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Taylor Ch. 3: Momentum ($\vec{p}$) + Angular momentum ($\vec{L}$)

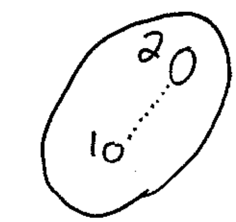
Let's start by looking back (Taylor 1.5) at the connection between Newton's Laws + momentum:

Newton 2 : $\vec{F} = \dfrac{d\vec{p}}{dt}$ for "pointlike objects"

Newton 3 : $\vec{F}_{12} = -\vec{F}_{21}$ " " " .

We're now interested in real objects, (built from bits).

E.g. a body made of 2 connected pieces



$\underbrace{\vec{F}_{on\ 1} = d\vec{P}_1/dt}_{\text{Newton 2}} = \underbrace{\vec{F}_{on\ 1\ by\ 2} + \vec{F}_{on\ 1\ by\ \text{anything else}}\,!}_{\text{(just explicating the L.H.S. of this eq'n)}}$

and $\vec{F}_{on\ 2} = d\vec{P}_2/dt = \vec{F}_{on\ 2\ by\ 1} + \vec{F}_{on\ 2\ by\ \text{"outsiders"}}$

Adding : $\dfrac{d(\vec{P}_1 + \vec{P}_2)}{dt} = \sum_{i=1}^{2} \vec{F}_{on\ i\ by\ \text{outsiders}} + \underbrace{\vec{F}_{12} + \vec{F}_{21}}_{\text{this} = 0,\ \text{by N-3}}$

$(\vec{P}_{Total} \equiv \vec{P}_1 + \vec{P}_2)$

so $\boxed{\dfrac{d\vec{P}_{Total}}{dt} = \sum_i \vec{F}_{external,\,i}}$: Newton 2 for real objects.

$\equiv \vec{F}_{net,\ external}$

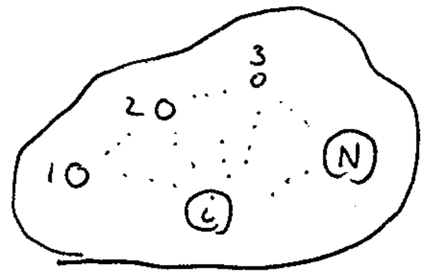
Internal forces don't appear!

Important: If $\vec{F}_{net,ext} = 0$, $\dfrac{d\vec{P}_{tot}}{dt} = 0 \Rightarrow \vec{P}_{Total}$ is <u>conserved</u>.

(So conservation of momentum comes from Newton's Laws.)

What if we have N particles, lots of them? Same logic!

$$\frac{d\vec{P}_1}{dt} = \vec{F}_1 = \vec{F}_{ext\ \underline{\underline{on\,1}}} + \vec{F}_{on\,1\,by\,2} + \vec{F}_{13} + \vec{F}_{14} + \dots + \vec{F}_{1N}$$



$$= \vec{F}_{1,ext} + \underbrace{\sum_{all\ \beta \neq 1}}_{\text{No "self forces" here!}} \vec{F}_{on\,1\,by\,\beta}$$

$$\frac{d\vec{P}_2}{dt} = \vec{F}_2 = \vec{F}_{2,ext} + \sum_{all\ \beta \neq 2} \vec{F}_{on\,2\,by\,\beta}$$

$\vdots$

$$\frac{d\vec{P}_i}{dt} = \vec{F}_i = \vec{F}_{i,ext} + \sum_{all\ \beta \neq i} \vec{F}_{on\,i\,by\,\beta}$$ (<u>shorthand</u>, $\vec{F}_{i,\beta}$)

$\vdots$

etc.

Defining $\vec{P}_{tot} = \vec{P}_1 + \vec{P}_2 + \dots + \vec{P}_N = \sum_{i=1}^{N} \vec{P}_i$, just <u>add</u> the eq'ns:

$$\frac{d\vec{P}_{tot}}{dt} = \sum_{i=1}^{N} \vec{F}_{i,ext} + \underbrace{\sum_{\beta \neq 1} \vec{F}_{1,\beta} + \sum_{\beta \neq 2} \vec{F}_{2,\beta} + \dots + \sum_{\beta \neq i} \vec{F}_{i,\beta} + \dots \sum_{\beta \neq N} \vec{F}_{N,\beta}}_{\text{It's a "sum of sums"}}$$

Looks ugly? It simplifies!

