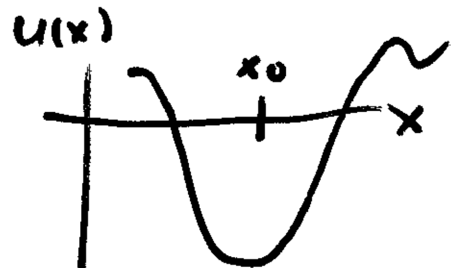
Perhaps the most ubiquitous motion in the universe: "oscillatory motion". Results when any system is moved away from stable equilibrium.

Last chapter, we noted <u>any</u> $U(r)$ can be Taylor expanded

$$U(x) \approx \underbrace{U(x_0)}_{\text{irrelev. constant}} + \underbrace{U'(x_0)}_{=0 \text{ if } x_0 = \text{equilib}}(x-x_0) + \frac{U''(x_0)}{2!}(x-x_0)^2 + \ldots$$



Motion around <u>here</u> is simple!

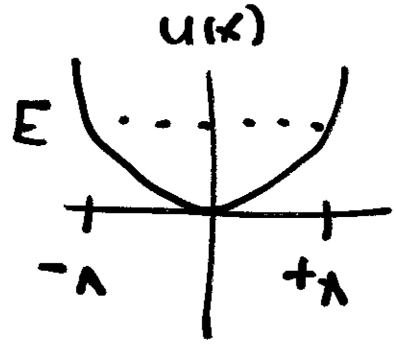
So $U \sim cx^2$ and $F = -kx$

It's "mass on a spring", Hooke's law. A good <u>model</u> for ~~many~~ any situation!

We'll start here, + then add physics (e.g. <u>damping</u> or time-dependent <u>driving</u>) to get ever richer models

Examples: Molecules, quarks in nuclei, RLC "oscillator" circuits crystals, ...

When $F = -Kx$, $U(x) = -\int_{x_0}^{x} \vec{F}\cdot d\vec{x} = -\int_0^x (-Kx')\,dx' = \frac{1}{2}Kx^2$



This is SHM, simple harmonic motion

"A" is the amplitude (turnaround pt)

At endpoints, it's clear $E = \frac{1}{2}KA^2$.

Newton: $F = ma$ so $m\ddot{x} = -Kx$, or

$$\ddot{x} = -\omega^2 x \quad \text{with } \omega \equiv \sqrt{K/m}$$

We've seen this ODE many times (Ch.1 "pendulum", e.g.)

It's a 2nd order linear, homogenous ODE.

So, expect 2 linearly independent sol'ns.

We know the sol'n, it's:

$x(t) = B_1 \cos \omega t + B_2 \sin \omega t$

<u>Check</u> (plug it in + see that it works).

sin & cos are independent fns, B_1 & B_2 are any constants (determined by <u>initial</u> <u>conditions</u>)

There are other ways to write / think about this sol'n, like e.g $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

So, to see this, let's take a brief digression for

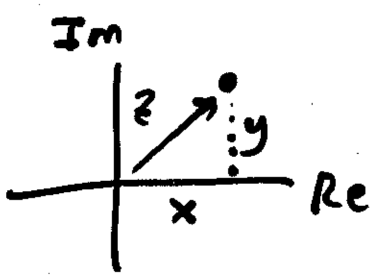
$\boxed{\text{COMPLEX \#'S}}$ (See Boas ch 2, + Taylor 2.6.)

These arise from $\sqrt{\text{neg \#'s}}$. We define $i \equiv \sqrt{-1}$

A general complex # is $z = x + iy$

$$\equiv Re(z) + i\, Im(z)$$

you can "draw" any complex # in the "complex plane"



This reminds me of polar coords, + we define

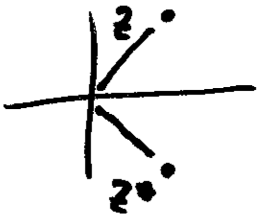
$$|z| = \sqrt{x^2+y^2} = \sqrt{Re(z)^2 + Im(z)^2}$$

Modulus of z, or magnitude.

Note that x^2 & y^2 are both positive (the "i" is not part of y, it's been pulled out)

Definition: If $z = x + iy$, then

$z^* = x - iy$ = "complex conjugate"



<u>Note</u> $z z^* = x^2 - i^2 y^2 = x^2 + y^2$, so

$$|z| = \sqrt{z z^*}$$

Can also divide, <u>e.g</u> $\frac{2+i}{3-i} = \frac{2+i}{3-i} \cdot \left(\frac{3+i}{3+i}\right)$ ← the trick!

$$= \frac{6 + 5i + i^2}{9+1} = \frac{5+5i}{10} = \frac{1}{2} + \frac{i}{2}$$

↳ careful!

Recall our Taylor series:

$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ for <u>any</u> x

$\sin x = x - x^3/3! + \cdots$ "

$\cos x = 1 - x^2/2! + \cdots$ "

Now check it out: $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \underbrace{\frac{(i\theta)^3}{3!}}_{= -i\theta^3/3!} + \cdots$

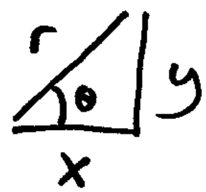
$$= 1 - \frac{\theta^2}{2!} + \cdots + i\left(\theta - \theta^3/3! + \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler's Formula.

<u>Very</u> useful!!

just look at picture!

Using Euler:



$$z = x + iy = r(\cos\theta + i\sin\theta)$$
$$= |z|\, e^{i\theta}$$

Multiplying + Dividing is very easy with this notation:

If $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = (r_1 r_2)\, e^{i(\theta_1 + \theta_2)}$$

$$z_1 / z_2 = \left(r_1/r_2\right) e^{i(\theta_1 - \theta_2)}$$

Power $(z_1)^\alpha = r_1^\alpha\, e^{i\alpha\theta_1}$

Convince yourself! $i = e^{i\pi/2}$

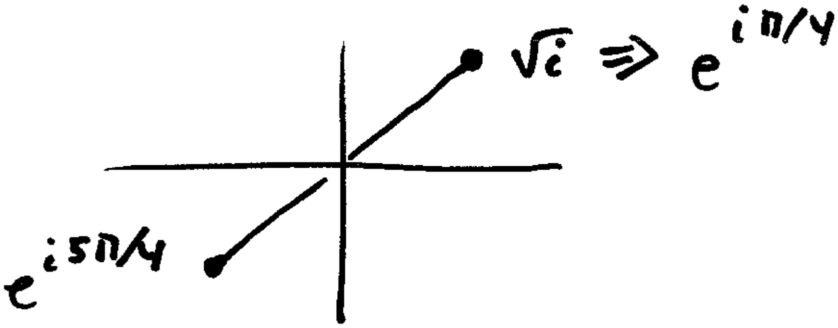
so e.g $\sqrt{i} = (i)^{1/2} = \left(e^{i\pi/2}\right)^{\frac{1}{2}} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

Actually there's an<u>other</u> root, because $i = e^{i5\pi/2}$ also!

