So far, we've focused on physics problems involving ODE's: Equations for $y(x)$, functions of 1 variable. But many (most!) physics problems are richer than this. The unknown function you're after may depend on many variables.

E.g. Electric field $\vec{E}(x, y, z)$ or even (x, y, z, t)
(or Voltage, or temperature, or force, or velocity, or ...)

The eq'ns describing this function will thus involve partial derivatives, $\dfrac{\partial f(x,y,z,t)}{\partial t}$ for instance. (A partial differential eq.!)

Examples of these include:

$$\frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0.$$

we abbreviate $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \equiv \nabla^2$, and write this

$$\nabla^2 V_{(x,y,z)} = 0$$ this is called "Laplace's equation".

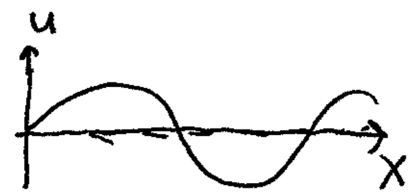
It holds for Voltage in charge-free regions
or Temperature in steady-state with no sources
+ many other physical systems.

PDE - 2.

In E+M, you'll derive that eq'n (from Gauss' Law!), as well as

$$\nabla^2 V(x,y,z) = -\frac{1}{\varepsilon_0}\rho(x,y,z)$$ "Poisson's Equation"

ρ ⟶ charge density



If $u(x,t)$ is the sideways displacement of a little string at position x, time t,

then

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

The wave equation

See Taylor Ch. 16 for a derivation

in 3-D, you can have waves, the eq'n is

$$\nabla^2 u(x,y,z,t) = \frac{1}{v^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2}$$ ⟷ 3D wave eq'n

In Quantum, the wave function $\Psi(x)$ satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(x,y,z,t) + U(x,y,z)\,\Psi(x,y,z,t) = i\hbar\frac{\partial\Psi(x,y,z,t)}{\partial t}$$

Schrödinger's Eq'n.

Each has a story, you'll solve these (+ more) many times over in upcoming classes. The sol'n is not as easy as ODE's, the "boundary condition" can strongly impact how (+whether!) you can solve it.

We're going to pick an example to see a common, general approach that works for many of the above PDE's.

The Heat Equation (or "Diffusion eq'n") is

$$\nabla^2 T(x,y,z,t) = \frac{1}{\alpha^2}\frac{\partial T(x,y,z,t)}{\partial t}$$

T = temperature.

Derivation sketched on next page.

It's a PDE, those are all partial derivatives. written out, ...

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t} \longleftrightarrow \text{in 3-D}$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T(x,t)}{\partial t} \longleftrightarrow \text{in 1-D}$$

We follow Boas ch.13 here, + use a common sol'n approach called SEPARATION of VARIABLES. *(we used that same name for a method of solving 1st order ODE's. This is totally different!)*

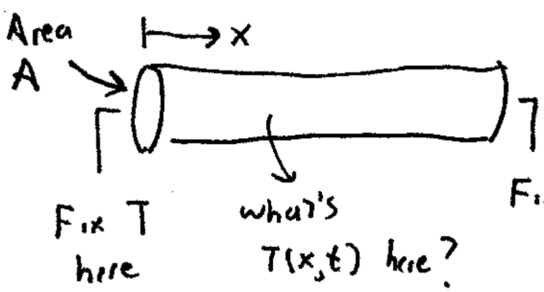
The heat equation describes how temperature varies over time and also over different positions for an ordinary solid object.

→ The eq'n can represent other physics too, like neutrons diffusing through a material. (If you look carefully, it's mathematically very similar to the Schrodinger eq'n too!) Anything that diffuses will obey an ODE like this, (here it's heat that diffuses)

As a simple concrete example, consider a 1-D solid rod. If the ends are held at different temperatures, there will be a distribution $T(x, t)$

$x \to$ depends on position

$t \to$ depends on time



I won't derive the heat eq'n, but will motivate it, + then we'll solve it (in a couple of pages from here...)

