



Electrodynamics HW Problems

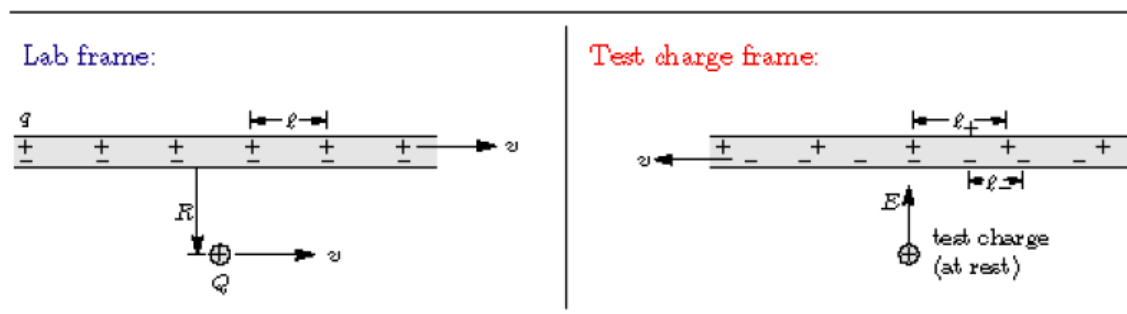
10 – Special Relativity

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10.01. Magnetic field of a wire [Dubson SP12, Pollock FA11]

One of the best texts for EM is Purcell's advanced freshman text. In it, he shows a wonderful application of the Lorentz contraction, to begin to understand how magnetism and electricity are related to one another. Let's step through the basics! (This exercise comes from an AAPT talk by Dan Shroeder of Weber State.)



Shown above is a model of a wire with a current flowing to the right. To avoid minus signs we take the current to consist of a flow of positive charge carriers, each with charge $+q$, separated by an average distance of ℓ . The wire is electrically neutral in the lab frame, so there must also be a bunch of negative charges, at rest, separated by the same average distance ℓ in this frame. Be aware that charge is Lorentz invariant: a charge Q has the same value in every inertial frame.

(a) Using Gauss' law, what is the electric field outside this wire in the lab frame? Suppose there is a test charge $+Q$ outside the wire, a distance R from the center of the wire, moving to the right (For simplicity, let's say the velocity is the same as that of the moving charges in the wire, i.e. v , as shown in the figure.)

- Given your answer for the E-field, what is the electrostatic force on this charge, in this frame?
- Using Ampere's (and the Lorentz force) law – what is the magnetic force on the moving test charge Q ?
- Put it together, what is the direction of the net force on the test charge, and what "causes" it?

(b) Now consider how all this looks in the reference frame of the test charge, where it's at rest.

- In THIS frame, what is the magnetic force on the test charge Q ? In this frame, it's the negative charges in the wire that are moving to the left. Because they're moving, the average distance between them is length-contracted. Meanwhile the positive charges are now at rest, so the average distance between them is now longer than ℓ .

(cont...)

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10.01. (cont.)

- What is the average distance (ℓ_+) between the positive charge carriers in this frame? Both of these effects give the wire a non-zero charge density.
- Compute the charge density (charge per length) in this frame, with the correct overall sign.
- Use Gauss's Law to compute the electrostatic force on the test charge.
- In THIS frame, what is the magnitude and direction of the force on the test charge, and what "causes" it?

(c) For normal currents, $\beta = v/c$ is about 10^{-13} . (Drift velocities are small!) Given this, show that the forces you computed in parts (a) and (b) are the same size.

Hint: expand in a Taylor series.

10.02. Star Wars with relativity [Dubson SP12, Pollock FA11, Rogers SP09]

(a) Space Probe #1 passes very close to earth at a time that both we (on earth) and the onboard computer on Probe 1 decide to call $t = 0$ in our respective frames. The probe moves at a constant speed of $0.5c$ away from earth. When the clock aboard Probe 1 reads $t = 60\text{sec}$, it sends a light signal straight back to earth.

- At what time was the signal sent, according to the earth's rest frame?

- At what time in the earth's rest frame do we receive the signal?

- At what time in *Probe 1's* rest frame does the signal reach earth?

(b) Space Probe #2 passes very close to earth at $t = 1\text{sec}$ (earth time), chasing Probe 1. Probe 2 is only moving at $0.3c$ (as viewed by us). Probe 2 launches a proton beam (which moves at $v = 0.21c$ relative to Probe 2) directed at Probe 1. Does this proton beam strike Probe 1? Please answer twice, once *ignoring* relativity theory, and then again using Einstein!

