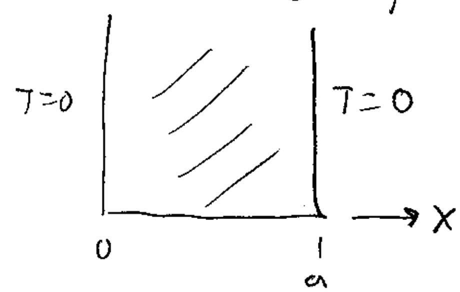
Fourier Transforms

We first encountered Fourier Series when we had a periodic

$f(t) = \sum_n b_n \sin n\omega_0 t \qquad (\omega_0 = 2\pi/T)$

We encountered them again, just now when solving Laplace's Eq'n with a rectangular plate:



The periodic bounary condition of our example problem led to solutions of the form $X(x) = \sin\frac{n\pi x}{a}$, and thus to a Fourier series.

In Fourier series, there is a "base" frequency ω_0, and an infinite set of other higher frequencies $n\omega_0$, but not all frequencies are present.

- What if that plate was not finite in x, but extended forever?
- Or similarly, what if our oscillator is driven by an $f(t)$ that has "infinite period" (i.e, it is not periodic)

How do we deal with such situations? Since $\omega_0 = 2\pi/T$, then if $T \to \infty$, it hints that perhaps there is no "base ω_0", + we will need to start from $\omega = 0$, and include all ω's.

This leads to an integral over ω, not a sum, Fourier Transforms

(See Boas 7.12)

COMPARISONS:

Fourier Series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 x + b_n \sin n\omega_0 x$$

or, using $e^{ix} = \cos x + i \sin x$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 x}$$

- Fourier's trick tells us

Note this sign

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-in\omega_0 x}\, dx$$

- Given $f(x)$, you can (thus) find $\{C_0, C_1, C_{-1}, C_2, C_{-2}, \dots\}$

(This ∞ set of #'s "represents" f, and vice versa)

- n is a dummy index. (could use any name for it)
- The C_n's are called the "Fourier Coefficients for $f(x)$"

Fourier Transform

think of letting $\omega_0 \to 0$, + this sum over discrete $n\omega_0$'s becomes an integral over continuous α's:

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha$$

Analogy to Fourier's trick gives

Note this sign

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

- Given $f(x)$, you can (thus) find $g(\alpha)$

(This ∞ set of #'s, i.e. this function, "represents" f, and vice versa)

- α is a dummy variable. (could use any name for it)
- $f(\alpha)$ and $g(\alpha)$ are called "Fourier Transforms" of each other

(or, $g(\alpha)$ is the Four. Transform of f and $f(x)$ is the inverse " " of g.)

In Fourier TRANSFORMS:

So $g(\alpha)$ corresponds to C_n ← continuous function rather than discrete set

α corresponds to n (or maybe $n\omega_0$) ← continuous frequencies now, rather than just integers

$\int_{-\infty}^{\infty}$ corresponds to $\sum_{-\infty}^{\infty}$ ← continuous integration, rather than discrete sums.

The factor $\frac{1}{2\pi}$ in front of the Fourier Transform is subtle, I didn't derive it for you. Some texts have different conventions! E.g., sometimes you see $\frac{1}{\sqrt{2\pi}}$ in both formulas for $f(x)$ + $g(\alpha)$

Note: If $f(x)$ is an odd function, we get only $b_n \sin n\omega_0 x$ terms.

With TRANSFORMS: $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$ Use Euler's theorem

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(x)}_{odd}\ \underbrace{\cos\alpha x}_{even}}_{\text{vanishes!}} - \frac{i}{2\pi}\int_{-\infty}^{\infty} \underbrace{f(x)}_{odd}\ \underbrace{\sin\alpha x}_{odd}\, dx$$

so replace with $2\int_0^{\infty}\ldots$

Pure sin integral.

