch" the spike.

I argue $\delta(t - t_0)$ looks like this



← area is still 1! It's just shifted.

Now consider $\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt$. The integrand is zero for all t, except the blip at $t=a$. So $f(t)$ is irrelevant except at $t = a$!

$$\text{so} \int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = \int_{-\infty}^{\infty} f(a)\, \delta(t-a)\, dt = f(a) \int_{-\infty}^{\infty} \delta(t-a)\, dt = f(a)$$

Integrating $f(t) * \delta(t-a)$ "catches" the value of $f(a)$!

Physics: In 1-D, the charge density λ of an electron at $x=1$ would be $\lambda(x) = -e\, \delta(x-1)$.

Why? Physically, $\lambda = 0$ everywhere except $x=1$, + is infinite there (electrons are points!). Total charge, however, is

$$Q = \underbrace{\int_{-\infty}^{\infty} \lambda(x)\, dx}_{\text{as usual!}} = -e \underbrace{\int_{-\infty}^{\infty} \delta(x-1)\, dx}_{\text{one!}} = -e, \text{ as it should be!}$$

What is $\delta(Kt)$, where K is, say, a positive constant?

Think about $\int_{-\infty}^{\infty} f(t)\,\delta(Kt)\,dt$ for <u>any</u> function $f(t)$ at all!

Do a "u-sub", $u = Kt$, so $du = K\,dt$, and this integral is just

$$\int_{-\infty}^{\infty} f\left(\frac{u}{K}\right)\delta(u)\,\frac{du}{K} = \frac{1}{K} f\left(\frac{0}{K}\right)\int_{-\infty}^{\infty}\delta(u)\,du = \frac{f(0)}{K}$$

This is <u>identical</u> to $\int_{-\infty}^{\infty} f(t)\cdot\frac{1}{K}\delta(t)\,dt = \frac{f(0)}{K}$.

Since these integrals are equal for <u>any</u>/<u>all</u> functions, we equate the integrand!

So $\delta(Kt) = \frac{1}{K}\delta(t)$ if $K > 0$.

If $K < 0$, the u-sub changes the <u>limits</u> to $\int_{+\infty}^{-\infty}$, this is a sign flip!

so $\delta(Kt) = -\frac{1}{K}\delta(t)$ if $K < 0$.

or, $\boxed{\delta(Kt) = \frac{1}{|K|}\delta(t)}$

Units of $\delta(t)$? Well, $1 = \int_{-\infty}^{\infty} \underbrace{\delta(t)}_{\text{units?}}\,\underbrace{dt}_{\text{time}}$. Clearly $[\delta(t)] = \frac{1}{\text{time}}$!

We'll explore $\delta(t)$ many times in future classes!