The 1st term is just $\vec{F}_{net,ext}$.

The (messy) rest are all internal forces. They all cancel out!

Proof: Let's organize all those "interactions" like this

$$
\begin{array}{lllllll}
\boxed{\text{Force}} & \text{by } 1 & \text{by } 2 & \text{by } 3 & \cdots & & \text{by } N \\
\sum_{\beta\neq 1}\vec{F}_{1,\beta}\to \text{ on } 1 & 0 & + F_{12} & + F_{13} & + & \cdots & + F_{1N} \\
\sum_{\beta\neq 2}\vec{F}_{2,\beta}\to \text{ on } 2 & F_{21} & + 0 & + F_{23} & + & \cdots & + F_{2N} \\
\vdots \quad \text{on } 3 & F_{31} & + F_{32} & + 0 & + & \cdots & + F_{3N} \\
\vdots & & & & & & + 0 \\
\text{on } N & F_{N1} & + F_{N2} & + F_{N3} & + & \cdots &
\end{array}
$$

Adding all these is what we had, $\sum_{\beta\neq 1}\vec{F}_{1,\beta}+\sum_{\beta\neq 2}\vec{F}_{2,\beta}+\dots$

But look. The "diagonal terms" are absent

+ all the rest cancel in pairs by N-III. The sum is 0!

Taylor argues it like this, that "sum of sums" is

$$\sum_{\text{all } i}\ \sum_{\text{all } \beta\neq i}\vec{F}_{i,\beta} \longrightarrow$$

Regroup, to add up the top of the grid matched w. partner in bottom

$$= \sum_{\text{all } i}\ \sum_{\text{all } \beta>0}\underbrace{\left(F_{i\beta}+F_{\beta i}\right)}_{=0 \text{ By N-III}} = \sum_i 0 = 0.$$

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Conclusion: For any object (or collection of objects, could be solid, or drop of liquid, or gas-filled chamber...)

$$\frac{d\vec{P}_{tot}}{dt} = \vec{F}_{net,\,ext}$$

all internal forces cancel out.

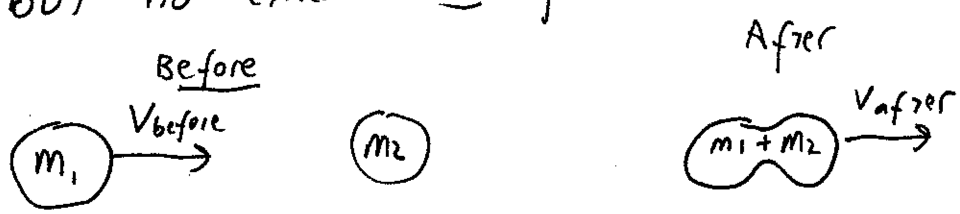
Now, $\vec{P}_{tot} = \sum_i \vec{P}_i = \sum_i m_i \vec{v}_i$. If $\vec{F}_{net,\,ext} = 0$,

$\vec{P}_{tot}$ is conserved, it doesn't change over time.

Irrespective of nature or complexity of internal interactions.

Very useful for "collision" or "explosion" problems, with complex internal interactions but no external net force.

E.g. Car crashes

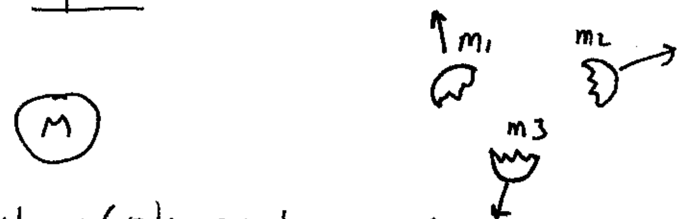


By observing (road marks? skid " ?) the motion after the crash,

Cons of $\vec{p}$ $\Rightarrow$ can deduce $\vec{V}_{before}$ (Ticket, or no?)