Also, this $g(\alpha) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx$ is itself an odd function (convince yourself, let $\alpha \to -\alpha$!)

$$\text{so } f(x) = \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha = \int_{-\infty}^{\infty} \underbrace{\underbrace{g(\alpha)}_{odd}\,[\underbrace{\cos\alpha x}_{even}}_{\text{vanishes}} + i\underbrace{\sin\alpha x}_{odd}]\, d\alpha = 2i\int_0^{\infty} g(\alpha)\sin\alpha x\, d\alpha$$

Summarizing, for ODD functions, we get Fourier sine Transform

$$f(x) = 2i \int_0^\infty g(\alpha) \sin(\alpha x)\, d\alpha$$

and the Fourier sine transform of $f(x)$ is

$$g(\alpha) = -\frac{i}{2\pi} \int_0^\infty f(x) \sin \alpha x \, dx$$

As before, there are various conventions regarding the constants.

For instance, if I define $g_s(\alpha) \equiv i\sqrt{2\pi}\, g(\alpha)$ ← Just a simple rescaling

then (convince yourself!) $$f_{odd}(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \sin \alpha x \, d\alpha$$

and $$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_{odd}(x) \sin \alpha x \, dx$$

Looks nicer, more symmetric, and all real!

Similarly, if you have an even function $f(x)$, you'll get

$$f_{even}(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\alpha) \cos \alpha x \, d\alpha$$

and $$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_{even}(x) \cos \alpha x \, dx$$

Lastly, if f is a function of time $f(t)$, the story is basically the same (just swap $t \leftrightarrow x$). In this case, it's quite common to rename α back to "ω", which seems very natural!

Summarizing that last point

$$f(t) = \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega$$ ← Looks a lot like Fourier series!

where $$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$ ← Looks a lot like Fourier coefficients

here, $g(\omega)$ is the Fourier transform of $f(t)$

and $f(t)$ is the inverse fourier transform of $g(\omega)$.

As mentioned, different texts have different conventions which amount to multiplying through by a convenient constant.

For instance, we can make the eq'ns look more symmetric if we redefine $\tilde{g}(\omega) \equiv \sqrt{2\pi}\, g(\omega)$. (or, $g(\omega) = \frac{1}{\sqrt{2\pi}} \tilde{g}(\omega)$)

Plug this in at the top of the page + convince yourself that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(\omega)\, e^{i\omega t}\, d\omega$$

where $$\tilde{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

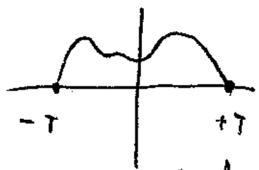
Nice + symmetric!

PDE - 24.5 - ~~23~~ - <u>Optional</u> <u>SIDE</u> <u>NOTE</u>

Here's a crude sketch (not proof!) of the Fourier transform formulas

Recall

For Fourier <u>series</u>:



$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}, \quad \omega_0 = \frac{2\pi}{2T} \text{ see figure!}$$

and, as usual, $C_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-in\omega_0 t} dt$

As $T \to \infty$, let's define a variable $\alpha \equiv n\omega_0 = n\pi/T$

As $T \to \infty$, this α because continuous, + as n marches from $-\infty$ to $+\infty$,

α continuously marches from $-\infty$ to ∞ too.

C_n should now be considered $C(n)$, or better yet, $C(\alpha)$, (a <u>function</u>.)

Finally, note that $\Delta\alpha \equiv \alpha_{n+1} - \alpha_n = (n+1)\omega_0 - n\omega_0 = \omega_0 = \pi/T$

so, $T \frac{\Delta\alpha}{\pi} = 1$. OK, we're set, go back to the usual expressions above

---

$$f(t) = \sum_{\alpha=-\infty}^{\infty} C(\alpha) e^{i\alpha t} = \sum_{\alpha=-\infty}^{\infty} C(\alpha) e^{i\alpha t} \overbrace{\frac{\Delta\alpha T}{\pi}}^{\text{this is just 1!}}$$

As $T \to \infty$, $\Delta\alpha \to d\alpha \to 0$, + this is basically

$$f(t) = \int_{-\infty}^{\infty} \frac{T}{\pi} C(\alpha) e^{i\alpha t} d\alpha.$$

C_n from top of page

Let's <u>define</u> $g(\alpha) \equiv \frac{T}{\pi} C(\alpha) = \frac{T}{\pi} C_n = \frac{T}{\pi} \cdot \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\alpha t} dt$

Look at what we have:

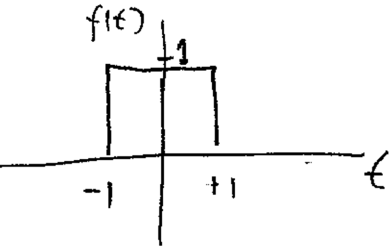
$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt$$

and, recapping,

$$f(t) = \int_{-\infty}^{\infty} g(\alpha) e^{+i\alpha t} d\alpha$$

OUR expressions we <u>claimed</u> earlier, with the 2π's now at least somewhat justified!