This is $e^{i(2\pi + \pi/2)}$

and $\left(e^{i5\pi/2}\right)^{\frac{1}{2}} = e^{i5\pi/4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

(There are no different answers, though.)



Osc - 6.

Note: $e^{i\theta} = \cos\theta + i\sin\theta \quad = \quad \mathrm{Re}\, e^{i\theta} + i\, \mathrm{Im}\, e^{i\theta}$

$\underline{e^{-i\theta} = \quad " \quad - i \quad "} \quad = \quad \underline{" \quad - i \quad "}$

$$\Rightarrow \cos\theta = (e^{i\theta} + e^{-i\theta})/2 \quad = \quad \mathrm{Re}\, e^{i\theta}$$

$$\sin\theta = (e^{i\theta} - e^{-i\theta})/2i \quad = \quad \mathrm{Im}\, e^{i\theta}$$

These will prove handy to us soon when we have integrals like

$$\text{e.g.} \int_{-\pi}^{\pi} \cos 3x \cos 2x \, dx = \int_{-\pi}^{\pi} \frac{e^{i3x} + e^{-i3x}}{2} \cdot \frac{e^{i2x} + e^{-i2x}}{2} \, dx$$

$$= \int_{-\pi}^{\pi} \frac{e^{i5x} + e^{-ix} + e^{+ix} + e^{-i5x}}{4} \, dx = \int_{-\pi}^{\pi} \frac{\cos 5x + \cos x}{2} \, dx$$

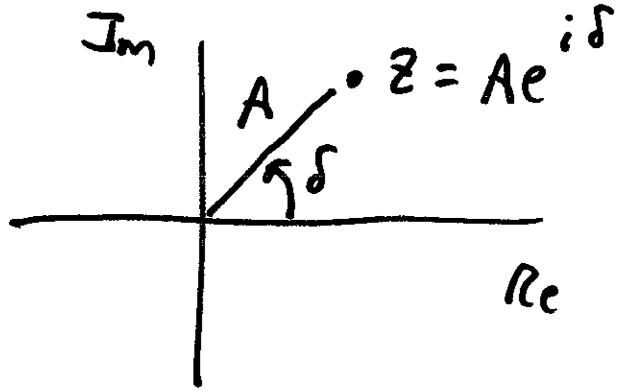
$$= \frac{1}{10} \sin 5x \Big|_{-\pi}^{\pi} + \frac{1}{2} \sin x \Big|_{-\pi}^{\pi} = 0 + 0 = 0 \; !$$

(we'll return to this soon)

Bottom line

Picture complex #'s

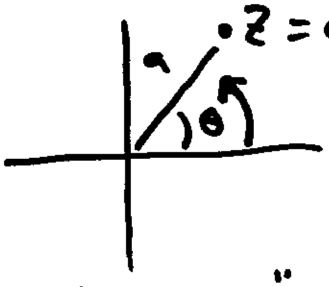
as points in complex plane



What's this got to do with our 2^{nd} order ODE?

SHM is very closely connected to simple rotations.

Consider a point rotating (ccw) in the complex plane



$z = ae^{i\theta}$ with steady rate, $\theta = \omega t$

i.e. $z(t) = ae^{i\theta(t)} = a\cos\omega t + i\,a\sin\omega t$

So "$a\cos\omega t$" & "$a\sin\omega t$" are Re & Im parts of this $z(t)$

Return to $\ddot{x} = -\omega^2 x(t)$

we know $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$ works (check! ↳ plug it in)

(2 constants, 2 independent fns, as there should be).

But we can relate these to our familiar sin's + cos's:

$$x(t) = C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t)$$

$$= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$$

$$= B_1\cos\omega t + B_2\sin\omega t$$

So these 2 different "forms" of sol'n are mathematically equivalent.

$B_1 = C_1 + C_2$, $\quad B_2 = i(C_1 - C_2)$

or $C_1 = \frac{B_1 - iB_2}{2}$ $\quad C_2 = (B_1 + iB_2)/2$

Consider a <u>specific</u> initial condition, $x(0) = A$, $\dot{x}(0) = 0$.

$$X(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \quad \overline{\overline{\text{or}}} \quad x = B_1 \cos\omega t + B_2 \sin\omega t$$

$$X(0) = A = c_1 + c_2 \quad \overline{\overline{\text{or}}} \quad A = B_1 = 0$$

$$\dot{x}(t) = C_1 i\omega e^{i\omega t} - C_2 i\omega e^{-i\omega t} \quad \overline{\overline{\text{or}}} \quad \dot{x}(t) = -B_1\omega\sin\omega t + B_2\omega\cos\omega t$$

$$\text{so } \underbrace{\dot{x}(0) = 0 = C_1(i\omega) - C_2 i\omega}_{} \quad \overline{\overline{\text{or}}} \quad \underbrace{0 = B_2\,\omega}_{}$$

conclusion $C_1 = C_2$ $\qquad B_2 = 0$

and, $C_1 = C_2 = A/2$ $\qquad$ and $B_1 = A$

so

$$x(t) = \frac{A}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \quad \overline{\overline{\text{or}}} \quad x = A\cos\omega t$$

$$= A \,\mathrm{Re}\, e^{i\omega t}$$

of <u>course</u>, these are the <u>same</u> sol'n!

It's powerful, in general, to <u>think</u> of SHM as Re (complex sol'n)

$e^{i\omega t}$ is called a "phasor". Think of it as a complex representation of a sin wave. (And, convince yourself that if $X(0) = 0$, $\dot{x}(0) = +V_0$, then $X(t) = \frac{V_0}{\omega}\sin\omega t$)

Recall, we're solving $\ddot{x} = -\omega^2 x$, + found general sol'n

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \quad \underline{\underline{\text{or}}} \quad B_1 \cos\omega t + B_2 \sin\omega t$$

these are <u>equivalent</u>, $B_1 = C_1 + C_2$, $B_2 = i(C_1 - C_2)$

In <u>either</u> case, these are periodic functions, with

Period $T = 2\pi/\omega$ $\left(= 2\pi\sqrt{m/k} \text{ for mass on spring}\right)$

Proof for $e^{i\omega t}$: $e^{i\omega(t+T)} = e^{i\omega t} e^{i\omega T} = e^{i\omega t} e^{i\omega 2\pi/\omega}$

$$= e^{i\omega t} \cdot e^{i 2\pi}$$

<u>definition</u> of a period!

$$= e^{i\omega t}$$

<u>Period</u> is <u>independent</u> of B's or C's!

