



## **Electrodynamics HW Problems**

### **09 – Radiation**

1. Classical instability of the H-atom
2. Radiation from a falling point charge
3. EM waves from an oscillating electric dipole
4. Small antenna array
5. Radio signals, revisited
6. Dipole radiation from Larmor formula

## 09 – Radiation

### 9.01. Classical instability of the H-atom [Dubson SP12, Pollock FA11]

In the Bohr Model of the H-atom, an electron orbits the nucleus in a circle of radius  $r_0$ . One serious problem with this model is that a charged particle moving in a circle is accelerating, and an accelerating charge radiates away EM energy according to the Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

So, the H-atom will lose energy and the electron will quickly spiral into the nucleus, according to classical electrodynamics and mechanics.

**(a)** Derive a formula for the total energy  $[U(r) = KE+PE]$  of an electron in circular orbit at radius  $r$ . (As usual, let's set the zero of PE at  $r = \infty$ .) Start with  $F=ma$  to get the KE. You can use your introductory physics knowledge to write down the formula for the PE.

**(b)** Using your result from part (a) and the Larmor formula for radiated power, derive the differential equation that describes  $r(t)$ .

**(c)** Derive an expression for  $r = r(t)$ . *Hint: Your formula for  $r(t)$  should be of the form  $r(t) = (r_0^3 - Ct)^{1/3}$ , where  $C$  is a constant that you will figure out.*

**(d)** Assume that the initial radius  $r_0$  is the Bohr radius ( $\sim 0.05$  nm); put in numbers and solve for the time for the electron to crash into the nucleus. How does this compare to the period of the ground state orbiting electron in the Bohr model?

### 9.02. Radiation from a falling point charge [Kinney SP11]

An electron is released from rest and falls under the influence of gravity. In previous mechanics courses we have probably completely ignored the fact that this charge is losing energy by radiation as it accelerates down! So here we will investigate how large an error we have been making; of all the charged particles in nature, the effect will be largest for an electron, so we'll take that as our worst case. First, assume that the effect is negligible so that you can simply find the position as a function of time, then calculate the power radiated in the first centimeter of falling assuming the acceleration only is caused by gravity. What fraction of the change in gravitational potential energy is this? Is this radiation effect negligible? If the effect is large, we need to go back and include the radiation reaction force in Newton's 2nd law, in order to find the correct trajectory.

## 09 – Radiation

### 9.03. EM waves from an oscillating electric dipole [Kinney SP11]

In class we found that the leading terms in the radiation “multipole” expansion for the electric and magnetic fields were

$$\vec{E}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \left[ \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p}(t_0)) \right] \quad \vec{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi cr} \left[ \hat{\mathbf{r}} \times \dot{\vec{p}}(t_0) \right]$$

**(a)** Consider the case of two point charges,  $+q$  and  $-q$  located on the  $z$ -axis, symmetrically about the origin. The separation of the charges is given by  $a \cos(\omega t)$ . Find the resulting  $E$  and  $B$  fields in spherical coordinates and basis vectors.

**(b)** Check whether the fields that you found satisfy Maxwell’s equations in vacuum, and find the time-averaged Poynting vector  $S$ . Make a polar plot of the magnitude of the time-averaged Poynting vector as a function of azimuthal angle  $\phi$  for  $\theta = \pi/2$  ( $r = \text{constant}$  in polar plots), as well as a polar plot of the magnitude as a function of  $\theta$  for  $\phi = 0$ .

### 9.04. Small antenna array [Kinney SP11, Munsat FA10]

Two sinusoidally oscillating dipoles are located in the  $xy$ -plane, on the  $y$ -axis at  $y = a$  and  $y = -a$ . Their dipole moments are oriented in the  $z$ -direction. Assume that our observation point  $r$  is located at a distance far away,  $r \gg a$ . Also assume the dipole moments have equal amplitude  $p_0$  and they oscillate at the same frequency  $\omega$ .

**(a)** Find the total electric field  $\mathbf{E}$  in the  $xy$ -plane using the standard complex notation for the fields.

**(b)** Assume the dipoles are a half-wavelength apart ( $\lambda = 4a$ ). Make a polar plot of  $|\mathbf{E}|$  as a function of azimuthal angle  $\phi$  for the case that the oscillations of the two dipoles are in phase.

**(c)** For the same wavelength, now make a polar plot of  $|\mathbf{E}|$  as a function of azimuthal angle  $\phi$  for the case that the oscillations of the two dipoles are  $180^\circ$  out of phase.

**(d)** Compare your polar plots with that of a single dipole located at the origin (and oriented in the  $z$ -direction) of twice the amplitude, i.e.,  $2p_0$ . What is the advantage of the two-dipole array over the single dipole setup?

## 09 – Radiation

### 9.05. Radio signals, revisited [Munsat FA10]

We earlier made a crude approximation of the electric field which we receive at our radio, located 10 km from the transmitting antenna ( $f = \omega/2\pi = 1190$  kHz with a 7800 watt transmitter)

**(a)** Assume that the antenna is a half-wave transmitter (oriented vertically with respect to the ground). How tall should it be?

**(b)** Find the amplitude of the electric field, assuming our radio is located at the midplane of the radiation pattern. This is much more realistic than our previous attempt.

**(c)** Half of the radiated intensity is emitted in the angular range  $\frac{\pi}{2} - \frac{\alpha}{2} \leq \theta \leq \frac{\pi}{2} + \frac{\alpha}{2}$ , i.e., in a sector with angular measure  $\alpha$  centered at the  $xy$ -plane.

Determine  $\alpha$ , known as the “half-intensity angle”.

### 9.06. Dipole radiation from Larmor formula [Munsat FA10]

The average power radiated by an oscillating dipole, with dipole moment  $p(t) = p_0 \cos(\omega t)$  is:

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

Derive this result from the Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

by treating the dipole as an oscillating pair of charges  $\pm q_0$ , that oscillate  $180^\circ$  out of phase with an amplitude of oscillation  $d/2$ . (Note that  $p_0 = q_0 d$ .)

**Hint:** The waves radiated by  $+q_0$  and  $-q_0$  interfere, so in the Larmor formula you must add the “ $qa$ ”s and then square, rather than adding the squares of the “ $qa$ ”s