<u>Example</u>: we initially suggested that Fourier transforms would be useful if we have a non-periodic (i.e. one-time-only) impulsive force. Let's work this out using our new Fourier transform formulas.



Let's suppose $f(t) = \begin{cases} 1 & \text{for } |t|<1 \\ 0 & \text{for } |t|>1 \end{cases}$

This is <u>not</u> periodic, it's a single blip.

It's <u>even</u>, so we can use our Fourier-cosine expressions, or we can just stick with the full complex form (it's all the same). Let's do the latter, & we'll stick with Boas's "2π" convention:

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{2\pi}\int_{-1}^{1} e^{-i\omega t}\, dt$$

Because $f(t)$ vanishes everywhere else

$$= \frac{1}{2\pi}\,\frac{1}{-i\omega}\, e^{-i\omega t}\Big|_{t=-1}^{t=+1} = \frac{1}{-2\pi i\omega}\left(e^{-i\omega} - e^{+i\omega}\right)$$

By Euler, this is $-2i\sin\omega$

$$g(\omega) = +\frac{\sin\omega}{\pi\cdot\omega}$$

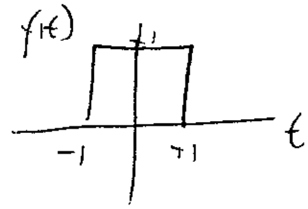
Remember, ω is a dummy index. This function is the continuous version of our old C_n's. <u>Before</u>, we had $f(t) = \sum_n C_n e^{in\omega_0 t}$

<u>now</u> we have $f(t) = \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha t}\, d\alpha$

<u>Before</u>, we had $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-in\omega_0 t}\, dt$

<u>Now</u> we have $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\alpha t}\, dt$

Continuing with this example:

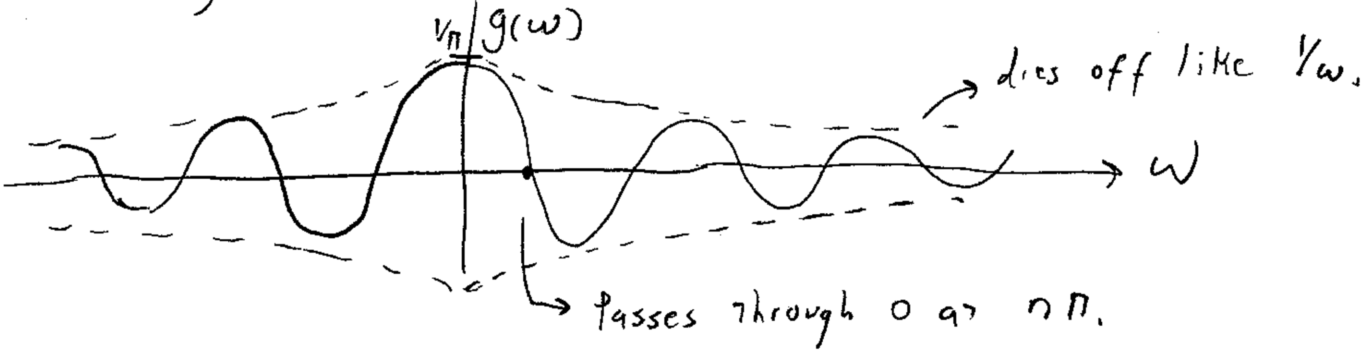


we found $g(\omega) = \frac{\sin \omega}{\pi \omega}$. Let's sketch this.