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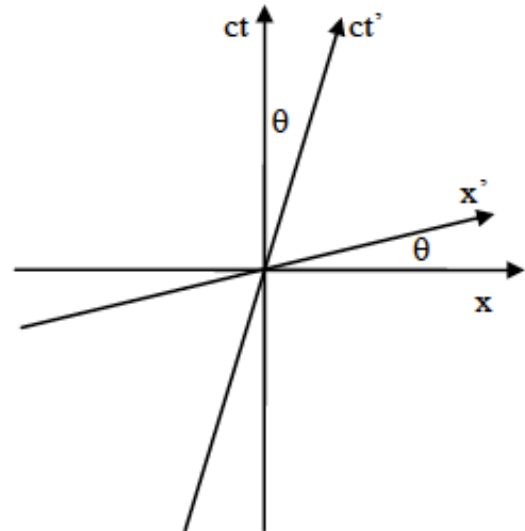
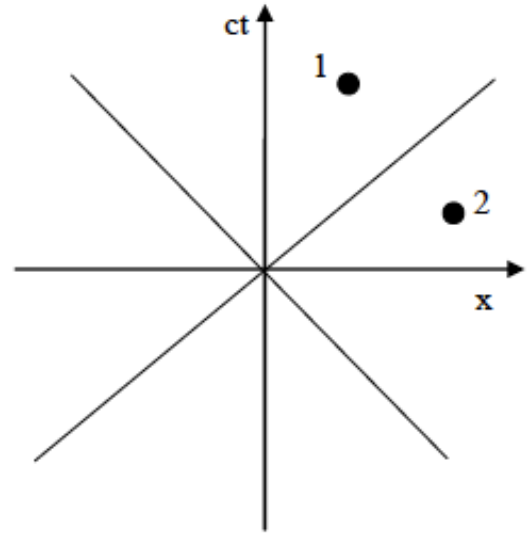
10.03. Spacetime diagrams [Dubson SP12]

Consider the two points labeled 1 and 2 in this spacetime diagram. Notice that in the Lorentz frame represented by this diagram, both events are in the future: $t_1 > 0$ and $t_2 > 0$. (The diagonal lines with 45° slope represent the light cone.)

(a) Prove that, regardless of the value of $\beta = v/c$, when you boost to a different Lorentz frame, event 1 remains in the future, that is, $t'_1 > 0$ always. Also prove that event 2 can be either in the future, $t'_2 > 0$, or in the past, $t'_2 < 0$, depending on the value of $\beta = v/c$.

(b) In a different Lorentz frame, the axes of the primed frame are tilted relative to the axes of the unprimed frame, as shown.

- Prove that for the usual boost of $\beta = v/c$ in the $+x$ -direction, the primed axes are tilted in the directions shown and both are tilted by the same size angle θ , as shown. What is the relation between $\beta = v/c$ and the tilt-angle θ ?
- Draw appropriate diagrams that illustrate clearly why event 2, in the primed frame, can be either in the past or the future, depending on the value of β .



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10.04. Faster-than-light neutrinos(?) [Pollock FA11]

Let's define the LHC lab in Switzerland to be located at $x_1=0$ in the earth's rest frame, and the Italian Gran Sasso neutrino detection facility is located at $x_2=+730$ km. (Ignore any motion of the earth in this problem.)

(a) Suppose that a beam of photons travels from LHC (starting at $x_1 = t_1 = 0$) heading to Gran Sasso in a straight line (through vacuum). At what time t_2 will the photons arrive at the detector in Gran Sasso (in the earth's rest frame)? (Neglect general relativity ☺)

Now consider the photon arrival time as measured in an inertial frame that is moving past LHC towards Gran Sasso at high speed v . (But of course, not greater than c !) This frame has been synchronized so that $x_1' = t_1' = 0$ when the photons leave LHC on their way towards Gran Sasso (i.e. the origins of the two inertial frames ($x_1 = x_1' = 0$) coincide at $t_1 = t_1' = 0$). In the limit that v approaches c (from below), what happens to the photon arrival time t_2' as measured in the moving frame? Show that t_2' is NEVER less than or even equal to 0 in an inertial frame that could contain a physical observer (one that is moving at less than c). Interpret this result in words (think about "causality").

(b) Now SUPPOSE that a beam of neutrinos could travel from LHC (starting at $x_1 = t_1 = 0$) heading to Gran Sasso in a straight line with a speed $v = c(1 + 2.5 \times 10^{-5})$. You read that right, let's suppose for the sake of argument that neutrinos somehow travel FASTER than c by 25 ppm.

- At what time t_2 will these neutrinos arrive at the detector in Gran Sasso (in the earth's rest frame)? How much sooner do they arrive than the photons of part (a)?