Note that if $\ddot{z} = -\omega^2 z$, with $z = \text{Re}\,z + i\,\text{Im}(z)$,

then $\text{Re}\,\ddot{z} = -\omega^2\,\text{Re}\,z$; so when you <u>find</u> a complex sol'n, you can always "take the real part" + you <u>still</u> have a sol'n! This is one way to understand why we use complex sol'ns to "real" physics problems, the <u>math</u> of $e^{i\omega t}$ is <u>easier</u> than sin's + cos's!

there's yet a 3rd way to write / think of our general sol'n (It's particularly useful when $x(0)$ is nothing special, i.e. neither "A" nor "0") I claim

$$x(t) = \underbrace{A \cos(\omega t - \delta)}_{\text{2 independent constants, as needed!}}$$

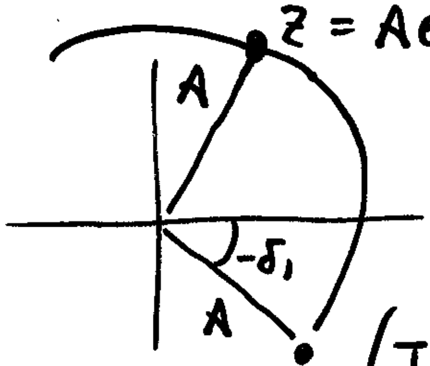
is again equivalent, a general sol'n to $\ddot{x} = -\omega^2 x$

Just use $\cos(a+b) = \cos a \cos b - \sin a \sin b$, so

$$x(t) = \underbrace{(A\cos\delta)}\cos\omega t + (A\sin\delta)\sin\omega t$$

$$= B_1 \cos\omega t + B_2 \sin\omega t,$$ ah ha, it is

our exact same general sol'n! Picture this



$z = Ae^{i(\omega t - \delta)}$, and $\mathrm{Re}\, z = A\cos(\omega t - \delta)$

This is just simple rotation in complex plane. This sol'n is again equivalent. (It's useful when $x(0) = A \cos\delta$ is not at the "extreme" spot)

(See Taylor p.167 for more!)

In this form, $x(t) = A\cos(\omega t - \delta)$, it's also particularly easy to look at <u>energy</u>.

Note: $\dot{x}(t) = -A\omega \sin(\omega t - \delta)$

so $KE = \frac{1}{2} m \dot{x}^2(t) = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t - \varphi)$

$PE = \frac{1}{2} k x^2(t) = \frac{1}{2} k A^2 \cos^2(\omega t - \varphi)$

But $\omega^2 = k/m$, so $m\omega^2 = k$, and thus

$$T + U = \frac{1}{2} k A^2 \left(\sin^2(\ldots) + \cos^2(\ldots) \right) = \frac{1}{2} k A^2$$

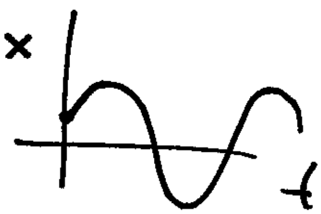
Energy is conserved, with the value we noted back on p.1!

In general, for SHM "Simple Harmonic Motion"

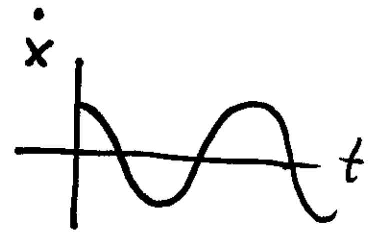
- Periodic (sinusoidal) motion, $x(t) = A\cos(\omega t - \delta)$
- $F = -kx$ Hooke's law, force opposes motion
- $U \propto x^2$ Quadratic potential energy
- Period is independent of Amplitude
- there are 2 independent "constants" determined by initial conditions (e.g $x(0)$ & $\dot{x}(0)$)

- The 2 "conditions" can also be $x(t_0)$ & $\dot{x}(t_0)$ at some time, (or T and U at some time) or A and δ ("Amplitude and phase shift")
- If you know $x(t_0)$ & $\dot{x}(t_0)$, you know x at all times!

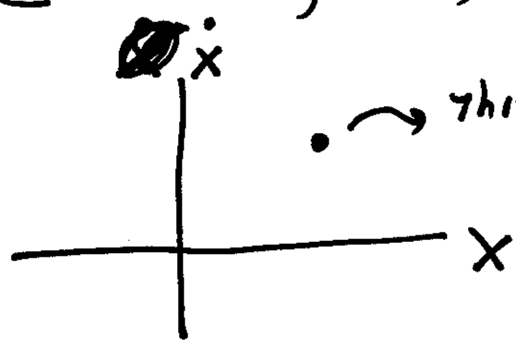
you can visualize



or

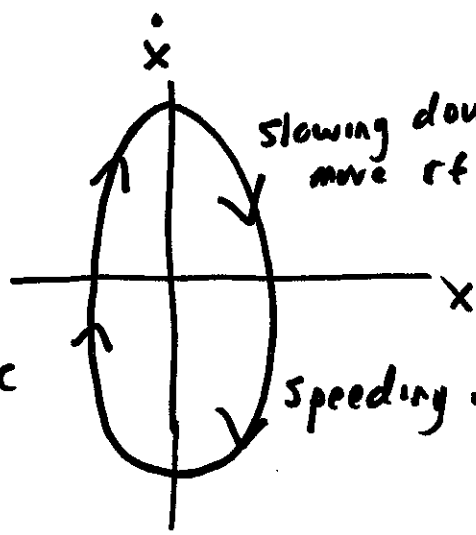


or something new, a "phase space diagram":



this point tells you x & $\dot{x}$ at one time, + then you can watch this point move around as time goes by

So e.g



slowing down, as we move rt-wards

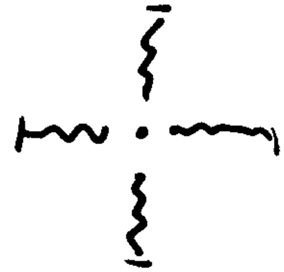
ex

speeding up (neg), moving leftwards

2-D Harmonic Motion: If you have restoring forces

in 2-D, $m\ddot{x} = -k_x x$

$m\ddot{y} = -k_y y$

e.g 

If $k_x = k_y = m\omega^2$, this is isotropic

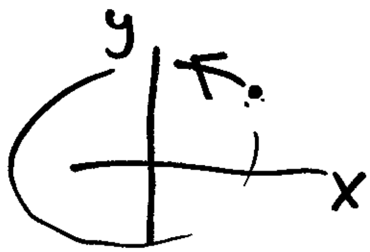
If $k_x \neq k_y$, this is anisotropic

$\boxed{\text{If isotropic,}}$: $x(t) = A_x \cos(\omega t - \delta_x)$

$y(t) = A_y \cos(\omega t - \delta_y)$

2 constants needed for each dimension

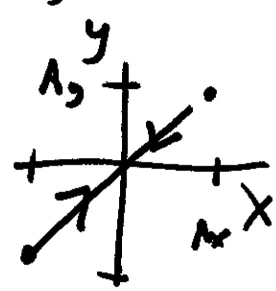
Here it's nice to plot $y(t)$ vs $x(t)$. As time goes by, we "map out" the trajector



← this is not a phase space plot. It's a real path (in real space, over time)

If $\delta_x = \delta_y$, both are "in phase", + $x = \frac{A_x}{A_y} y$

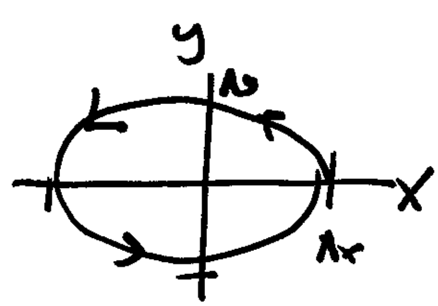
convince yourself!