As $\omega \to 0$, need L'Hopital's rule, $g(\omega)\Big|_{\omega \to 0} = \frac{\frac{d}{d\omega}\sin\omega}{\frac{d}{d\omega}\pi\omega}\Bigg|_{\omega\to 0} = \frac{\cos\omega}{\pi}\Bigg|_{\omega \to 0} = \frac{1}{\pi}$

(that's a surprise!)

As $\omega \to \infty$, the $1/\omega$ kills us off. So here it is

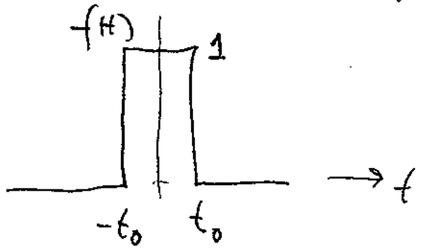


Our $f(t)$ was localized in time. But when we think of it as a sum of sinusoids $e^{i\omega t}$, with strength $g(\omega)$ i.e. $\int g(\omega) e^{i\omega t}$

We see that $g(\omega)$ is spread out, you need many ω's (all, in fact!)

Now, $g(\omega)$ is dominated (big amplitude) for ω up to, oh, the 1st or 2nd zero, i.e. out to $\omega \sim$ couple $* \pi$, before it fades away.

Let's pursue this by considering an $f(t)$ that's more limited in time.

So if 

In this case, $g(\omega) = \frac{1}{2\pi}\int_{-t_0}^{t_0} e^{-i\omega t}\,dt$

$= \frac{1}{\pi\omega}\sin \omega t_0$

Nearly same as before, but $g(\omega)$ hits zero first when $\omega = \pi/t_0$

So if t_0 gets smaller, $g(\omega)$'s first zero gets bigger, i.e. $g(\omega)$ is getting wider.

Mathematically (or physically!) this says you need more ω's, more "strength at higher frequency" to build up a narrower $f(t)$.

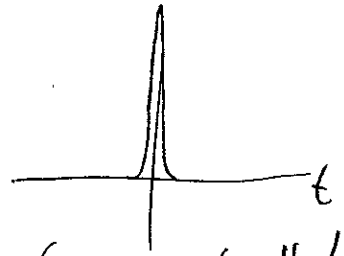
It's a very general feature of Fourier TRANSFORMS that

$$\text{(width of } f(t)) \propto \frac{1}{(\text{width of } g(\omega))}$$

This has many consequences!

To get a signal that's a quick blip, you need many frequencies (so e.g. short laser pulses have many "colors" involved)

As we'll see, the Heisenberg uncertainty principle also arises from this!

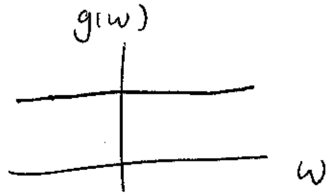


Example: Consider $f(t) = f_0\, \delta(t)$

This is the skinniest, most localized function of all!

Let's find $g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_0\, \delta(t)\, e^{-i\omega t}\, dt = \frac{f_0}{2\pi}$

Aha!



This is a constant!

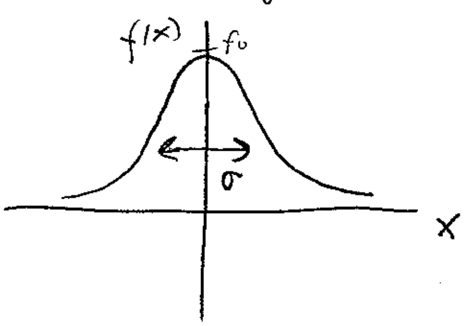
Infinitely narrow $f(t)$ $\Longleftrightarrow$ infinitely wide $g(\omega)$!

you need all frequencies (equally strong) to "build up" a δ function.

$\Rightarrow$ Short laser pulses have no definite color (!)
Short percussive claps have no definite pitch.

Example: Consider the Gaussian function $f(x) = f_0\, e^{-x^2/2\sigma^2}$

- Note I'm going back to $f(x)$, + thus will revert to $g(\alpha)$



σ is called the standard Deviation, + it directly tells you the width of this function.

you can compute $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

This would make a nice homework problem! I'll show you some tricks to help find the answer $g(\alpha) = \frac{\sigma f_0}{\sqrt{2\pi}}\, e^{-\alpha^2\sigma^2/2}$

PDE - 28.5 - -Aside-

The trick to this integral is called "completing the square".

