PHYSICS 2210: Non-Inertial Reference frames

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See also Taylor book Chapter 9.

Recall that Newton's laws are valid in an inertial frame (i.e non-accelerated).

However not always it is possible to be in an inertial frame. For example is this classroom an inertial or a noninertial reference frame? why?

Actually to a very good approximation a reference frame fixed to the earth is inertial, but nevertheless the earth is rotating and it is a non-inertial reference frame. Our task in these two coming classes is try to understand how we describe the motion of an object in a non inertial reference frame.

We will start by describing the situation in which the frame has a linear acceleration and then we will consider the case of a rotating frame which is relevant of the case of the earth.

Case 1: Frame with a linear acceleration

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I will use Taylor's convention and denote:

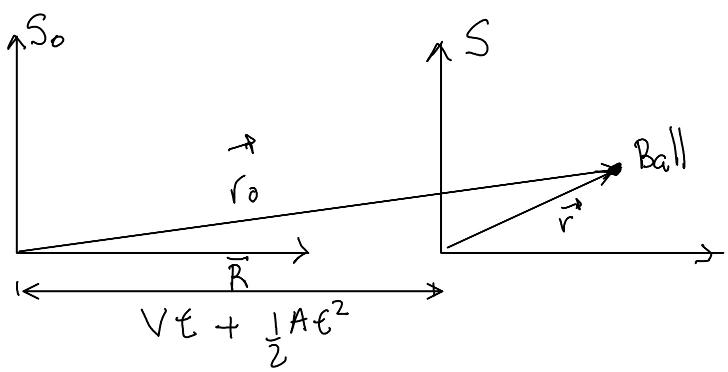
S_0: An inertial frame

S: A frame that is accelerating relative to S_0.

V: velocity of frame S with respect to S_0

A: Acceleration of frame S with respect to S_0

Example: S could be a car in motion.



$$\vec{r}_0 = \vec{R} + \vec{r}$$

differentiate it:

$$\frac{d\vec{r}_0}{dt} = \underbrace{\frac{d\vec{R}}{dt}}_{\vec{V}} + \frac{d\vec{r}}{dt}$$

$$\frac{d\vec{r}_0}{dt} = \vec{V} + \frac{d\vec{r}}{dt}$$

$\vec{r}$: position of the ball with respect to S

$\vec{r}_0$: position of the ball with respect to S_0

$\vec{R}$: position of S with respect to S_0

or

Velocity of ball respect to S_0 = Velocity of S with respect to S_0 + Velocity of the ball respect to S

differentiate it once again

$$\frac{d^2 r_0}{dt^2} = \frac{d\vec{V}}{dt} + \frac{d^2\vec{r}}{dt^2}$$

$$\ddot{\vec{r}}_0 = \vec{A} + \ddot{\vec{r}}$$

We also know that Newton's law hold in an inertial frame so

$$m\ddot{\vec{r}}_0 = \vec{F} \qquad \vec{F}: \text{net force on the ball}$$

Then

Inertial force

$$\vec{F} = m\vec{A} + m\ddot{\vec{r}} \Rightarrow m\ddot{\vec{r}} = \vec{F} - \vec{A}m \quad (1)$$

Eq. (1) is exactly Newton's second law in a non inertial reference frame

The extra term $\vec{F}_{inertial} = -m\vec{A}$ is a fictitious force we need to add to properly account for the physics in a non inertial accelerating frame.

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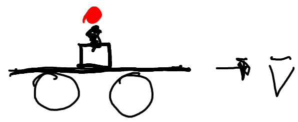
Example

case i) 

car is moving with constant velocity $\vec{V}$
A ball is shoot out of cart. Where does the ball land?

Answer in car.

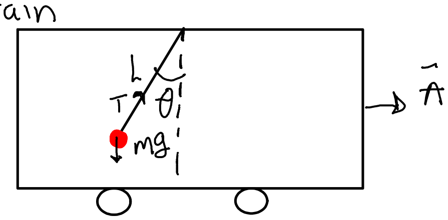
case ii) car now is accelerating with acceleration $\vec{A}$, where the ball lands?