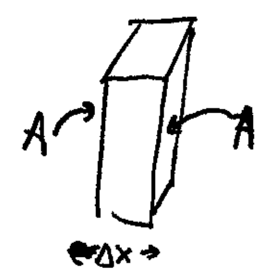
I want to start with a claim about heat flow in steady state

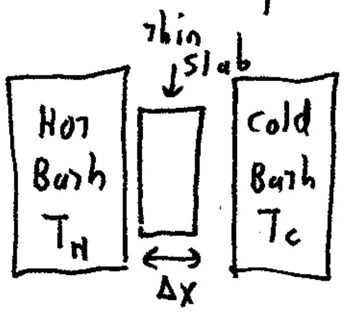
Define $H(x,t) \equiv \dfrac{\text{Amount of thermal energy passing by}}{\text{sec}}$, (In 1-D)

then I claim $H(x,t) = -K A\, \partial T/\partial x$.

$K \to$ Some "thermal conductivity constant"

This is an experimental eq'n. A is the cross sectional area of our slab





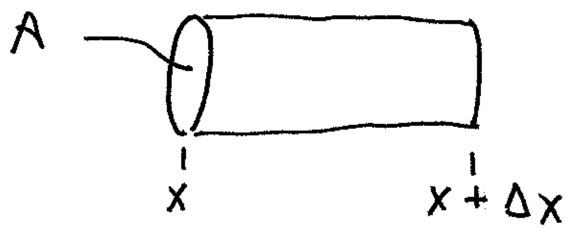
Here, $H(x,t) = \dfrac{\text{Joules passing}}{\text{sec}} = -KA\dfrac{\Delta T}{\Delta x}$

Makes sense! Big Area or $\Delta T \Rightarrow$ big heat flow.

Small/thin slab, $\Delta x \to 0 \Rightarrow$ big heat flow

Minus sign just says heat flows opposite ΔT, i.e. hot towards cold!

Now, consider our 1-D rod not yet in steady state:



Thermal energy flows in the left, that's $H(x)$ each sec

Thermal energy flows out the right, that's $H(x+\Delta x)$ each sec

Since $H = \dfrac{\text{thermal energy}}{\text{sec}}$, then $H\,\Delta t$ = energy passing through.

I claim $\underbrace{H(x)\,\Delta t}_{\text{energy in at left}} - \underbrace{H(x+\Delta x)\,\Delta t}_{\text{Energy out at right}}$ = energy buildup inside, in time Δt.

How can this be? If this is not zero, there's a net inflow of energy, + the rod must get hotter! Remember heat capacity?

Heat Capacity $C = \dfrac{\text{Joules input}}{\text{mass} * (\text{change in temperature})} = \dfrac{J}{Kg \cdot {}^\circ C}$

or $\Delta T = \dfrac{1}{C} * \dfrac{\text{"Joules in"}}{\text{mass}}$. For our rod, above,

mass = ρ * Volume = $\rho * A * \Delta x$ (see figure)

and "Joules in" = $[H(x) - H(x+\Delta x)]\,\Delta t$ (see above)

(↳ I'm assuming no other heat sources!)

so $\Delta T = \dfrac{1}{c\rho A \Delta x} \cdot \Delta t \cdot \Big(H(x) - H(x+\Delta x) \Big)$

$$\Delta T = -\frac{\Delta t}{c\rho A} \cdot \left(\frac{H(x+\Delta x) - H(x)}{\Delta x} \right)$$

(This is just $\partial H / \partial x$)

Diving out Δt, + using $H = -KA\,\partial T/\partial x$ from 2 pages ago

so $\partial H/\partial x = -KA\,\partial^2 T/\partial x^2$

$$\frac{\partial T}{\partial t} = -\frac{1}{c\rho A} \cdot -KA \frac{\partial^2 T}{\partial x^2}$$

This is what we wanted, an eq'n for T, (eliminating the function H!)

$$\boxed{\frac{\partial T}{\partial t} = \frac{K}{\rho c} \frac{\partial^2 T}{\partial x^2}} \equiv \alpha^2 \frac{\partial^2 T}{\partial x^2} \qquad \text{with } \alpha = \sqrt{\frac{K}{c\rho}}$$

In 3-D, you can guess the generalization, $\dfrac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$

If you let things settle down + reach steady state, $\dfrac{\partial T}{\partial t} = 0$

+ you get

$$\underline{\underline{\nabla^2 T = 0}}$$

(steady state)

← Laplace's eq'n, again.