Ex: Explosions

Before | After



No matter what the details of the explosion, if you can track all outgoing pieces, cons. of momentum relates the individual $\vec{p}_i$'s. (So measuring all but one lets you deduce motion of the unobserved one)

For Particle Physicists looking at cloud or bubble chamber tracks, curvature in a $\vec{B}$ field tells you $\vec{p}$ (for charged objects)
Intensity gives info on ENERGY, (so $\Rightarrow$ can deduce m and $\vec{v}$)
But neutrals are typically "invisible", so cons. of $\vec{p}$ lets you deduce info on the missing objects, "seeing the invisible".

This was how we first "discovered" neutrinos, missing energy + momentum in radioactive decays. (Details of the "weak interaction" responsible for the decay was irrelevant!)

Ex: $\xrightarrow{\text{Laser}}$ $\leftarrow$ O Atom   If atom absorbs light (which has $\vec{p}$!)
cons of $\vec{p}$ $\Rightarrow$ Atom slows down. (!) "Laser cooling"

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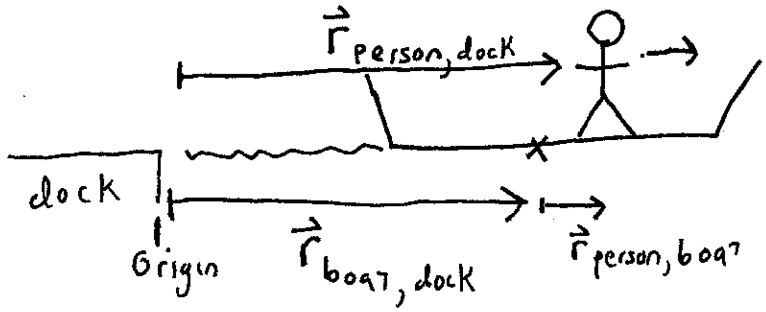
Conservation of $\vec{p}$ example: you are walking in a boat.
How does the boat respond? (Neglect boat-water drag)
so $\vec{F}_{net, external} = 0$!



Define $\vec{r}_{person, dock} \equiv$ "Position of person relative to the dock" etc

By inspection, $\vec{r}_{p,dock} = \vec{r}_{p,b} + \vec{r}_{b,dock}$

Look at the subscripts, make sense of this.
So take d/dt of both sides.

$$\vec{V}_{person, dock} = \vec{V}_{person, boat} + \vec{V}_{boat, dock}.$$

If you walk at speed $\vec{V}_0$, that's $\vec{V}_{person, boat}$, it's how fast you walk w.r.t. the floor. Simplifying to $\vec{V}_b = \vec{V}_{boat, dock}$

$\vec{V} = \vec{V}_{person, dock}$

$\vec{V}_0 = \vec{V}_{person, boat}$ = "walking speed"

$$\vec{V} = \vec{V}_0 + \vec{V}_b$$

If Boat starts at rest, $\vec{P}_{tot} = 0$ is conserved for all time!
+ Person

so $\vec{P}_{tot} = m_b \vec{V}_b + m_p \vec{V} = 0$ for all times. ~~At late times,~~

$\cancel{0 = m_b \vec{V}_b + m_p(\vec{V}_0 + \vec{V}_b)}$

The key here is $\vec{V}_0$ is known + fixed, (~ couple mph for normal walking)

So at any time

$$m_b \vec{V}_b + m_p (\vec{V}_0 + \vec{V}_b) = 0$$

so $$\vec{V}_b = -\frac{m_p \vec{V}_0}{m_b + m_p} = -\frac{m_p}{m_{tot}} \vec{V}_0$$ ← boat speed.

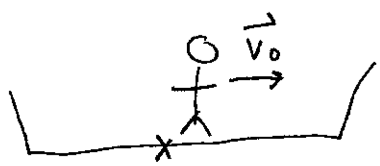
and $$\vec{V} = \vec{V}_0 + \vec{V}_b = +\frac{m_b}{m_{tot}} \vec{V}_0,$$ ← your speed

- Light boat, $m_{tot} \approx m_p$, $\vec{V}_b \approx -\vec{V}_0$, $\vec{V}_\bullet \approx \frac{\text{small}}{\text{big}} V_0 \approx 0$

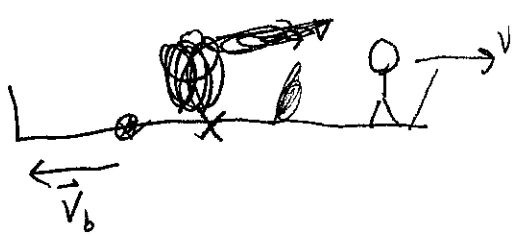
  the boat moves backwards, you practically stay put!