Answer: Behind the cart

for a person in the car there is a fictitious force $-m\vec{A}$ acting on the ball.

For a person outside this make sense since the cart is accelerating but not the ball

Another example: a pendulum inside an accelerating train



For an observer in an inertial frame, Newton's law state

$mg = T\cos\theta$

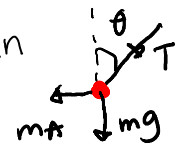
$T\sin\theta = mA$

or $\tan\theta = \frac{A}{g}$

$m\ddot{\vec{r}}_0 = \vec{T} + m\vec{g}$

$m\vec{A} = \vec{T} + m\vec{g}$

Inside the train



$m\cancel{\ddot{\vec{r}}} = \vec{T} + m\vec{g} - m\vec{A}$

ball is not moving

$\Downarrow$

$mg = T\cos\theta$

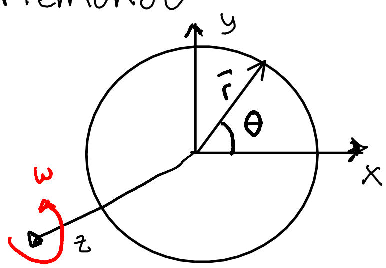
$T\sin\theta = mA$

$\tan\theta = \frac{A}{g}$

SAME CONCLUSION :

Before going into rotating frames let's briefly study what we mean by angular velocity

Remember



$\omega = \frac{d\theta}{dt}$

$\downarrow$
angular velocity

we can represent ω as a vector

$\vec{\omega} = \omega\hat{u}$

$\downarrow$ axis of rotation

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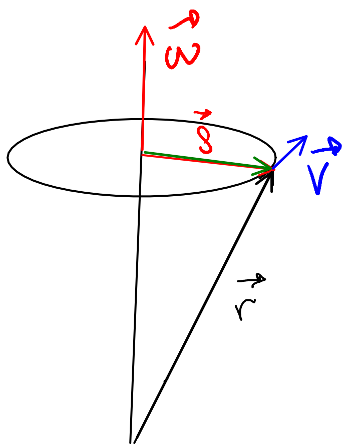
Q? In the prior example $\hat{u} = \hat{z}$. why not $-\hat{z}$

A. We choose the direction of $\hat{u}$ or $\vec{\omega}$ by using the right-hand rule: curl right hand fingers around rotation and determine direction by using your thumb

Note the magnitude and direction of $\vec{\omega}$ can change with time. Here however we will restrict to the case in which $\vec{\omega}$ is constant.

Q Is there any relationship between $\vec{\omega}$ and the velocity of an object fixed in the rotating frame?

A: Yes



$\rho = r\sin\theta$

$V = \rho \frac{d\theta}{dt} = r\sin\theta\,\omega$

Result of a cross product

$\vec{V} = \vec{\omega} \times \vec{r}$

use right hand rule

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Conclusion

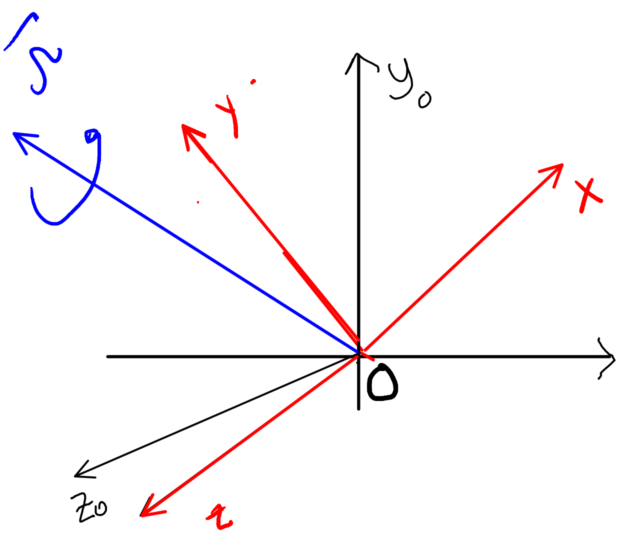
$$\frac{d\vec{r}}{dt} = \bar{\omega} \times \vec{r}$$