We have $g(\alpha) = \frac{f_0}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2} - i\alpha x}\, dx = \frac{f_0}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma^2} + i\alpha x\right)} dx$

The thing is, I happen to know $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

you can look this up, or ask me for a clever proof!

So the trick is to "complete the square" + do a u-sub. First

Write $\left(\frac{x^2}{2\sigma^2} + i\alpha x\right) = \left(\frac{x}{\sqrt{2}\sigma} + \text{something}\right)^2 - \text{same something}^2$

By inspection (check for yourself!) this is

$$\left(\frac{x}{\sqrt{2}\sigma} + \frac{i\alpha\sigma}{\sqrt{2}}\right)^2 + \frac{\alpha^2\sigma^2}{2}$$

So letting $u = \left(\frac{x}{\sqrt{2}\sigma} + i\frac{\alpha\sigma}{\sqrt{2}}\right)$, so $du = \frac{dx}{\sigma\sqrt{2}}$, you get $g(\alpha)$ as claimed

So $g(\alpha) = \frac{\sigma f_0}{\sqrt{2\pi}} e^{-\alpha^2\sigma^2/2}$

This is also a Gaussian. The standard deviation is $\frac{1}{\sigma}$ (!!)

So this is consistent with our pattern: narrow in $f(x)$ $\Leftrightarrow$ wide in $g(\alpha)$ (+ vice versa)

In quantum mechanics, we will find that

$$\underbrace{\Psi(x)}_{\text{The wave fn}} \propto \int_{-\infty}^{\infty} \underbrace{\Phi(p)}_{} e^{2\pi i p x/h} \, dp$$

This looks mathematically just like "g", the fourier transform (except for some unit sneakiness)

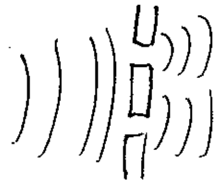
$\Phi(p)$ is the wave function too, in terms of momentum.

Narrow $\Psi(x)$ (well defined x) $\Leftrightarrow$ wide $\Phi(p)$ (poorly defined momentum)

+ vice-versa, the Heisenberg Uncertainty principle!

Physics is filled with applications of Fourier transforms!

when light scatters from an object



e.g a slit, or pair of slits, or lattice ...

The transmitted light is a sum of waves (this is Huygen's principle)

The intensity of detected light $\propto$ $\left| \text{Fourier transform of } f(x) \right|^2$

where $f(x)$ describes the spacial distribution of scatterers /holes ...

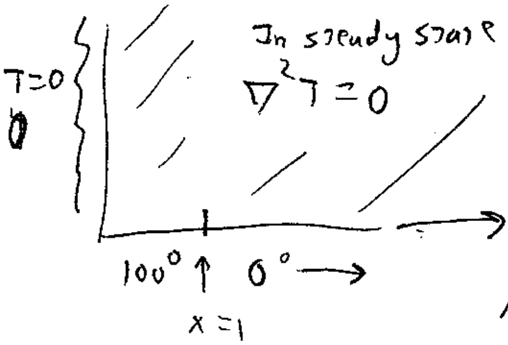
So you see the fourier transform of the crystal lattice or slits!

(For a crystal, this requires a 2-D generalization of what we've done)

But again, a smaller hole / slit / scatterer $\Rightarrow$ a wider pattern (more diffraction!)

Example Back to our square plate. Let it be ∞ big now!

Let's set $T(x, y=0) = f(x) = \begin{cases} 100^\circ & 0<x<1 \\ 0^\circ & x>1, \text{ \underline{all} the way} \ldots \end{cases}$



$T(x=0, y) = 0^\circ$.

As before, try $T(x) = X(x)\, Y(y)$

As before, $\nabla^2 T = 0 \Rightarrow X''(x) = -k^2 X(x) \Rightarrow A\sin kx + B\cos kx$

$Y''(y) = +k^2 Y(y) \Rightarrow Ce^{ky} + De^{-ky}$

As before, ambiguity of sign choice on k^2, but this will work out.