If $\delta_y = \delta_x + \pi/2$, $\longrightarrow$ $x(t) = A_x \cos(\omega t - \delta_x)$

$y(t) = A_y \sin(\omega t - \delta_x)$

so $\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1$. $\left(\frac{\text{ccw}}{\omega > 0} \text{ if}\right)$



$\boxed{\text{If anisotropic}}$: ω's are different. Pictures get more complicated, "Lissajous Patterns"

If $\frac{\omega_x}{\omega_y} = \frac{n}{m}$ = ratio of integers then it "repeats" (closes on itself)

If $\frac{\omega_x}{\omega_y}$ is irrational, it does <u>not</u> ever repeat/close

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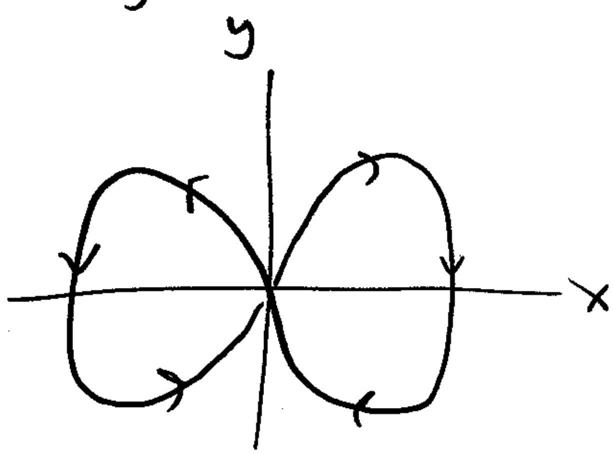
<u>Ex</u>: $\omega_y = 2\omega_x$, so it oscillates <u>twice</u> in y direction for every one in x direction:



Back to 1-D, let's add <u>DAMPING</u>

Consider <u>linear</u> drag, $m\ddot{x} = -kx - b\dot{x}$

(Quadratic drag $\Rightarrow$ <u>non</u>-linear 2^{nd} order ODE, this gets much more complicated). Rewrite Newton as

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$ } A 2^{nd} order, linear, homogeneous ODE with constant coefficients

Many physical systems obey this ODE (mechanical, electrical. ← RLC circuits, e.g.) Let's take a brief digression here to look at 2^{nd} order ODE'S (Boas 8.5) The most <u>general</u> linear 2^{nd} order ODE

$$y'' + P(t)y' + Q(t)y = R(t)$$

↳ Homogeneous if $R = 0$

- Should find 2 linearly independent sol'ns
- We'll deal with $R(t) \neq 0$ <u>later</u>. (soon!)

What does "linear" imply? If <u>also</u> homogeneous,

1) If $y_1(t)$ solves 17, so does $c\,y_1(t)$

(Convince yourself! c comes thru d/dt ...)

2) If $y_2(t)$ also solves 17, so does $y_1(t) + y_2(t)$

(Again, just plug it in to check)

- y_1 and y_2 are "linearly independent" if you <u>cannot</u> find <u>any</u> constants that make $C_1 y_1(t) + C_2 y_2(t) = 0$ for all times (<u>except</u> the trivial $C_1 = C_2 = 0$)

(Think of 2 <u>vectors</u> being independent similarly, $C_1\vec{V}_1 + C_2\vec{V}_2$ can't be zero)

There is a nifty quick tool to <u>test</u> if functions are linearly independent, you form the <u>Wronskian</u> <u>determinant</u>

$$W = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix}$$

If $W = 0$, they are dependent. If $W \neq 0$, independent

Ex: $\begin{vmatrix} \cos \omega t & \sin \omega t \\ -(\sin \omega t)\cdot \omega & (\cos \omega t)\cdot \omega \end{vmatrix} = \omega \cos^2 \omega t + \omega \sin^2 \omega t = \omega \neq 0$

So $\sin \omega t$, $\cos \omega t$ are independent (unless $\omega = 0$!)

Ex: $\begin{vmatrix} e^{i\omega t} & e^{-i\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} \end{vmatrix} = -2i\omega \neq 0$, again independent (unless $\omega = 0$!)

Ex: Can generalize to more fns,

$$\begin{array}{l} \text{fn's} \rightarrow \\ 1^{\text{st}} \text{ deriv} \rightarrow \\ 2^{\text{nd}} \text{ deriv} \rightarrow \end{array} \begin{vmatrix} \cos \omega t & \sin \omega t & e^{i\omega t} \\ -\omega \sin \omega t & \omega \cos \omega t & i\omega e^{i\omega t} \\ -\omega^2 \cos \omega t & -\omega^2 \sin \omega t & -\omega^2 e^{i\omega t} \end{vmatrix} = \begin{array}{l} \cos\left[-\omega^3 \cos e^{i\omega t} + i\omega^3 \sin e^{i\omega t}\right] \\ -\sin\left[-\omega^3 \sin e^{i\omega t} + i\omega^3 \cos e^{i\omega t}\right] \\ +e^{i\omega t}\left[-\omega^3 \sin^2 + \omega^3 \cos^2\right] \end{array}$$

$$= \omega^3 e^{i\omega t}\left[-\cos^2 + i \sin\cos + \sin^2 - i\sin\cos - \sin^2 + \cos^2\right]$$

$= 0$! So they are not all 3 linearly independent,

(that's why we have $A\cos + B\sin$ or $Ae^{i\omega t} + Be^{-i\omega t}$

but not some combo of 3 or more of these!)

Back to our ODE: $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

Consider a simple, special case, just as an example

e.g. $\ddot{y} + 3\dot{y} + 2y = 0$

Here's the trick: Consider the operator $D \equiv \frac{d}{dt}$

It's linear: $D\,3y = 3\,Dy$ and $D(y_1 + y_2) = Dy_1 + Dy_2$

So $D^2 y + 3Dy + 2y = 0$ or $(D^2 + 3D + 2)y = 0$

Now treat this like it was algebra $(D+1)(D+2)y = 0$

or $(D+2)(D+1)y = 0$

This works if $(D+1)y = 0$

or $(D+2)y = 0$.

These are 1^{st} order ODE's, + I know their sol'n -

$(D+1)y = 0 \Rightarrow \dot{y} = -y \Rightarrow y = C_1 e^{-t}$

$(D+2)y = 0 \Rightarrow \dot{y} = -2y \Rightarrow y = C_2 e^{-2t}$

Check Wronskian

$$\begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = e^{-3t}(-2+1) \neq 0\,!$$

So $y = C_1 e^{-t} + C_2 e^{-2t}$ is the general sol'n here.