- Heavy boat, $m_{tot} \approx m_b$, $\vec{V}_b \approx 0$, $\vec{V}_\bullet = \vec{V}_0$

  you're just walking forward at normal walking speeds.



Rest frame of boat



Rest frame of dock

If $\vec{V}_0 = v_0 \hat{x}$, $\vec{V}^\bullet = v\hat{x}$, $\vec{V}_b = -V_b \hat{x}$, i.e. if we switch to speeds,

then $\vec{V}_\bullet = \vec{V}_0 + \vec{V}_b$ becomes $V = V_0 - V_b$ Note the minus sign

Your speed V is reduced from normal walking speed V_0, by boat's backward motion

Center of mass (Review from 1110?)

Given a system of particles (mass m_i, positions $\vec{r}_i$)

$$\vec{P} = \vec{P}_{TOT} = \sum_i m_i \vec{v}_i = \frac{d}{dt} \sum_i m_i \vec{r}_i$$

It's convenient to define $\vec{R}_{CM} = \frac{1}{M_{TOTAL}} \sum_i m_i \vec{r}_i$

(where $M_{TOT} = \sum_i m_i$)

This way, $M_{TOT} \vec{R}_{CM} = \sum_i m_i \vec{r}_i$, so $\underline{\vec{P} = \frac{d}{dt} M_{TOT} \vec{R}_{CM}}$

or $\vec{P} = M_{TOT} \vec{V}_{CM}$ $\left(\text{with } \vec{V}_{CM} \equiv \frac{d\vec{R}_{CM}}{dt} = \frac{1}{M_{TOT}} \sum_i m_i \dot{\vec{r}}_i\right)$

Also, $\frac{d\vec{P}}{dt} = M_{TOT} \frac{d\vec{V}_{CM}}{dt} = \sum_i m_i \ddot{\vec{r}}_i \underset{\text{By Newton 2}}{=} \sum_i \vec{F}_i = \vec{F}_{net}$

i.e. $\vec{F}_{NET} = M_{TOT} \ddot{\vec{R}}_{CM}$

So complex objects/systems "look like" a single pointlike object, mass M_{TOT}, located at $\vec{R}_{CM}$,

with velocity $\vec{V}_{CM} = d\vec{R}_{CM}/dt$

momentum $\vec{P}_{TOT} = M_{TOT} \vec{V}_{CM}$

satisfying $\vec{F}_{NET} = M_{TOT} \ddot{\vec{R}}_{CM}$.

(Simplifies a lot of complexity!)

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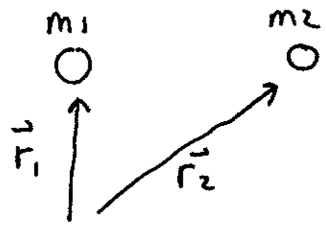
$\vec{R}_{CM} = \sum_i \frac{m_i}{M_{TOT}} \vec{r}_i$ is a "weighted average position"



$\vec{R}_{CM} = \frac{m_1}{m_1+m_2} \vec{r}_1 + \frac{m_2}{m_1+m_2} \vec{r}_2$

if $m_1 = m_2$, $\vec{R}_{CM} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, the "average position"

It's a vector, so e.g $x_{CM} = (\vec{R}_{CM})_x = \frac{m_1}{m_1+m_2} x_1 + \frac{m_2}{m_1+m_2} x_2$

Claim: $\vec{R}_{CM}$ lies on the line from $\vec{r}_1$ to $\vec{r}_2$ (if there are only 2 objects)

e.g, if $\vec{r}_1 \equiv 0$ (i.e, define m_1 to be at the origin)

then $\vec{R}_{CM} = 0 + \frac{m_2}{m_1+m_2} \vec{r}_2$, which points in the $\vec{r}_2$ direction, straight along the line from m_1 to m_2.

CM is closer to more massive object.