- Now consider the neutrino arrival time, as measured in an inertial frame that is moving past LHC towards Gran Sasso at high speed v . (But of course, not greater than c !) This frame has been synchronized so that $x_1' = t_1' = 0$ when the neutrinos leave LHC on their way towards Gran Sasso (i.e. the origins of the two inertial frames coincide at $t_1 = t_1' = 0$). What speed (with respect to the earth frame) does an inertial frame need, in order for a local observer in that frame to measure the neutrinos hitting the detector at $t_2' = 0$ (i.e., at EXACTLY the same time as they left in that frame?) What is γ for this inertial frame?

- If this inertial frame were moving just a little FASTER than the above speed (but still less than c , always!) what would be the SIGN of t_2' , the time in that frame at which the neutrinos arrive? Is there any physical principle forbidding you from being an observer in this moving inertial frame? (In such an inertial frame, I claim the neutrinos are measured to arrive at the detector before they are emitted from the source. What do you think of this result?)

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(c) Sketch the world line of a neutrino emitted from LHC and ending at Gran Sasso. On the same diagram, with a dashed line, indicate the world line of a photon making the same trip. Given such (claimed) data, is the space-time interval between emission and detection for individual neutrinos space-like, time-like, or light-like?

In the inertial frame that flips the time-ordering, again tell us about the character of the space-time interval (space-like, time-like, or light-like).

- Suppose that, upon detection of a neutrino event at Gran Sasso, the detector sends back a light signal to LHC to indicate they “got it”. Add the world line of that return signal to your diagram. Given the claim that the neutrinos traveled faster than c , some people in the media have claimed this means we have “backwards time travel”. What do you think they mean by that? E.g., does this mean that this “return signal” arrives before the original signal was sent? (Use your diagram to answer this question unambiguously.)

10.05. Lorentz transformation for general velocity vector [Kinney SP11]

Textbooks tend to only give you the Lorentz transformation along a single coordinate axis, but it is not always convenient to keep redefining the coordinate system for problems with several different velocities. To derive a more general formula using vector notation, use the idea that the part of a position vector \mathbf{r} that is parallel to the velocity is the part that is changed by the transformation, while the part that is perpendicular to the velocity is unchanged. Assume that you wish to transform from your inertial frame [the (\mathbf{r}, ct) frame] to the “primed” inertial frame (\mathbf{r}', ct') moving with velocity $\vec{v} = c\vec{\beta}$ that points in some arbitrary direction (e.g., it has an x, y and z component). You should find the following:

$$ct' = \gamma(ct - \vec{\mathbf{r}} \cdot \vec{\beta})$$

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r} \cdot \hat{\beta})\hat{\beta} - \gamma ct\vec{\beta}$$

Show that in the case that the velocity is in the x -direction, you get back the usual transformation.

10.06. Transformation of acceleration [Munsat FA10]

Find the transformation laws for the components of acceleration ($a_x = \frac{dv_x}{dt}$, etc...).

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10.07. Rapidity [Pollock FA11, Kinney SP11]

It is common in nuclear physics to talk about “rapidity” of a particle, defined as an angle $\phi = \cosh^{-1} \gamma$ (here γ is the usual relativistic gamma factor, and that’s an inverse hyperbolic cosh).

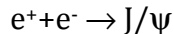
(a) Prove that the usual relativistic $\beta = v/c$ is given by $\beta = \tanh \phi$, and then show $\beta\gamma = \sinh \phi$. With these, rewrite the Lorentz transformations in matrix form entirely in terms of the rapidity angle. The result you get might remind you of a rather different *kind* of transformation, please comment!

(b) Suppose that observer B has rapidity ϕ_1 as measured by observer A, and C has rapidity ϕ_2 as observed by B (both velocities are on the x-axis). Show that the rapidity of C as measured by A is just $\phi_1 + \phi_2$, i.e. rapidities “add” (unlike velocities, which do not “properly” add in relativity!)

Hint: There is a hyperbolic identity you might find useful:

$$\tanh(a+b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}$$

(c) One way to create exotic heavy particles at accelerators is to simply smash particles together. Consider the energy needed to produce the famous “J/ψ” particle by the reaction in which a positron and electron annihilate to produce it:



The mass of the J/ψ particle is 3.1 GeV/c², so you might naively think it takes about 3.1 GeV (??) Lets see!

- First, move to the “center of momentum frame”. In this frame, electron and positron (which are equal in mass) crash together head on. In this frame, what kinetic energy do the e^+ and e^- need, to *just barely* produce J/ψ’s? (This is the minimum, or “threshold” energy) Is your answer reasonable; is it what you might naively expect?

- Now consider the “lab frame”, in which a target electron is at rest, and you smash the positron into it. What kinetic energy does the positron need in this frame, to just produce a J/ψ ?

Use your answers above to explain why particle physicists go to the trouble and expense of building colliding-beam experiments instead of “fixed target” experiments.