In general, $(D-r_1)(D-r_2)y = 0$ is solved by

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ (as long as $r_1 \neq r_2$, see bottom of page *)

Indeed, if r_1 and r_2 are roots of the "auxiliary algebraic eq'n"

(i.e. From $a\ddot{y}(t) + b\dot{y}(t) + cy = 0$, auxiliar eq'n is $aD^2 + bD + c = 0$,) then roots of this

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ generate our sol'n,

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

---

* If roots are equal, I claim

$y(t) = (c_1 + c_2 t)e^{-rt}$ will solve the ODE.

Can check by plugging it in, + use Wronskian to convince yourself e^{-rt} and te^{-rt} are independent.

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<u>Derivation</u> of that previous claim:

$$(D-r)(D-r)y = 0.$$

clearly $y = c_1 e^{rt}$ is <u>one</u> sol'n.

Let $u(t) = (D-r)y$. So $(D-r)u = 0 \Rightarrow u = c_2 e^{rt}$

thus, $(D-r)y = u = c_2 e^{rt}$. Recall, Boas has a <u>trick</u> to solve this, remember?

$\dot{y} + Py = Q$ (here, $P = -r$, and $Q = c_2 e^{rt}$)

then $I = \int P\,dt = -rt$

and $y = e^{-I} \int Q e^{I}\,dt + Ce^{-I} =$

$$= e^{-rt} \int c_2 e^{rt} e^{-rt}\,dt + Ce^{-rt}$$

$$= e^{-rt} \cdot c_2 t + ce^{-rt}$$

$$= (c + c_2 t)\,e^{-rt}.$$

Example: $\ddot{y} + \omega_0^2 y = 0$. Sol'n?

Auxiliary eq'n is $(D^2 + \omega_0^2) = 0$

roots are $\pm i\omega_0$, so general sol'n is

$$y = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

Precisely our sol'n we've used before!

What about $\ddot{y} + \frac{b}{m}\dot{y} + \omega_0^2 y = 0$, which is what we've been after? Let's define a damping constant $\underbrace{\beta \equiv \frac{b}{2m}}$

units of $[\beta]$ = units of $[\omega]$

So we're solving $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0$

auxiliary eq'n $D^2 + 2\beta D + \omega_0^2 = 0$

roots are $r_{1,2} = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

Assuming $\beta \neq \omega_0$, these roots are distinct, + we have a sol'n $y(t) = e^{-\beta t}\left(C_1 e^{+\sqrt{\beta^2-\omega_0^2}\, t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\, t}\right)$

(If $\beta = \omega_0$, we use the trick from prev page, we'll come back to this)

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0 \qquad \text{(Damped SHM)}$$

There are 3 possible cases

1. $\beta < \omega_0$, "weak damping", $\sqrt{\beta^2 - \omega_0^2}$ is imaginary!
2. $\beta > \omega_0$, "strong damping", " " real.
3. $\beta = \omega_0$, "critical damping", use "double root" trick.

Case 1. $\beta < \omega_0$, also called UNDERDAMPED.

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} \equiv i\,\omega_1$$

(Defines ω_1. For small β, $\omega_1 \approx \omega_0$)

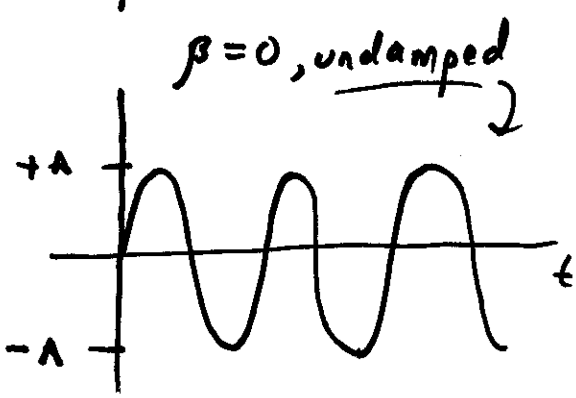
$$y(t) = e^{-\beta t}\left(C_1 e^{+i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

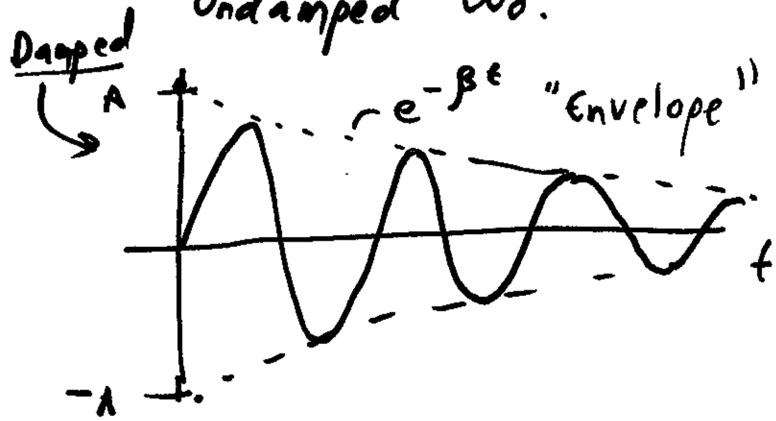
or, looking back a few pages, can also rewrite as

$$y(t) = \underbrace{e^{-\beta t} A}_{\text{"Amplitude" is decaying}} \cos(\omega_1 t - \delta).$$

Very much like SHM

freq is a bit below "natural" undamped ω_0.





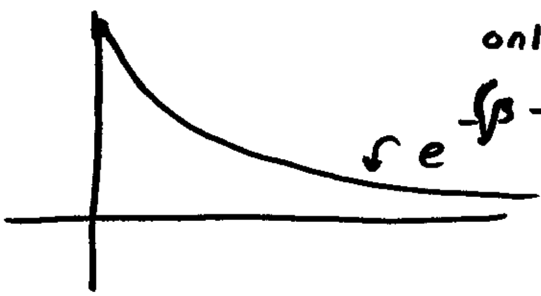
Case 2, $\beta > \omega_0$, "over damped"

$$y(t) = C_1 e^{-\beta t + \sqrt{\beta^2 - \omega_0^2}\, t} + C_2 e^{-\beta t - \sqrt{\beta^2 - \omega_0^2}\, t}$$

Both terms are dying exponentials. (Convince yourself the C_1 term is decaying, the + part is smaller than the − part!)

No oscillations, too much damping! It might cross the axis (once) if you pick C_1 & C_2 cleverly, but no "ringing".

Note: C_2 term has a much more negative coefficient for t so it $\to 0$ faster. At large t, C_1 term dominates.



only bit left is $e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t}$

if $C_1, C_2 > 0$

so $y(0) > 0$,

again, $e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t}$

if eg $C_2 = -C_1$.