The same eq'n that appears in electrostatics, (The math is identical!)

(For our 1-D rod in steady state $\Rightarrow H(x) = H(x+\Delta x) \Rightarrow \dfrac{\partial H}{\partial x} = 0$ so $\dfrac{\partial^2 T}{\partial x^2} = 0$)

using $H = -KA\dfrac{\partial T}{\partial x}$

So the PDE we must solve is $\nabla^2 T(x, y, z) = 0$ in steady state.

Looks simple. But, it's very rich + complicated! In fact, there is no generic one-size-fits all sol'n to this PDE! $T(x,y,z)$ depends not just on the eq'n $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$, but also on boundary conditions! (This is very different from ODE's!)

Let's look in lower dimensions first to learn something.

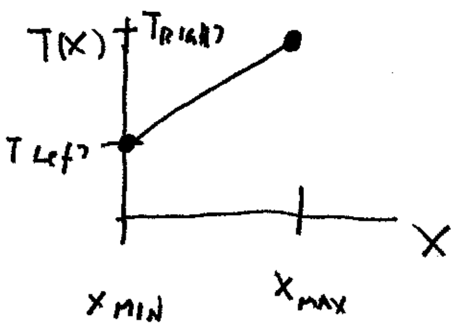
In 1-D, we have $\frac{d^2 T(x)}{dx^2} = 0$.

- Because we're in steady state, time t is gone.
- Because we're in 1-D, only x is variable

This is actually kind of trivial, because now there is only 1 variable, + we're back to an ODE!

Here, there is a generic sol'n, $T(x) = a + bx$ ← 2 undetermined constants

For ODE's, the form of sol'n is known, and Boundary conditions simple tell you what constants are. Here,



Rod temperature varies smoothly (linearly) from cold side to hot side.

[Voltage between 2 capacitor plates is same story! $\frac{d^2 V}{dx^2} = 0$]

What about 2-D? Imagine e.g a metal plate, with the edges set at fixed temperature distributions. In steady state, what is $T(x,y)$ everywhere else in the plate?

well, $\nabla^2 T(x,y) = 0$, i.e $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$

There is no generic form to solve this! you could imagine all sorts of complicated functions to try out, but if the boundary conditions are nice, we may find that

$$T(x,y) = (\text{some fn of } x) * (\text{some other fn of } y)$$

Like, say, $e^x * \cos y$ or something. (This works, it's just not very general!)

If that fails, maybe a simple linear combination of such functions like, say, $e^x \cos y + .337\, e^{2x} \cos 2y - 1.7\, e^{3x} \cos 3y + \dots$?

↳ This way, we could "build up" a lot of very complex functions!

So this will be our procedure, the Method of Separation of Variables, where we guess (hope!) that perhaps

$$T(x,y) = \mathcal{X}(x) * \mathcal{Y}(y) \quad \text{or, some sum of such fns}$$

a fn of x only * a fn of y only.

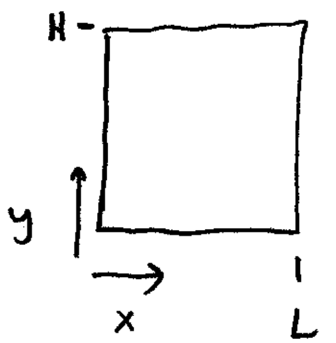
Remember, our goal is 2-fold:

a) Find $T(x,y)$ such that $\nabla^2 T(x,y) = 0$

b) $T(x,y)$ must satisfy our particular, given boundary condition.