As before, $T(y \to \infty) = 0 \Rightarrow C$ vanishes, no growing exponential

As before, $T(0,y) = 0 \Rightarrow B = 0$, only $\sin kx$ vanishes at $x=0$.

But now, no second B.C. in x, no period, all possible k's are OK!

So, instead of summing over sol'ns, we integrate over all k

$$T(x,y) = \int_0^\infty B(k) \cdot \sin kx \, e^{-ky} \, dk$$

The B.C at $y=0$ will help, $T(x,0) = f(x) = \int_0^\infty B(k) \sin kx \, dk$

This is just our Fourier sin expression, + from early (p.23) we can read off the sol'n for $B(k)$.

Letting "α" become "K", and using the "g_s" convention (p.23)

If $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(K) \sin Kx \, dK$

Then $g_s(K) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin Kx \, dx$

Here, letting $B(K) = \sqrt{\frac{2}{\pi}}\, g_s(K)$ we get $f(x) = \int_0^\infty B(K) \sin Kx \, dK$,

+ so

$$B(K) = \sqrt{\frac{2}{\pi}}\, g_s = \underline{\underline{\frac{2}{\pi} \int_0^\infty f(x) \sin Kx \, dx}}$$

That's it! I can find $B(K)$ for $\underline{\underline{\text{any}}}$ $f(x)$

For the given one, I get

$$B(K) = \frac{2}{\pi} \int_0^1 100^\circ \sin Kx \, dx = \frac{200}{\pi} \left. \frac{\cos Kx}{-K} \right|_{x=0}^{x=1} = \frac{200}{\pi K}(1 - \cos K)$$

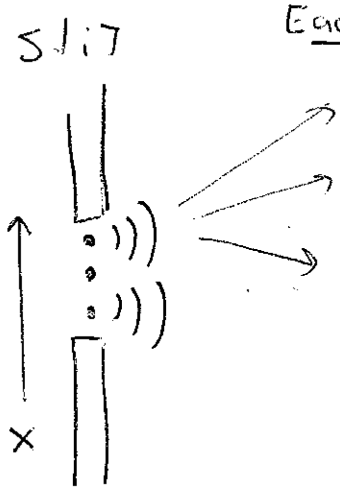
+ thus,

$$T(x,y) = \int_0^\infty \frac{200}{\pi} \left(\frac{1 - \cos K}{K} \right) e^{-Ky} \sin Kx \, dK.$$

It's a 1-D integral, just compute it. If you can't do it analytically, it's a definite integral, so you can always compute it numerically for any desired x, y.

This ends our (first!) treatment of Fourier Transforms; their power is clear, Fourier series limit us to periodic functions, transforms do not!

Fourier supplement #1



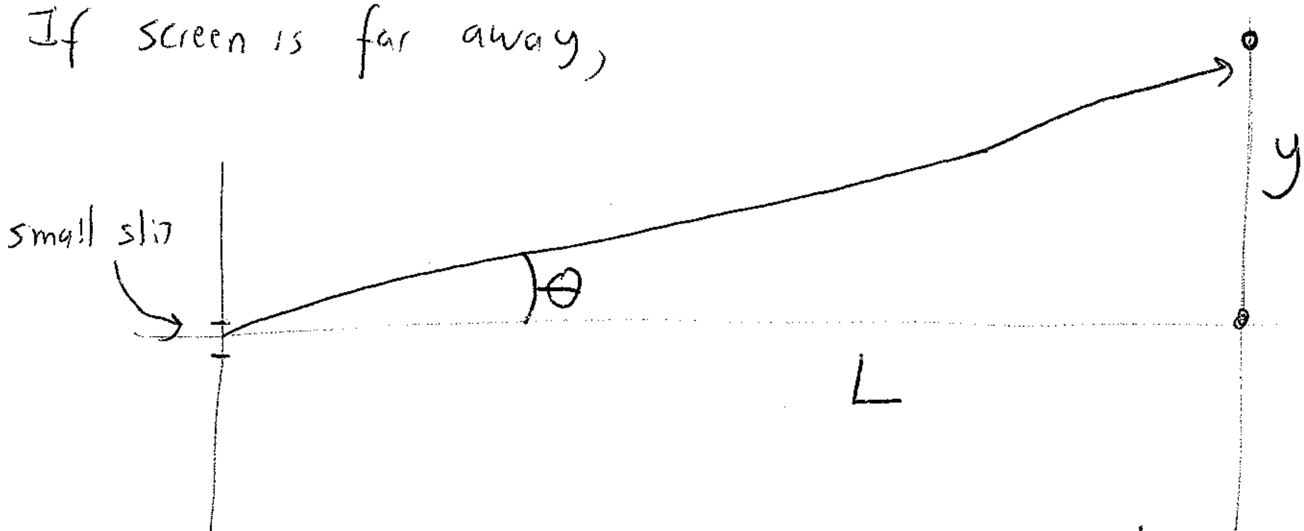
Each point x is a source of outgoing rays then superpose!
"Huygen's principle"