One could show that this relashionship holds for any vector fixed on a rotating body. For example if **e** is a unit vector fixed in the body, then its rate of change as seen in the non-inertial frame is

$$\frac{d\hat{e}}{dt} = \bar{\omega} \times \hat{e}$$

with this result we are now ready to discuss our second case:

Case 2: Frame rotating with angular velocity Ω



S_0: Inertial frame
S: Non inertial frame, rotating with angular velocity Ω
O: common origin

Let's now consider an arbitrary vector $\vec{Q}$:
$\vec{Q}$ could be velocity, position, acceleration etc

We want to understand how the rate of change of Q as measured in So relates to the rate of change measured by someone in the rotating frame. i.e

$$\left(\frac{d\vec{Q}}{dt}\right)_{So} \quad ? \quad \left(\frac{d\vec{Q}}{dt}\right)_{S}$$

We just need to define

$$\vec{Q} = Q_1\hat{e}_1 + Q_2\hat{e}_2 + Q_3\hat{e}_3 = \sum_{i=1}^{3} Q_i\hat{e}_i$$

Here $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ are unit vectors fixed in the S frame.
$\hat{e}_i$ are fixed in S, but they ARE NOT fixed in So.

As seen in S $\quad \left(\frac{d\vec{Q}}{dt}\right)_S = \sum_i \frac{dQ_i}{dt}\hat{e}_i$ since those are fixed in S

As seen in So $\quad \left(\frac{d\vec{Q}}{dt}\right)_{So} = \sum_i \frac{dQ_i}{dt}\hat{e}_i + Q_i\frac{d\hat{e}_i}{dt}$

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The last term can be easily evaluated from our prevuos relation

$$\left(\frac{d\hat{e}_i}{dt}\right)_{so} = \vec{\Omega} \times \hat{e}_i \quad \text{and therefore}$$

$$\left(\frac{d\vec{Q}}{dt}\right)_{so} = \sum_i \frac{dQ_i}{dt}\hat{e}_i + \sum_i Q_i(\vec{\Omega}\times\hat{e}_i)$$

$$= \left(\frac{d\vec{Q}}{dt}\right)_s + \vec{\Omega}\times \sum_i Q_i\hat{e}_i \rightarrow \vec{Q}$$

$$\left(\frac{d\vec{Q}}{dt}\right)_{so} = \left(\frac{d\vec{Q}}{dt}\right)_s + \vec{\Omega}\times\vec{Q}$$

Great! we did it. We now know how to relate the time derivative in the rotating frame to the one in an inetial frame :)

With this we are now ready to derive Newton's law in a rotating frame

Newton's second law in a rotating frame

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In an inertial frame $\vec{F} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{SO}$

but we know now that

$$\left(\frac{d\vec{r}}{dt}\right)_{SO} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$$

If we differentiate one more time

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{SO} = \frac{d}{dt}\Bigg(\underbrace{\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}}_{\text{Let's call this } \vec{Q}}\Bigg)_{SO}$$

$$= \frac{d}{dt}\left(Q\right)_{SO}$$

$$= \left(\frac{d}{dt}Q\right)_S + \vec{\Omega}\times\vec{Q}$$

$$= \frac{d}{dt}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]_S + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_S + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_S + 2\vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Then Newton's second law in the rotating frame is

$$m\ddot{\vec{r}} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} - 2m(\vec{\Omega}\times\dot{\vec{r}}) - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\ddot{\vec{r}} \equiv \left(\frac{d^2\vec{r}}{dt^2}\right)_S \quad \text{and} \quad \dot{\vec{r}} = \left(\frac{d\vec{r}}{dt}\right)_S$$

$$m\ddot{\vec{r}} = \underset{①}{\vec{F}} + \underset{②}{2m(\dot{\vec{r}}\times\vec{\Omega})} + \underset{③}{m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}$$