This method (postulating $T = X(x)\,Y(y)$) won't always work, but it often will, + is quite general + powerful. This approach can also be used for all the ODE's I listed on pp. 1-2)

Let's pick a concrete example where this works.



Consider our 2-D metal plate, $\nabla^2 T(x,y) = 0$

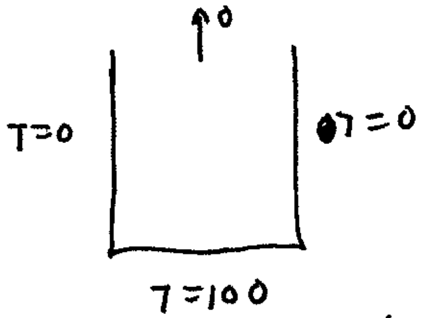
we want $T(x,y)$ (Steady state).

As a first example, let us refrigerate ~~Three~~ Two sides (left, + right,), fix $T = 0^\circ$ on those edges. Let's also fix $T = f(x)$ along the bottom. This could be anything, it's our choice. A simple case might be to put boiling water along this edge, so $T(x, y=0) = 100^\circ$. (But we can solve more complicated cases too.)

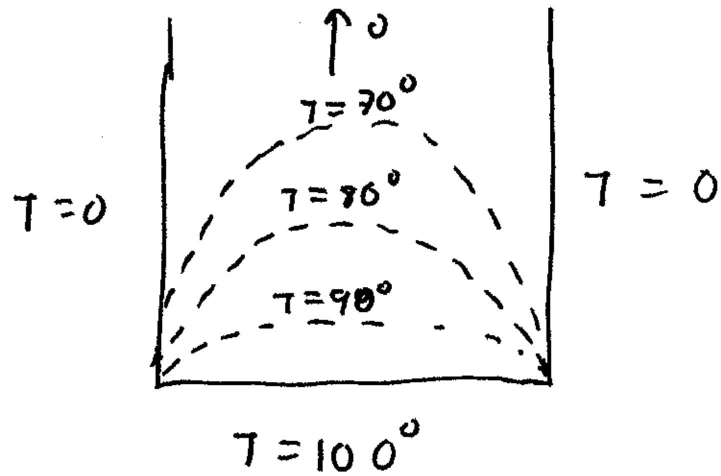
For the top, let's begin with a tall plate, so Height $H \to \infty$. (I'll assume the top is like the 2 sides, at 0°, just very far away)

PDE - 9.5

Before we proceed, what do we expect, physically?



For large y, I expect $T \approx 0$ everywhere. It's refrigerated all around! But, near the bottom, heat will flow from the hot base towards the cool walls. In steady state, I expect strong temp variation down there. Just guessing I predict some "equi-temperature" lines that might look like this:

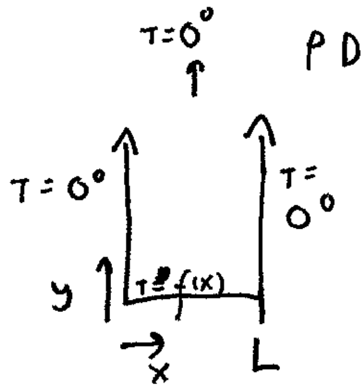


- I expect left/right symmetry
- I expect slow cooling as y increases up the middle
- Rapid variation near corners

The 2 bottom corners are odd, some sort of discontinuity there. (This is an artifact of my choice of $T = 100°$ at base + $T=0$ on sides, it is discontinuous at the corner! In real life, we wouldn't have this, and I expect to see some artifact of the discontinuity in our sol'n)

So, let's now do the math to get a formula for $T(x,y)$

Such that $\begin{cases} \nabla^2 T = 0 \text{ everywhere} \\ T \text{ (at Boundaries)} = \text{what we specified here.} \end{cases}$



$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

what is $T(x,y)$? Let's try separation of v'bles

so assume (hope!) $T(x,y) = \mathcal{X}(x)\,\mathcal{Y}(y)$ + see what happens!