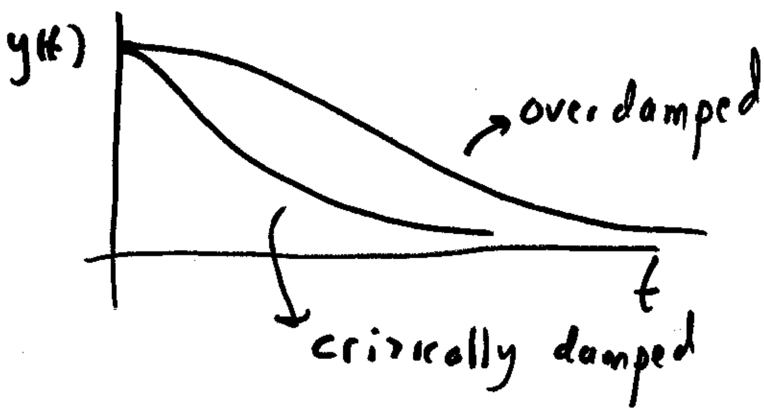
so $y(0) = 0$, $\dot{y}(0) > 0$

Note: As β gets bigger, $\beta - \sqrt{\beta^2 - \omega_0^2} \to 0$, so somewhat surprisingly, large damping in fact slows the rate of decay!

Case 3 : Critical damping , $\beta = \omega_0$

Double root of $-\beta$, so $y(t) = (C_1 + C_2 t)e^{-\beta t}$

The dying exponential always "wins" eventually, so this does die away, very much like $e^{-\beta t}$. This is a faster die-off than the $e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t}$ we had in the overdamped case.



Think of the pneumatic tube on a screen door, or shocks on your car:

Too little damping $\Rightarrow$ oscillations (or banging)

Too much damping $\Rightarrow$ very slow to settle down

critical " $\Rightarrow$ fastest return to equilib w/o oscillations.

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What if we <u>drive</u> / <u>force</u> our oscillator?

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = \underbrace{R(t)} \quad \leftarrow \text{inhomogeneous}$$

From Newton, this is $\frac{F(t)}{m}$

(or, e.g. in an RLC circuit, it might be a voltage source)

The approach for inhomogeneous linear ODE's is:

Solve the homogeneous case to get $\underset{\text{Complementary}}{y_c} = C_1 y_1(t) + C_2 y_2(t)$

+ then find <u>any</u> particular sol'n y_p.

then $y = y_c + y_p$ is your fully general sol'n w. 2 constants!

We know how to find y_c, so we just need y_p.

Sometimes: "by inspection" works! e.g $\ddot{y} + 4\dot{y} + 3y = 5$

I claim $y_p = 5/3$ is <u>clearly</u> a sol'n. (Do you see why? Try it!)

General sol'n: Roots of $D^2 + 4D + 3$ are 1 & 3, so

$\underline{y = Ae^{t} + Be^{3t} + 5/3}$ is the fully general sol'n

(So if $R(t) = R_0 =$ <u>constant</u>, use $y_p = R/\omega_0^2 \ldots$)

What if $\underline{R(t) = f_0 e^{ct}}$ (where c $\underline{\text{can}}$ be imaginary!)

This is really much more general than it looks, because $e^{i\omega t}$ as we've seen has $\cos(\omega t)$ and $\sin \omega t$ "built in", + we can b<u>uild</u> <u>up</u> any oscillatory $R(t)$ by summing up * sin's + cos's with different ω's. We'll come back to do that, but this is why this example is so fundamental + ess<u>ential</u>.

* [Note that if we find y_{p1} for $R_1(t)$ on the right side and y_{p2} for $R_2(t)$, then $y_{p1} + y_{p2}$ will solve the ODE with $R = R_1(t) + R_2(t)$

We can <u>derive</u> y_p for $R(t) = f_0 e^{ct}$, but the nice thing about y_p is you just need to find anything that works, so the method of "guess and check" is just fine!

We're solving $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 e^{ct}$

+ looking for any y_p. Let's try $y_p(t) = C_3 e^{ct}$!

[This doesn't always work (e.g. if c happens to be one of the roots of the auxiliary eq'n, we'll need to try $C_3 t e^{ct}$, + if it equals a double root, $C_3 t^2 e^{ct}$, But those are really special cases)]

Important: C_3 is not a new "unknown, arbitrary" constant! This is y_p, we need to plug it in, + the ODE itself will require a very particular value for C_3, it's not set by "initial conditions"! Let's do it: $\ddot{y}_p = c^2 y_p$

$\dot{y}_p = c\, y_p$

so $(c^2 + 2\beta c + \omega_0^2) C_3 e^{ct} = f_0 e^{ct}$

So $C_3 = \dfrac{f_0}{c^2 + 2\beta c + \omega_0^2}$ is required, it's fixed!

(This is all good as long as c isn't accidentally a root of the ~~aux~~ auxiliary eq'n, (then the denom $\to 0$)

Important example: $R(t) = f_0 e^{i\omega t}$, (oscillating driver)
(here, $c = i\omega$)

so $y_p = \dfrac{f_0 e^{i\omega t}}{(-\omega^2 + 2\beta i\omega + \omega_0^2)} = \dfrac{f_0 e^{i\omega t}}{(\omega_0^2 - \omega^2) + 2\beta i\omega}$

$y_p = c_3 e^{i\omega t}$

If the driving force is real, say $f_0 \cos\omega t$, no problem, just take the real part of this sol'n !!

Our $c_3 = f_0 / (\omega_0^2 - \omega^2) + 2\beta i\omega$ is complex. Any complex # can always be written in the form $c_3 = A e^{-i\delta}$

amplitude (A), phase (δ)

(Note that "taking the real part" is much easier in this form, $Re(c_3) = A\cos\delta$.)

Let's pause to work out c_3, then:

To get $A = |c_3|$, use $\left|\dfrac{1}{a+bi}\right| = \left|\dfrac{1}{a+bi}\,\dfrac{a-bi}{a-bi}\right|$

$$= \frac{|a-bi|}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}}$$

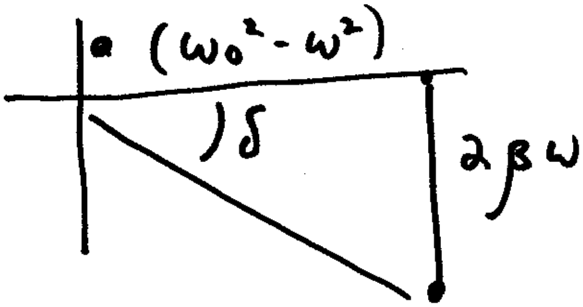
(To get δ, rewrite c_3 in the form $Re + i\,Im$)

Our $\quad C_3 = \dfrac{f_0}{(\omega_0^2 - \omega^2) + i \cdot 2\beta\omega}$; so $\boxed{|C_3| = \dfrac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}}$

(use $\left|\frac{1}{a+bi}\right| = 1/\sqrt{a^2+b^2}$

also, $C_3 = \dfrac{f_0}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \cdot \left[(\omega_0^2 - \omega^2) - 2\beta\omega i\right]$

so



$\boxed{\delta = \tan^{-1} \dfrac{2\beta\omega}{\omega_0^2 - \omega^2}}$

from the picture!