Here I used $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

① Net force in the inertial frame

② $2m(\dot{\vec{r}}\times\vec{\Omega})$ = Coriolis force = $\vec{F}_{co}$

③ $m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$ = Centrifugal force = $\vec{F}_{ce}$

Our task is trying to make sense of these two new fictitious forces

Note coriolis force is zero for an object at rest and it is small for objects moving slowl

$F_{co} \sim mv\Omega \qquad F_{ce} \sim m\Omega^2 r$

$\frac{F_{co}}{F_{ce}} = \frac{v\Omega}{\Omega^2 r}$ At earth surface $r=R$ $\frac{F_{co}}{F_{ce}} = \frac{v}{\Omega R}$, $\Omega R \sim 1000\frac{m}{h}$ for earth

only for objects moving $v > 1000\frac{m}{h}$ coriolis is important

The centrifugal Force

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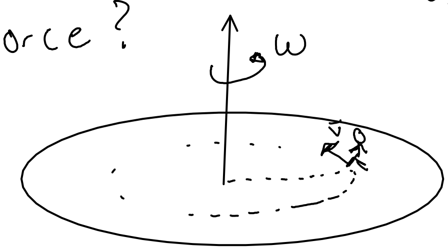
We just learned that Coriolis force is not important for slow objects. for the earth case for objects moving $V < 1000$ mi/h.
we will first then study the centrifugal force which is the dominant one in those cases,

To warm up let's first imagine that you are at rest on a Merry-go-round that is rotating with angular velocity ω

What is the direction of the centrifugal force?



$$\vec{F}_{ce} = m\underbrace{(\vec{\Omega}\times\vec{r})}_{\omega r\hat{\theta}}\times\vec{\Omega}$$

$$= m\omega^2 r\,\hat{r}$$
outwards

$$\vec{F}_{ce} = m\frac{v^2}{r}\hat{r} \qquad v = \omega r$$

For an observer in $\vec{S}$ $\vec{F}_{ce}$ cancels out the centripetal force which goes in the inward direction

Let's now study the role of the centrifugal force on an object dropped near the surface of the earth.

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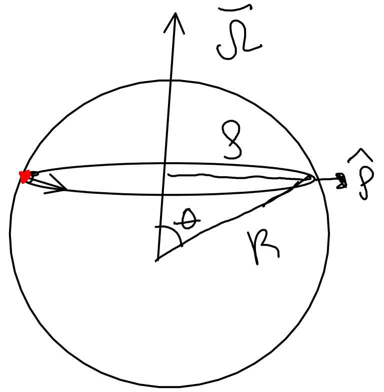
Again at the earth surface.



$$\vec{F}_{ce} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$
$$= m\Omega^2\rho\;\hat{\rho}$$
(outwards)

$$\rho = R\sin\theta$$

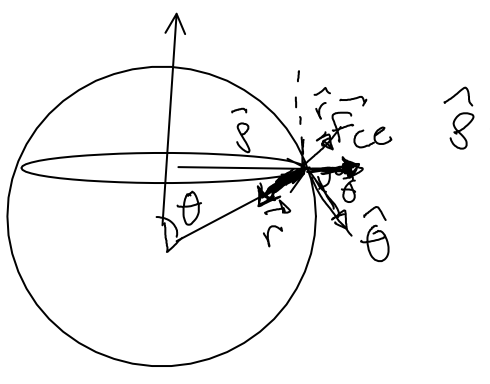
If an object is released near the earth surface

$$m\ddot{\vec{r}} = \vec{F}_{grav} + \vec{F}_{ce}$$

(for an observer moving with the earth)

$$\vec{F}_{grav} = -\frac{GMm}{R^2}\hat{r} = m\vec{g}_0 \qquad \vec{g}_0 = \underbrace{\frac{-GM}{R^2}\hat{r}}_{\text{local acceleration}}$$

$$\vec{F}_{ce} = m\Omega^2 R\sin\theta\,\hat{\rho}$$



$$\hat{\rho} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

$$\vec{F}_{net\ radial} = m\vec{g}_0 + m\Omega^2 R\sin^2\theta\,\hat{\rho} = m\vec{g}_{eff}$$

$$\vec{g}_{eff} = (g_0 - \Omega^2\sin^2\theta\,R)(-\hat{r})$$

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$\vec{g}_{eff} = \vec{g}_0$ at poles

$\Omega^2 R \simeq 0{,}034\ m/s^2$: Small but measurable

$\vec{F}_{net}\,\hat{\theta} = \cos\theta \sin\theta\, \Omega^2 R$ again vanishes at the poles but also at the equator.