1) $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{d^2 \mathcal{X}(x)}{dx^2}\,\mathcal{Y}(y)$ → $\mathcal{Y}$ is a constant as far as $\dfrac{\partial}{\partial x}$ is concerned

↳ This is a regular deriv, since $\mathcal{X}$ depends only on x

2) $\dfrac{\partial^2 T}{\partial y^2} = \mathcal{X}(y)\,\mathcal{Y}''(y)$ ← simpler notation, same as $\dfrac{d^2\mathcal{Y}(y)}{dy^2}$.

So my PDE is $\mathcal{X}''(x)\,\mathcal{Y}(y) + \mathcal{Y}''(y)\,\mathcal{X}(x) = 0$.

Key trick in sep. of v'bles: Divide both sides by $\mathcal{X}(x)\,\mathcal{Y}(y)$!

Leaving $\underbrace{\dfrac{\mathcal{X}''(x)}{\mathcal{X}(x)}}_{\text{a fn only of } x} + \underbrace{\dfrac{\mathcal{Y}''(y)}{\mathcal{Y}(y)}}_{\text{a fn only of } y} = 0$

a fn only of x + a fn only of y = 0. For all x and y!

Huh? This looks nuts! x and y are independent! I can pick an x and vary y, and this eq'n says I always get 0! How can that be? It cannot, unless these "functions" don't depend on x or y!

so $\dfrac{X''(x)}{X(x)} + \dfrac{Y''(y)}{Y(y)} = 0$

requires

this is some constant $+c$

this is some constant, must be $-c$.

(It can't depend on x!)

If both are true... we have a sol'n!

"c" is called the separation constant.

so $X''(x) = c\,X(x)$. Well, I recognize this, it's an ODE.

The sol'n is familiar, $X(x) = a_1 e^{\sqrt{c}\,x} + a_2 e^{-\sqrt{c}\,x}$

If $c > 0$, pure exponentials

If $c < 0$, pure sin's + cos's.

To have $\nabla^2 T = 0$, need

Same constant, opposite sign!

At same time, $Y''(y) = -c\,Y(y)$

I know this ODE too,

$Y(y) = a_3 e^{\sqrt{-c}\,y} + a_4 e^{-\sqrt{-c}\,y}$

Here, if $c > 0$, pure sin's + cos's

If $c < 0$ pure exponentials.

Now, remember our particular problem. Boundary conditions are needed to proceed!

For our specific problem, we said $T(x,y) \to 0$ as $y \to +\infty$

sin's + cos's (y) wiggle, they don't settle down to 0.

$e^{-(\text{something})y}$ does what we want, it goes to 0 as $y \to +\infty$.

So for this specific problem with these particular boundary conditions, looks like we need $C<0$ to give us the exponential fn in y.

So let's rename our constant $C = -K^2$
↳ so it's obviously neg!

(This works out nicely, because it gives sin's + cos's in $\mathfrak{X}(x)$, which is needed to get T to vanish at two sides!)

So $\mathfrak{X}(x) = a_1 \sin Kx + a_2 \cos Kx$

$\mathcal{Y}(y) = a_3 e^{+Ky} + a_4 e^{-Ky}$

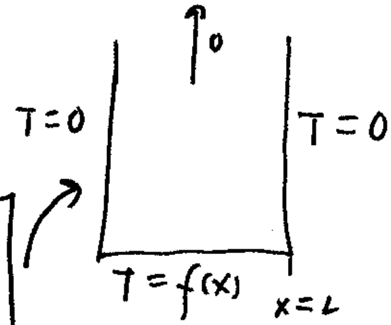
↳ our B.C. as $y \to \infty$ also tells us $a_3 = 0$!
otherwise $\mathcal{Y}(y)$ would blow up.

