CLASSICAL MECHANICS AND MATH METHODS, SPRING, 2011

Homework 8

(Due Date: Start of class on Thurs. March 3)

1. Look at the figure below, which shows a possible design for a child's toy. The designer (you!) wants to build a "weeble" (which wobbles, but doesn't fall down). It is a single, solid object, the shape is a perfect hemisphere (radius a) on the bottom, glued to a solid cylinder on top. The CM is shown, it is "h" above the base. You don't need to try to compute h, assume it is a given quantity. (As the designer, how might you control/change the position of h?)

(a) Write down the gravitational potential energy when the weeble is tipped an angle θ from the vertical, as a function of given quantities (m, a, h, g, and of course θ).
It might help to think about the height of the hemisphere's center O as the toy tilts.

(b) Assuming the toy is released from rest, determine the condition(s) for any equilibrium point(s). Then consider stability: what values of (or relations between) "a" or "h" ensure that the weeble doesn't fall down? Explain. Can this toy work as desired?

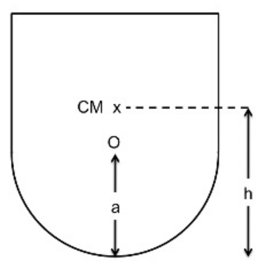

Figure 1:

2. A particle is under the influence of a force $\vec{F} = k^2(-x + x^3/\alpha^2)\hat{x}$, where k and α are constants.

(a) What are the units of k and α? Now, determine $U(x)$ assuming $U(0) = 0$ and sketch it. (Sketch means hand-drawn, not plotted with Mathematica. Include all features you consider interesting in your sketch, including e.g. zero crossings, behavior at large x, "scales" of your axes, etc)

(b) Find the equilibrium points, if any, and determine if they are stable or unstable.

(c) Qualitatively explain the motion of an object in this force field released from rest at $x = \alpha/2$. Then, qualitatively explain the motion of an object in this force field released from rest at $x = 3\alpha/2$

This potential, or small variants of it, occurs in various physics situations. (This is a more accurate form of the true force on a pendulum than our usual approximation, can you see why?) Another (famous!) case, albeit with signs flipped, gives the "Higgs potential" in particle physics, which is being actively investigated at CERN's LHC collider.

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3. A planet of mass M and radius R has a nonuniform density that varies with r, the distance from the center according to $\rho = Ar$ for $0 \leq r \leq R$.

(a) What is the constant A in terms of M and R? Does this density profile strike you as physically plausible, or is just designed as a mathematical exercise? (Briefly, explain)

(b) Determine the gravitational force on a satellite of mass m orbiting this planet. In words, please outline the method you plan to use for your solution. (Use the easiest method you can come up with!) In your calculation, you will need to argue that the magnitude of $\vec{g}(r, \theta, \phi)$ depends only on r. Be very explicit about this - how do you know that it doesn't, in fact, depend on θ or ϕ?

(c) Determine the gravitational force felt by a rock of mass m inside the planet, located at radius $r < R$. (If the method you use is *different* than in part b, explain why you switched. If not, just proceed!) Explicitly check your result for this part by considering the limits $r \to 0$ and $r \to R$.

4. Imagine (in a parallel universe of unlimited budgets) that NASA digs a straight tunnel through the center of the moon (see figure). A robot place a rock in the tunnel at position $r = r_0$ from the center of the moon, and releases it (from rest). Use Newton's second law to write the equation of motion of the rock and solve for r(t). Explain in words the rock's motion. Does the rock return to its initial position at any later time? If so, how long does it takes to return to it? (Give a formula, and a number.) Assume the moon's density is uniform throughout its volume, and ignore the moon's rotation.

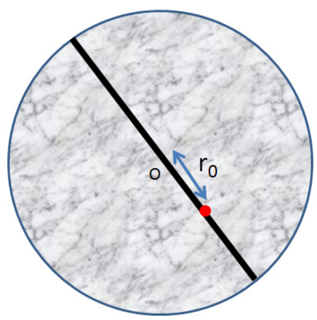

Figure 2:

5. Consider a very (infinitesimally!) thin but massive loop, radius R (total mass M), centered around the origin, sitting in the x-y plane. Assume it has a uniform linear mass density λ (which has units of kg/m) all around it. (So, it's like a skinny donut that is mostly hole, centered around the z-axis)

(a) What is λ in terms of M and R? What is the direction of the gravitational field generated by this mass distribution at a point in space a distance z above the center of the donut, i.e. at $(0, 0, z)$ Explain your reasoning for the direction carefully, try not to simply "wave your hands." (The answer is extremely intuitive, but can you justify that it is correct?)

(b) Compute the gravitational field, $\vec{g}$, at the point $(0, 0, z)$ by directly integrating Newton's law of gravity, summing over all infinitesimal "chunks" of mass along the loop.

(c) Compute the gravitational potential at the point $(0, 0, z)$ by directly integrating $-Gdm/r$, summing over all infinitesimal "chunks" dm along the loop. Then, take the z-component of the gradient of this potential to check that you agree with your result from the previous part.

(d) In the two separate limits $z << R$ and $z >> R$, Taylor expand your g-field (in the z-direction)out only to the first non-zero term, and convince us that both limits make good physical sense.

(e) Can you use Gauss' law to figure out the gravitational potential at the point $(0, 0, z)$? (If so, do it and check your previous answers. If not, why not?)

Extra credit: If you place a small mass a small distance z away from the center, use your Taylor limit for $z << R$ above to write a simple ODE for the equation of motion. Solve it, and discuss the motion