some E field pattern given by $A(x)$, the "spatial distribution of slits"

Screen, where you Detect $E(y)$ (or in fact, $|E(y)|^2$ = "brightness")

• At a given point y, where does $E(y)$ come from? It's the superposition, the sum, of all the E's arriving from all the different x's.

If screen is far away,

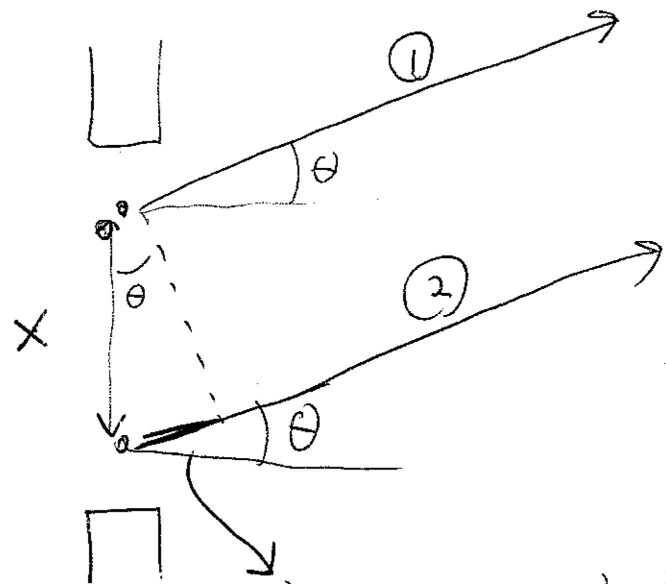


Big Distance $L \Rightarrow \theta \simeq y/L$

so y is just a measure of the ray direction θ

Fourier Supplement #2

Back to the slit, blown up.



If L is big
These two "parallel" rays both reach the same point y (!)
(parallel rays converge at ∞)

This extra path length is $x \sin\theta \approx x\theta$

It's the extra distance one ray goes compare to the other.

so the E fields of the two rays (which we add)

are $E_1 + E_2 = E_1(1 + e^{i\delta})$

a phase shift caused by that extra path length

I claim $\delta = 2\pi$ for each λ extra that the path length represents

so $\delta = \frac{2\pi}{\lambda} * (\text{extra distance})$

$= \frac{2\pi}{\lambda} x\theta$

so $E = E_1(1 + e^{i\delta_{x_2}} + e^{i\delta_{x_3}} + e^{i\delta_{x_3}} + \dots)$

Fourier supplement #3

Thus $$E(y) \propto \int \underset{\substack{\text{Sum over all} \\ \text{sources}}}{dx} \; \underbrace{A(x)}_{\substack{\text{the } \cancel{\text{strength}} \\ \text{spatial} \\ \text{distribution of} \\ \text{slits}}} \cdot e^{i\delta_\theta}$$

$$= \int dx \; A(x) \, e^{i \frac{2\pi x \theta}{\lambda}}$$

$$= \int dx \; A(x) \, e^{i \frac{2\pi x}{L} y}$$

$$= \int dx \; A(x) \, e^{i x \left(\frac{2\pi y}{L}\right)}$$

This is thus saying that $E(y)$ at the screen is the fourier transform (with constants, $2\pi/L$) of $A(x)$ (and vice versa)

The pattern (image) on screen ($E(y)$) is the F.T. of the slit pattern $A(x)$!

So e.g narrow slits $\Rightarrow$ wide patterns, + vice versa.