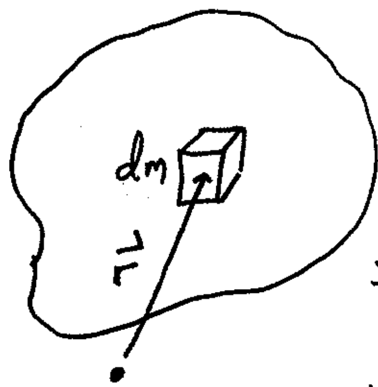
If $m_2 \to \infty$, $\vec{R}_{CM} \approx \vec{r}_2$

If $m_2 \to 0$, $\vec{R}_{CM} \approx \vec{r}_1$

If mass is distributed <u>continuosly</u>

$M_{TOT} \vec{R}_{CM} = \sum_i m_i \vec{r}_i$ becomes $\int \vec{r}\, dm$

What does this <u>mean</u>? Must split your volume up into little chunks:



and add them up.

<u>That</u>'s what integrals are, sums of little bits.

But, how do you integrate a vector?

Look at the components!

$$x_{CM} = \frac{1}{M_{TOT}} \int x\, dm$$

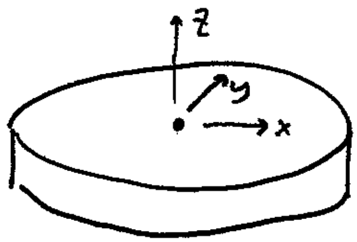
$$y_{CM} = \frac{1}{M_{TOT}} \int y\, dm$$

← Don't think of this as a "formal" or mathematical integral. It's a <u>sum</u> of little bits. You can't "do" the integral until you know more!

It'll make more sense when we do some examples.

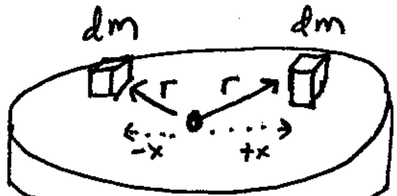
If your object has symmetry, you can often spot where $\vec{R}_{cm}$ is by eye, no formal integral needed.

E.g



Hockey puck: $\vec{R}_{cm}$ is at the center. So $X_{cm} = 0 = Y_{cm}$

In the formal expression $X_{cm} = \frac{1}{M_{tot}} \int x \, dm$, consider breaking it up into chunks dm



But by symmetry,

we can pair up the chunks, one with $+x\,dm$, the other with $-x\,dm$

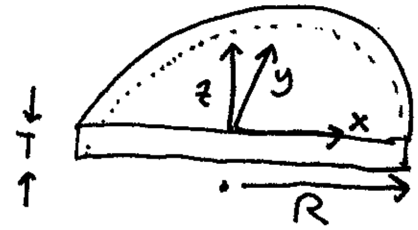
and so X_{cm} adds a bunch of canceling terms $= 0$!

(Same for Y_{cm}.)

If the puck is "T" thick, Z_{cm} will be $T/2$ from bottom.

(No real calculations needed here!)

Example: A <u>half</u>-hockey puck



By <u>inspection</u> (by symmetry)

$x_{CM} = 0$ (equal dm's on left + right)

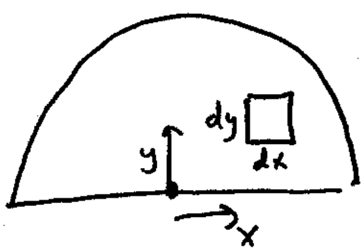
$z_{CM} = -T/2$ (halfway thru, looks like $z=0$ at top)

The only tough one is y_{CM}.

(I expect it to be <u>less</u> than $R/2$, because there's <u>more</u> mass near $y=0$ than near $y=R$.)

We want $y_{CM} = \frac{1}{M_{tot}} \int y\,dm$. (I assume density ρ is <u>uniform</u>)

Top view:



Could break into chunks "dx dy", with

mass $dm = \rho \cdot \underbrace{(dx\,dy\,T)}_{\text{volume of chunk}}$, but

surely this problem will be easier in <u>polar</u> coords.

Top view

<u>This</u> little mass chunk has

$dm = \rho \cdot \text{Volume} = \rho \cdot \text{Area} \cdot T$

$dm = \rho\,(r\,dr\,d\phi)\,\underbrace{T}$

(this T is clear - no need to worry about integrating over z, do you agree?)