It's gotten a little ugly, let's recap:

we're solving $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 e^{i\omega t}$

we know how to find $y_c(t)$ (solving "ancillary", homog eq'n)

we just found $y_p(t) = C_3 e^{i\omega t} = A e^{-i\delta} e^{i\omega t}$

(If we have a real driver, $f_0 \cos\omega t$, we'll simply take the Real part of our sol'n, giving $\mathrm{Re}(y_p) = A\cos(\omega t - \delta)$)

Let's do an example!

Underdamped, driven oscillator. (Like, resonating circuit in your stereo)

Recall, the homogeneous $y_c(t)$ sol'n has an overall $e^{-\beta t}$, so it dies away, it's transient. In general, that sol'n is

$$y_c(t) = A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr})$$

these are our 2 arbitrary coefficients, found from initial conditions

so $y(t) = y_p + y_c$

$$= A\cos(\omega t - \delta) + A_{tr} \underbrace{e^{-\beta t}}\cos(\omega_1 t - \delta_{tr})$$

dies off, hence label "transient"

where $A = \dfrac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$

and $\delta = \tan^{-1} 2\beta\omega/\omega_0^2 - \omega^2$

(β and ω_1 are as before, $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$)

After some time $\gg 1/\beta$, only $y_p = A\cos(\omega t - \delta)$ remains. (True also if overdamped, or critically damped, so this is quite general)

- If you drive an oscillator, it settles down by oscillating at the driving frequency (but, phase shifted)

Bottom line: $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 e^{i\omega t}$

large t sol'n: $y = A\cos(\omega t - \delta)$

A is fixed, it's $f_0 / \sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$, it's not initial conditions!

δ is fixed, it's $\tan^{-1} 2\beta\omega/\omega_0^2-\omega^2$, again " " ".

ω is the driver's ω, not the natural freq. (neither ω_0 nor ω_1)

The Amplitude $A \propto f_0$, so "strong drivers" $\Rightarrow$ large response

---

A tells you the "strength of response", the long-term amplitude of motion. Energy in an oscillator $\propto \frac{1}{2}KA^2$, so it also tells you how much energy the "final state" has. (For this reason, we're often interested in $|A|^2$.)

A depends (linearly) on f_0, the driving force, but also on ω_0 (natural freq), β (damping), + ω (driving freq)

when β is small, interesting things happen, let's investigate.

Resonance

1) In some situations, ω is set somehow, but you can vary or control ω_0. E.g, if PBS broadcasts a radio wave at 98.5 MHz, that *drives* your stereo. $\omega = 2\pi \cdot 98.5$ MHz

But by twisting a knob (thus varying a capacitancy or inductance), you can "tune" ω_0 at will! We'll come back to the details of "RLC circuits" soon.

2) In other situations, ω_0 is set somehow, but you can vary or control the driving frequency ω. (Consider a bridge or building with natural vibrational frequency, driven by a controllable or variable outside force)

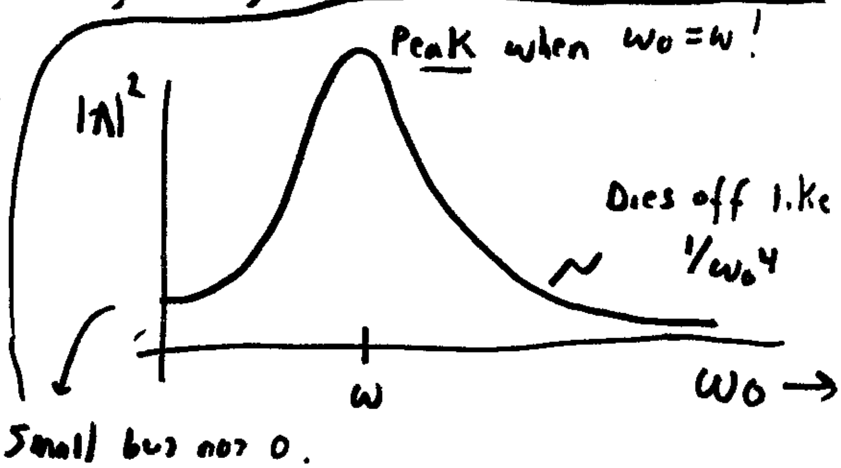
Situation 1 is slightly simpler mathematically, but for small drag ($\beta \ll \omega, \omega_0$) both are qualitatively similar.

So, let's 1st consider case 1), ω is *fixed*, and we are free to vary ω_0 at will. How does Amplitude, A, respond? (or, how does $|A|^2$ respond, since that's $\propto$ energy)

 (Case 1: Fix ω, vary ω_0)

$$|A|^2 = f_0^2 \Big/ \left((\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right)$$ Think of this as a

fn of ω_0: $\dfrac{1}{c^2 + (\omega_0^2-\omega^2)^2}$:



The max occurs when denom is min, + that's $\omega_0 = \omega$

If β is small, $|A|^2$ grows very large when $\omega_0 = \omega$, in fact.

$$|A|^2_{max} = f_0^2 / 4\beta^2\omega^2 \quad (\rightarrow \infty \text{ if } \beta \rightarrow 0)$$

This "Blowup" when $\omega_0 \approx \omega$ is Resonance.

The system responds strongly when you drive it at the natural frequency.

(If ω_0 is far from ω in either direction, the response is much weaker)

 (Case 2: Fix ω_0, vary ω)

$$|A|^2 = f_0^2 / \left((\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 \right)$$

Again, Max occurs when denominator is minimum, i.e.

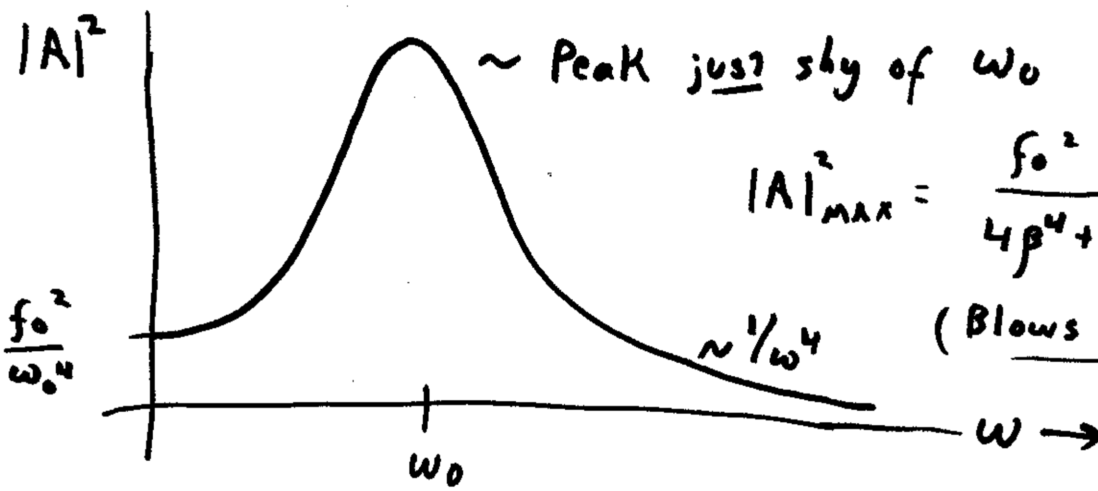
$$\frac{d}{d\omega}\left(\omega_0^4 - 2\omega^2\omega_0^2 + \omega^4 + 4\beta^2\omega^2\right) = 0$$

i.e. $-2\omega_0^2 \cdot 2\omega + 4\omega^3 + 8\beta^2\omega = 0$

[$\omega = 0$ solves this, but that turns out to be a max, not a min, of the denominator, at least if $\beta \ll \omega_0$.]