This component causes that the free fall of an object is not straight down.

The coriolis Force:

$\vec{F}_{co} = 2m\vec{v}\times\vec{\Omega}$

objects velocity relative to the rotating frame.

Note $\vec{F}_{co}$ is similar than the force experienced by a charged particle with charge q in a magnetic field

$\vec{F}_{mag} = q\vec{v}\times\vec{B}$

$2m \leftrightarrow q$
$\vec{\Omega} \leftrightarrow \vec{B}$

$\vec{F}_{co} \perp \vec{v}$ $\quad \vec{F}_{co} \perp \vec{\Omega}$ direction given by right hand side rule.

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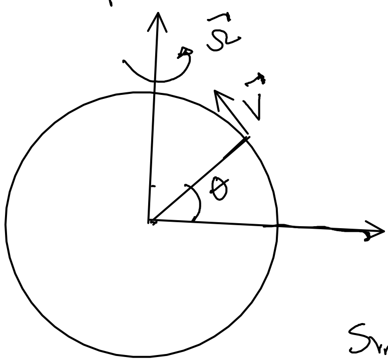
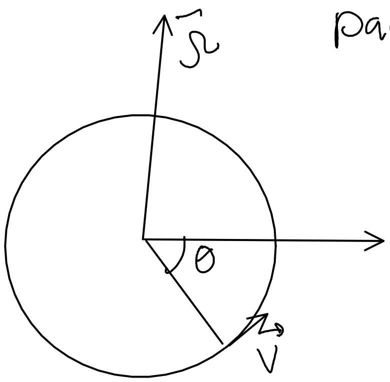
Let's look at some examples considering Coriolis force on the surface of earth



- If you fire a cannon at a latitude $\theta = 45°$ north what is the direction of F_{co}?

Since $\vec{F}_{co} = 2m\vec{v}\times\vec{\Omega}$ then in this case $\vec{F}_{co}$ is into the page $\otimes$ i.e East

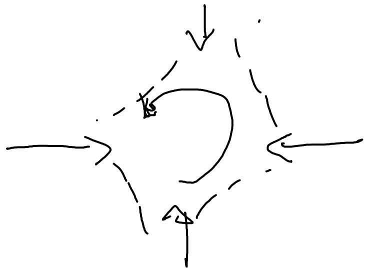


If you fire the same cannon in the southern hemisphere, what is the direction of $\vec{F}_{co}$?

$\vec{F}_{co} = 2m\vec{v}\times\vec{\Omega}$ Now $\vec{F}_{co}$ is out the page $\odot$ i.e. west

Conclusion: Direction changes in the southern hemisp. This fact cost the British gunners an important loss during WWI. They were surprised that their bombs, which used to be accurated, kept landing to the left of German ships. The designers forgot to consider the direction change of the coriolis force in the southern hemisphere. ! :(

Due to this reversal of direction, it was believe that water moving towards a central drain should move CC wise



in the northern hemisphere and C wise in the southern hemisphere. These would be reflected in your sink drain moving CC or C depending on your position

In reality this effect is very small and the direction in which your sink drains is dominated by your sink geometry.

However this effect is important in metrology The coriolis force is the reason why hurricanes tend to be C clockwise in the Northern hemisphere and clockwise in the southern hemisphere !!

Other systems such as the Foucault pendulum are affected in a noticiable way by Coriolis force. In this case even though the force is very small, it can act for a long time producing a large effect. There is a foucault pendulum in the Duane tower. Go and check it out !!