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So $M_{TOT} = \underbrace{\int dm}_{\text{generic "sum"}} = \iint \rho T \, r\,dr\,d\varphi$

$$= \rho T \int_{r=0}^{R} r\,dr \int_{\varphi=0}^{\pi} d\varphi = \rho T \cdot \frac{R^2}{2} \cdot \pi$$

Oh, duhh! Mass = density * Thickness * $\frac{1}{2}\pi R^2$, of course!

and $y_{CM} = \frac{1}{M_{TOT}} \int_{r=0}^{R} \cdot \int_{\varphi=0}^{\pi} \underbrace{y} \, \underbrace{\rho T \, r\,dr\,d\varphi}_{\text{this is my "dm", right?}}$

(we're just summing up $y\,dm$!)

$$y_{CM} = \frac{1}{M_{TOT}} \int_{r=0}^{R} \int_{\varphi=0}^{\pi} (r\sin\varphi)\,(\rho T \, r\,dr\,d\varphi)$$

This was key,  $y = r\sin\varphi$ for our chunk dm!

$$= \frac{(\rho T)}{M_{TOT}} \int_{r=0}^{R} r^2\,dr \cdot \int_{\varphi=0}^{\pi} \sin\varphi\,d\varphi = \frac{\rho T}{M_{TOT}} \cdot \frac{R^3}{3} \cdot \underbrace{-\cos\varphi\Big|_0^{\pi}}_{=+2}$$

$$= \frac{\rho T \cdot 2R^3/3}{\rho T \cdot \frac{1}{2}\pi R^2} = \frac{4}{3}\frac{R}{\pi} \approx \underbrace{0.42\,R}$$

(Yup, less than $\frac{1}{2}$!)

(Note: $X_{CM} = \frac{1}{M_{TOT}} \iint r\cos\varphi\,(\rho T r\,dr\,d\varphi) = \ldots \sin\varphi\Big|_0^{\pi} = 0$, NICE!)

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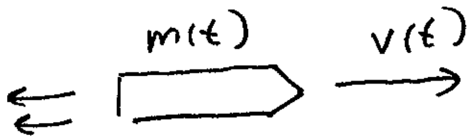
Applying conservation of Momentum - <u>Rockets</u>



No external forces here (!!)
The chemistry / explosions going on are all <u>internal</u> forces ... as long as we consider the ejected fuel to still be part of our system. <u>Must</u> keep track of it.

$\vec{F}_{ext} = 0$ means $\vec{P}_{Total, system}$ is conserved.

(ignoring e.g drag, which is tiny in space, but at launch, maybe we should add $-cv^2$... Let's not bother!)

<u>Note</u> : $\vec{F} = m\vec{a}$ is no good, because m is <u>not</u> constant.
This equation is <u>not</u> true, (Newton 2 is $\vec{F} = \frac{d\vec{p}}{dt}$.)

<u>Assume</u> in the "rocket frame", fuel is exhausted at speed $|V_{exh}|$

<u>Assume</u> this is constant in time (decent assumption)

<u>Assume</u> we lose mass, $\frac{dm}{dt}$, at a constant rate.

(This could depend on time, that's easy enough to include if <u>needed</u> to)

Note: $\frac{dm}{dt}$ is "rate of change of <u>rocket</u> mass", it's <u>not</u> "rate of fuel output"

so, it's manifestly <u>negative</u>, $\dot{m} < 0$ (M_{rocket} <u>decreases</u> with time as fuel leaves)

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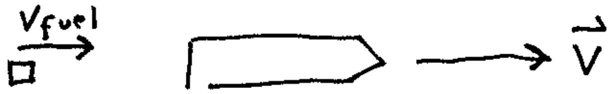
Look back at our "boat" problem. I claim, for an inertial observer on the ground:



$\vec{V}_{fuel} = \vec{V}_{exh} + \vec{V}$, or more carefully (with "wrt = with respect to")

$\vec{V}_{\text{fuel wrt observer}} = \vec{V}_{\text{fuel wrt rocket}} + \vec{V}_{\text{rocket wrt observer}}$

(Convince yourself, this is crucial!)

In our case, $\vec{V} = +V\,\hat{x}$

$\vec{V}_{exh} = -|V_{exh}|\,\hat{x}$ (fuel is ejected leftward)

So $\vec{V}_{fuel} = \vec{V}_{ex} + \vec{V}$

$\Rightarrow V_{fuel,x} = V_{ex,x} + V_x$

where these both mean "speeds" now!