Now remember, $T(x,y) = \mathfrak{X}(y)\,\mathcal{Y}(y)$, so we have

$T(x,y) = (a_1 \sin Kx + a_2 \cos Kx)\, e^{-Ky}$. This is our trial sol'n

No need to include a_4 any more, just absorb it into a_1 and a_2.

This satisfies $\nabla^2 T = 0$, and our Bound. Condition as $y \to +\infty$.



We have more BC's!

[we must have $T(x,y)=0$ for all/any values of y, whenever $x=0$]

the left edge B.C.

Plug in $x=0$, to get

$$T(x=0,y) = (a_1 \cdot \sin(0) + a_2 \cos(0))\, e^{-Ky} = 0$$

How can that be true for all y? $a_2 e^{-Ky} = 0$? for any y?!

only if $a_2 = 0$!

So if $a_2=0$, $T(x,y) = a_1 \sin Kx \, e^{-Ky}$ is our trial sol'n.

What about the right wall, where $x=L$? We need $T(x=L, \text{any } y)=0$

so $T(L,y) = a_1 \sin KL \, e^{-Ky} = 0$ for all values of y.

Could try $a_1 = 0$, but that's a FAIL, because then

$T(x,y) = 0$. Doesn't work at the bottom edge! And, is awfully trivial!

Are we stuck, did we fail? We're just trying to find something that works. Remember, K was a separation constant, we don't yet know what it is. Let's choose it, so that $\sin(KL)=0$.

i.e, pick K so that $KL = n\pi$. Any n (integer!) will work.

Let's label it $K_n \equiv \frac{n\pi}{L}$. Many different K's all work!

So check it out: Any trial function of the form

$$T_n(x,y) = a \sin K_n x \; e^{-K_n y} \quad , \text{ with } K_n \equiv n\pi/L$$

solves $\nabla^2 T = 0$ (by construction)

+ satisfies 3 of our 4 BC's. (Left, right, + "top" are all good)

Any integer n gives a different, yet valid sol'n.

we still need to satisfy our B.C. at bottom, $T(x, y=0) = f(x)$ ← given.

Useful observation:

If $T_1(x,y) = X_1(x)\, Y_1(x)$ solves $\nabla^2 T_1 = 0$

then so does $c_1 X_1 Y_1$ (for any c_1, ← this is linear!)

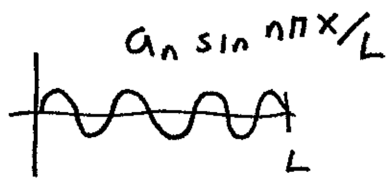
If $T_2(x,y)$ solves $\nabla^2 T_2 = 0$, then

$$\nabla^2 (C_1 T_1 + C_2 T_2) = \nabla^2 C_1 T_1 + \nabla^2 C_2 T_2 = 0 \quad \text{too!}$$

So, if we have multiple valid sol'ns T_n, we can always

form $\sum_{n=1}^{\infty} C_n T_n(x,y)$, + this too will satisfy $\nabla^2 T = 0$!

The $T_n(x,y)$ at the top of the page gives, all by itself

$T_n(x,0) =$ 

.

If our given $f(x) = T(x,0)$ were a pure sin fn, like $f(x) = \sin \frac{17\pi x}{L}$ we'd be done. (Just pick $n = 17$)

But if not, we're ok, build what we want!

$$T(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-\frac{n\pi y}{L}}$$

By linearity, this combo still satisfies $\nabla^2 T = 0$. And, convince yourself, it also still satisfies ALL 3 other boundary c's.

we can choose these at will!

we will pick them to satisfy B.C. #4,

$$T(x,0) = \underbrace{f(x)}_{\text{given}} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

since $y=0$, $e^{-\frac{n\pi y}{L}} = 1$ in every time!

This is a Fourier sum. I know how to find these constants.

Remember? $a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

Before, we called these the b_n's. Also, before we integrated from $-T/2$ to $T/2$, but this is fine, it's just a shift of origin.