The other sol'n: $\omega^2 - \omega_0^2 = -2\beta^2$ (Divided out ω)

so $\boxed{\omega_{peak}^2 = \omega_0^2 - 2\beta^2}$ gives "resonant ω"



$$|A|^2_{MAX} = \frac{f_0^2}{4\beta^4 + 4\beta^2(\omega_0^2 - 2\beta^2)} = \frac{f_0^2}{4\beta^2(\omega_0^2 - \beta^2)}$$

(Blows up as $\beta \to 0$)

Again, resonance when $\omega \approx \omega_0$.

Summarizing:

$\omega_0 = \sqrt{k/m}$ = natural undamped freq.

ω = driving freq

$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ = nat damped freq

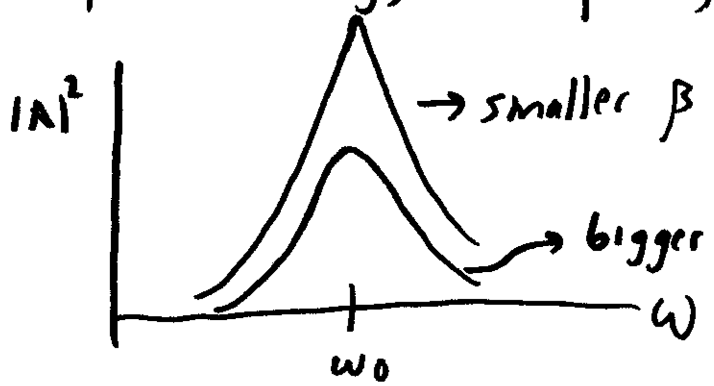
$\omega_{peak} = \sqrt{\omega_0^2 - 2\beta^2}$ = resonant freq (for fixed ω_0.)

$|A_{MAX}| \approx f_0/2\beta\omega_0$ (if $\beta \ll \omega_0$)

Kids on swings know all this! ω_0 is the "natural swing freq",

+ you get the best ride if you pump at that freq.

If there's drag, $|A|$ is finite, but can be quite big.



→ smaller $\beta \Rightarrow$ higher peak (nearly at ω_0)

→ bigger $\beta \Rightarrow$ lesser peak (+ wee bit "left" of ω_0)

Width of the resonance curve: Let's look for the ω

where $|A^2|(\omega) = \frac{1}{2}|A^2|(\text{Peak})$. That's "half max"

It will be (next page) $\omega_{\substack{\text{Half}\\\text{MAX}}} \approx \omega_0 \pm \beta$.

So, smaller $\beta \Rightarrow$ taller and narrower resonance.

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$$|A|^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$$

This is "half max" when the 1^{st} term = 2^{nd} term (if β is small))

So this is when $\omega_0^2 - \omega^2 \approx \pm 2\beta\omega_0$

if β is small, $\omega_0^2-\omega^2$ is thus small, so our

ω = "$\omega_{\text{Half MAX}}$" is in fact still very <u>near</u> ω_0.

In that case $\omega_0^2 - \omega^2 = (\omega_0+\omega)(\omega_0-\omega) \approx 2\omega_0(\omega_0-\omega)$

and $\pm 2\beta\omega \approx \pm 2\beta\omega_0$, so

$$(\omega_0-\omega) \approx \pm\frac{2\beta\omega_0}{2\omega_0} = \pm\beta, \text{ as claimed.}$$

It's common to define a "Quality Factor" $Q \equiv \frac{\omega_0}{2\beta}$

unitless!

Narrow resonance $\Leftrightarrow$ small β $\Leftrightarrow$ big Q.

- $$|A|^2_{peak} \approx \frac{f_0^2}{4\beta^2\omega_0^2} = \frac{f_0^2\,\omega_0^2}{4\beta^2\omega_0^4} = Q^2\,\frac{f_0^2}{\omega_0^4} \propto Q$$

- $$Q = \frac{\omega_0}{2\beta} = \frac{2\pi/T_{natural}}{2\beta}.$$ Recall that with no driver, amp $\sim e^{-\beta t}$ dies of in time $\tau \simeq 1/\beta$

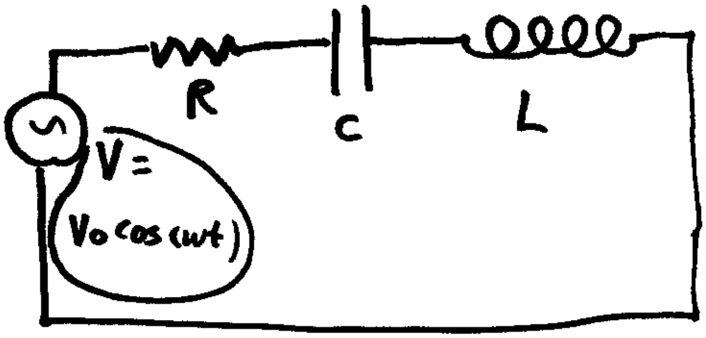
so $Q = \pi \frac{\tau}{T_{natural}}$. Big Q means "die-off time" is long compared to natural period.

So, Big Q means you will "Ring" many times without a driver

And, with a driver, you'll get strong resonance.

you can also show Q tells you $\dfrac{\text{Energy stored in oscillator}}{\left(\text{Energy lost to damping in one cycle}\right)}$

I've referred to RLC circuits. They might look like this:



Kirchhoff says $\Delta V_{loop} = 0$

$$+V_0 \cos\omega t - IR - \frac{Q}{C} - L\frac{dI}{dt} = 0$$

ΔV resistor, capacitor, inductor

But $I = dQ/dt$, so

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 \cos\omega t \Rightarrow \text{our usual ODE!}$$

$$\ddot{Q} + \underbrace{\frac{R}{L}}_{2\beta}\dot{Q} + \underbrace{\frac{1}{LC}}_{\omega_0^2} Q = \underbrace{\frac{V_0}{L}}_{f_0} \cos\omega t$$

R causes damping, makes sense!

$1/LC = \omega_0^2$ is the "natural frequency"

For a $\underbrace{\text{30 pF capacitor, 100 nH inductor}}_{\text{cheap, simple, ordinary values}}$ $\Rightarrow f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \approx 90.9\ \text{MHz}$