$\Rightarrow V_{fuel,x} = -|V_{ex}| + V = -V_{ex} + V$

<u>Sense making check</u>: If $V = 0$ (i.e. initially), this says

$V_{fuel,x}$ (initially) $= -V_{ex}$. Good, it's to left!

If $V = V_{exh}$, i.e. rocket moves <u>right</u> at <u>same speed</u> it spews fuel out, then $V_{fuel,x} = -V_{ex} + V_{ex} = 0$. <u>Yup</u>, fuel is "launched left" at V_{exh} out of a right-moving rocket going $+V_{exh}$, the fuel exhaust comes out at <u>rest</u> to the observer. Convince yourself!!

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Now we're all set to apply conservation of momentum:

$m = m_{rocket}$ at time t.

After short interval dt, dm = change of rocket's mass (negative!)

so $-dm$ = (positive, physical) mass of the chunk of ejected fuel.

and $m + dm$ is the <u>new</u> rocket mass (it's <u>less</u> than m!)

At t: $\boxed{m} \xrightarrow{v}$ $\qquad \xrightarrow{+x}$

At $t+dt$: $\boxed{-dm} \xrightarrow{v_{fuel}}$ $\qquad \boxed{m+dm} \longrightarrow v + dv$

See prev page. The chunk of fuel has mass $(-dm)$,
and in the observer frame, has velocity $\vec{v}_{fuel}$, (which could be left or right, depending on $\vec{v}$, but in any case is <u>always</u> $\vec{v}_{ex} + \vec{v}$)

$$\begin{pmatrix}\text{Momentum at } t \\ (\underline{\text{system}})\end{pmatrix}_x = \begin{pmatrix}\text{Momentum at } t + dt \\ (\text{system})\end{pmatrix}_x$$

$$mv = (-dm)\, v_{fuel,x} + (m+dm)(v+dv)$$

previous page

$$= (-dm)(v - |v_{ex}|) + \underbrace{(m+dm)(v+dv)}$$

$$mv = \underbrace{-v\,dm}_{\text{cancel!}} + |v_{ex}|\,dm + mv + \underbrace{v\,dm}_{\text{cancel!}} + m\,dv + \underbrace{dm\,dv}_{\text{tiny!!}}$$

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$$m\vec{v} = -dm\,\vec{V}_{fuel} + (m+dm)(\vec{v}+d\vec{v})$$

cancels

tiny

$$\therefore\ m\vec{v} = -dm\,(\vec{V}_{exh}+\vec{v}) + (m\vec{v} + \vec{v}\,dm + m\,d\vec{v} + dm\,d\vec{v})$$

Cancels

So $-\vec{V}_{exh}\,dm + m\,d\vec{v} = 0$

so $m\dfrac{d\vec{v}}{dt} = \vec{V}_{exh}\dfrac{dm}{dt}$

Canceling mdv, we're left with

$$0 = |V_{ex}|\, dm + m\, dV$$

Divide by dt to get

$$m \frac{dv}{dt} = -|V_{ex}|\, dm/dt \qquad \text{the rocket equation.}$$

Sign check: $\frac{dm}{dt}$ is negative, so RHS is overall +, good!

It does look like $\vec{F} = m\frac{dv}{dt}$ after all, but $m(t)$,

and $\vec{F} = -|V_{ex}|\frac{dm}{dt}$. This is called "thrust".

Big thrust $\Rightarrow$ Big $|V_{ex}|$ and/or large rate $|dm/dt|$

If V_{exh} is constant,

$$m\, dv = -|V_{ex}|\, dm \Rightarrow \frac{dm}{m} = -\frac{dv}{|V_{exh}|} \qquad \text{separates!}$$

so $\ln m \Big|_{m_0}^{m} = -\frac{1}{|V_{exh}|}(V - V_0)$

or $\Delta V = +|V_{exh}| \ln m_0/m_{final}$

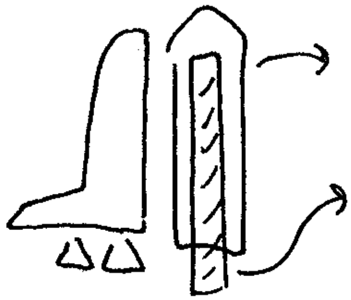
ln is "sluggish", you need $m_0 >> m_{final}$ to pick up speed!