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10.08. Lorentz transformation of the wave equation [Munsat FA10]

Show by direct substitution of the Lorentz transformations into the 1-D wave equation that the Lorentz transformation preserves the form of the wave equation.

Namely, show that if $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$, then $\frac{\partial^2 \psi}{\partial \bar{x}^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial \bar{t}^2}$.

10.09. Invariance of the space-time interval [Dubson SP12]

Prove that the interval between two events is Lorentz Invariant:

$$I = \Delta \bar{x}_\mu \Delta \bar{x}^\mu = \Delta x_\mu \Delta x^\mu$$

The Lorentz transform is $\Delta \bar{x}^\mu = \Lambda^\mu_\nu \Delta x^\nu$, or, in matrix notation:

$$\begin{pmatrix} c\Delta \bar{t} \\ \Delta \bar{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix}$$

For this proof, you can suppress the y and z coordinates, to save space.

10.10. Conserved vs. invariant quantities [Dubson SP12, Pollock FA11]

Which of the following quantities are conserved and which are Lorentz-invariant?

Proper time, rest mass, energy, 3-momentum, 4-momentum, charge.

10.11. Muon calculations [Munsat FA10]

The mean lifetime of muons is 2 μ s in their rest frame. Muons are produced in the upper atmosphere, as cosmic-ray secondaries.

(a) Calculate the mean distance traveled by muons with speed $v = 0.99c$, assuming classical physics (i.e. without special relativity).

(b) Under this assumption, what percentage of muons produced at an altitude of 10 km reach the ground, assuming they travel downward at $v = 0.99c$?

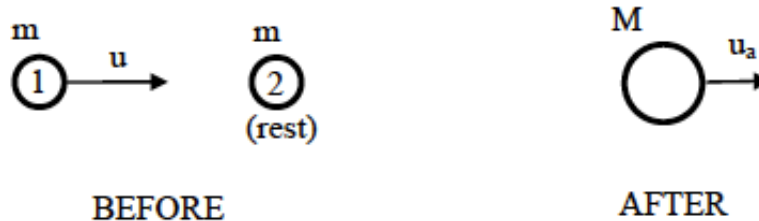
(c) Calculate the mean distance traveled by muons with speed $v = 0.99c$, taking into account special relativity.

(d) Under this assumption, what percentage of muons produced at an altitude of 10 km reach the ground, assuming they travel downward at $v = 0.99c$?

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10.12. Relativistic inelastic collision [Dubson SP12]

Consider the following 1D collision: object 1 with rest mass m , traveling with speed u , collides head-on with object 2, with identical rest mass m and initially at rest. The two objects merge and form a single object of mass M , moving with final speed u_a (“a” for “after”)



(a) Write down expressions for the 4-momentum of the system, showing clearly the time and space components of the 4-vectors, for both Before and After the collision. It will help to be clear about gamma-factors, so define γ_u to be the gamma factor for speed u , and γ_a the gamma factor for speed u_a .

(b) The center-of-mass (CM) frame is defined as the inertial frame in which the total momentum of the system is zero. Solve for the value of $\beta = v/c$ that transforms from the original reference frame in part (a) to the center-of-mass frame. Again, be careful with the various gamma factors: define γ_v as the gamma factor for speed v . Check that the answer becomes the correct non-relativistic answer in the low speed limit.

(c) Solve for the 3-momentum (the ordinary 3-momentum, not the 4-momentum) of particle 1 in the CM frame. (Careful to distinguish between γ_u and γ_v !) Simplify your answer as much as possible and verify that the answer is correct in the non-relativistic limit.

(d) Solve for the mass M after the collision. Simplify as much as possible and verify that the answer is correct in the non-relativistic limit.

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10.13. E & B transformations [Pollock FA11]

$$\begin{aligned}\bar{E}_x &= E_x & \bar{E}_y &= \gamma(E_y - vB_z) & \bar{E}_z &= \gamma(E_z + vB_y) \\ \bar{B}_x &= B_x & \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) & \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right)\end{aligned}$$

(a) Use these equations to show that both $\mathbf{E} \cdot \mathbf{B}$ and $(E^2 - c^2 B^2)$ are Lorentz invariants. We found earlier that \mathbf{E} and \mathbf{B} are mutually perpendicular for traveling EM waves. Given that this is true in *some* frame, can there be any *other* reference frame in which you would find \mathbf{E} and \mathbf{B} *not* perpendicular for traveling EM waves?

(b) Suppose $E > cB$ in some frame. Show that there is *no possible* frame in which $E = 0$.

- If $E = 0$ in some frame, do these relations mean that E is always equal to 0 in every other inertial frame?

- If $B = 0$ (but E is nonzero) in some frame, can you always (ever?) find another frame in which $E = 0$ (but B is nonzero)?