Check it out: We found our sol'n! $\nabla^2 T(x,y) = 0$, + we satisfy all B.C.'s. And, here's some joy, there is a uniqueness theorem that says if we solve Laplace's eq'n + our B.C's, there is no other sol'n. We're done!!

Suppose e.g. $f(x) = 100°$ along the base. So, we need

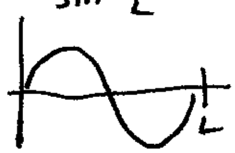
$$T(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = 100$$

thus
$$a_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L} \cdot 100\, dx = \frac{200}{L}\cdot\frac{L}{n\pi}\, - \cos\frac{n\pi x}{L}\Big|_0^L$$
$$= \frac{200}{n\pi}\left(1 - \cos n\pi\right)$$
$$= \frac{200}{n\pi}\begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd}\end{cases}.$$

Add up: $\sin\frac{\pi x}{L}$

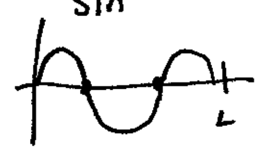
Lots of this, $a_1 = 400/\pi$

\+ $\sin\frac{2\pi x}{L}$



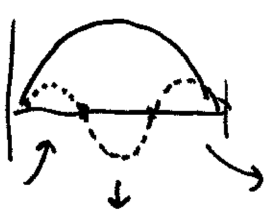
None of this! Bad symmetry, why would left + right half be different? $a_2 = 0$!

\+ $\sin 3\pi x/L$



Some of this, but less 'cause of $\frac{1}{n}$ factor

when add



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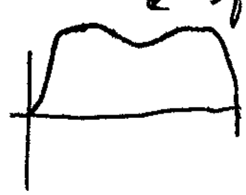
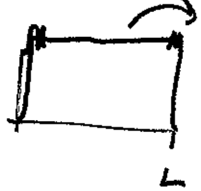


squared off the top a bit!

After many terms, we get



constant, $T = 100$.

Gibbs "rings" at edges, 'cause of our ugly discontinuous corners.

Don't forget, that was all just to find the a_n's by looking at the boundary, $y=0$. The full sol'n is

$$T(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\, e^{-n\pi y/L}$$

$$\boxed{T(x,y) = \sum_{\substack{n=1 \\ (\text{odd } n \text{ only!})}}^{\infty} \frac{400}{n\pi} \sin\frac{n\pi x}{L}\, e^{-\frac{n\pi y}{L}}}$$

Full sol'n $\forall\, x, y$.

Mustn't forget this!

This does produce the temp pattern we expected back on p. 9.5

- It's left-right symmetric because of $\sin\frac{(\text{odd } n)\pi x}{L}$
- It dies off (exponentially) as you climb in y.
- A few terms is probably enough to get a good approximation

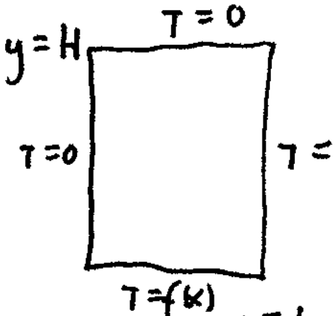
This solved $\nabla^2 T = 0$ for very specific, (artificial) boundary c's.

Let's consider some more cases.

- If $f(x)$ is something besides $100°$ along the base, no problem.

  Just recompute the $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$, that's all that changes.

- what if it had a finite height, $y = H$, with $T=0$ at the top?



Since $T(x=0) = T(x=L) = 0$, we must have

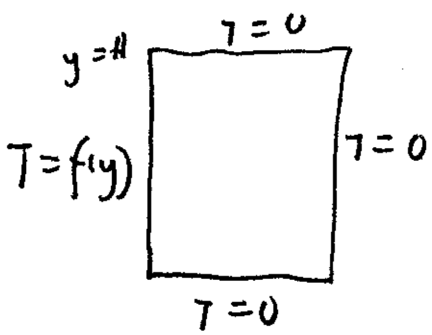
- $X(x)$ = sinusoidal. ($ae^{Kx} + be^{-Kx}$ never vanishes twice along x, for any a,b.)