$\ln(10)$ is just 2.3!

$$\frac{m_0}{m_{final}} = \frac{M_{empty\ rocket} + m_{fuel}}{M_{empty\ rocket}} = 1 + \frac{m_{fuel}}{M_{empty\ rocket}}.$$

So, better make m_{fuel} big, + $M_{empty\ rocket}$ small.

Space shuttle:



See wikipedia!

Stage 0: SRB's:
$|V_{exh}| \approx 2500$ m/s, thrust $\approx 10^7$ N each (*2) (~2 min)

Stage 2: Liquid:
$|V_{exh}| \approx 3600$ m/s, thrust $\approx 5 \cdot 10^6$ N (~ 8 min)

(Stage 3: Orbiter engines, thrust $\approx 5 \cdot 10^4$ N)

SRB burn is uncontrolled, can't stop it for ~2 min.

Drop tank (to reduce weight), pick up from Atlantic

External tank is released (+ burns up on reentry)

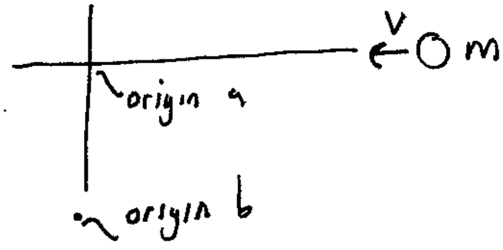
$$\frac{M_{SRB\ fuel}}{M_{SRB\ empty}} \simeq 6 \qquad \frac{M_{external\ tank,\ full}}{M\ "\ "\ ,\ empty} \simeq 30$$

2210 - 55

Wrapping up ch. 3, a brief discussion of angular momentum.

$\vec{L} \equiv \vec{r} \times \vec{p}$ for a point-like object.

Depends on origin! e.g:



From origin a perspective, $\vec{L} = \vec{r} \times m\vec{v} = 0$

" " b " , $\vec{L} = \vec{r} \times m\vec{v} \neq 0$!

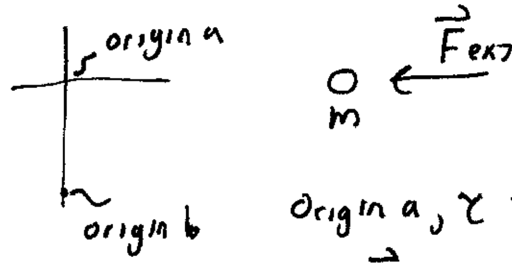
$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p} \quad \leftarrow \text{product rule.}$$

$$= \vec{r} \times \vec{F}_{ext} + \underbrace{\frac{\vec{p}}{m} \times \vec{p}}_{\text{this is } 0\,!}$$

$$\equiv \vec{\tau}_{ext} \quad \leftarrow \text{this too depends on origin!}$$

$\boxed{\text{So, if } \vec{\tau}_{ext} = 0 \quad \vec{L} \text{ is conserved}}$

Eg:



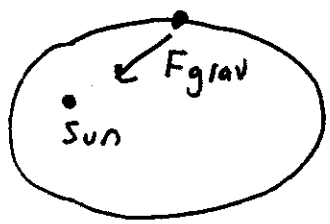
Origin a, $\tau = 0$, so $\vec{L}_{about\ a} = 0$ always.

Origin b: $\vec{\tau} \neq 0$, but see above, $\vec{L} = \vec{r} \times m\vec{v}$ is also changing with time!

2210 - 56.

Many applications! we won't pursue this, but just to mention 2:

① Orbits



with sun as origin,

$\vec{F}_{grav} \propto \hat{r}$, so

$\vec{\tau} = \vec{r} \times \vec{F} = 0$, so

$\vec{L}_{planet}$ is conserved.

$\vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = mvr_{\perp}$

As $r_{\perp} \downarrow$, $v \uparrow$

speed is not constant!

Kepler observed this first.

② Complex systems still conserve $\vec{L}$ if $\vec{\tau}_{net,ext} = 0$

This is more interesting than $d\vec{p}/dt$: think e.g. of ice-skater who pulls in her arms. r reduces $\Rightarrow$ $v_{rotation}$ increases.

or, a rotating star collapses, thus "spinning up" pulsars!