So, we still want our separation constant $= -K^2$

so $X''(x) = -K^2 X(x)$ gives $a_1 \sin Kx + a_2 \cos Kx$

But now, $Y(y) = a_3 e^{+Ky} + a_4 e^{-Ky}$, we need both terms to ensure $Y(H) = 0$. Otherwise, the process is the same, + we can solve this problem pretty much the same as before.

- What if we had a B.C. on the left wall that was the non-zero one?



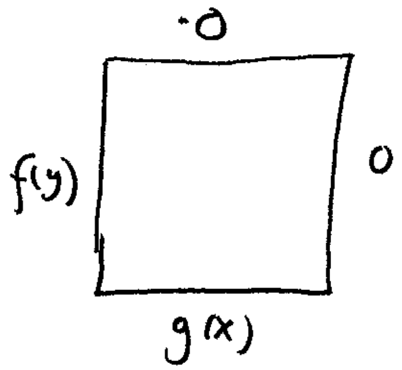
Now I want T to vanish for two y-values so I need the separation constant to have the other sign, giving

$X''(x) = +K^2 X(x) \rightarrow$ these give e^{Kx} and e^{-Kx}

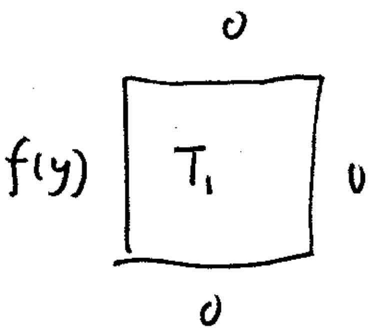
$Y''(y) = -K^2 Y(y) \longrightarrow \sin Ky$ and $\cos Ky$

It's like what we did, but swapping $x + y$!

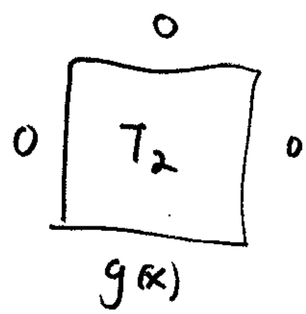
• What if



? Use superposition!! The idea would be to solve

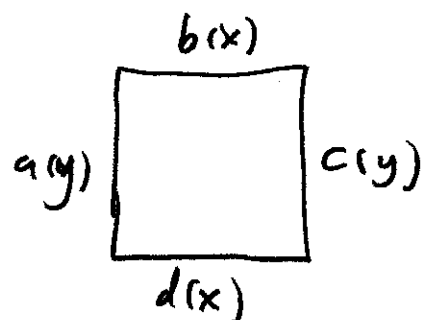


and add



, $T_1 + T_2$ will work!

So we can in fact solve



By summing 4 sol'ns!

Pretty general problem ↗ !

So our method is really more robust/general than it may have 1st appeared.

PDE -19-

Recap To solve $\nabla^2 T(x,y) = 0$ (or any other PDE!)

1) Hope that $T(x,y) = X(x)\, Y(y)$ (just try!)

2) Plug this into the ODE, (partials all become "total derivatives")

Divide through by T, + discover you have separate ODE's

with some new (as yet undetermined) separation constants

$X''(x) = +k^2 X(y)$ $\qquad$ $Y''(y) = -k^2 Y(y)$

a constant to be determined

the same constant, but opposite in sign, to make $\nabla^2 T = 0$.

Pick the sign of the constant depending on whether you B.C.'s need $X(x)$ to be sinusoidal, $Y(y)$ exponential, or vice versa

3) Solve the separate ODE's, + make sure $T(x,y)$ satisfies your B.C.'s (one by one)

This will fix your separation constant (there may be many options) and most other ODE constants

4) You can sum up valid sol'ns (superposing) if that helps!