



Electrodynamics HW Problems

08 – Potentials and Fields

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08 – Potentials and Fields

8.01. Maxwell's equations in general potential form [Kinney SP11]

Now that we have the full Maxwell's equations for \mathbf{E} and \mathbf{B} that describe static and non-static fields, we can see how Maxwell's equations look in terms of the scalar and vector potential.

(a) While we still have $\mathbf{B} = \nabla \times \mathbf{A}$, in general, $\mathbf{E} \neq -\nabla V$ for non-static fields. Explain why the formula for \mathbf{B} in terms of \mathbf{A} is unchanged, while the one for \mathbf{E} needs an additional term. Show that the additional term is $-\partial\mathbf{A}/\partial t$.

(b) Now substitute these relations for \mathbf{E} and \mathbf{B} in terms of V and \mathbf{A} into Maxwell's equations and show that the general equations for the potentials $V(r,t)$ and $\mathbf{A}(r,t)$ (without using any particular gauge condition) are

$$\begin{aligned}\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) &= -\rho/\epsilon_0 \\ \nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) &= -\mu_0 \mathbf{J}\end{aligned}$$

(c) The equations in a particular gauge can be obtained from the general equations by substituting in the gauge condition. Derive the differential equations for V and \mathbf{A} in the Lorentz gauge and in the Coulomb gauge by this method. In the Lorentz gauge

$$\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0$$

whereas in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$.

(d) The textbook proves that the general solutions to these equations in the Lorentz gauge are

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}',t_R)}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}',t_R)}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

which involve the retarded time $t_R = t - |\mathbf{r} - \mathbf{r}'|/c$; for this reason these are called the *retarded potentials* and satisfy our sense of causality. Prove that the advanced potentials in which t_R is replaced with the *advanced* time $t_A = t + |\mathbf{r} - \mathbf{r}'|/c$ are also solutions to Maxwell equations in potential form, in the Lorentz gauge. You are proving in fact that the laws of electrodynamics are invariant under time reversal!

These potentials have enormous impact in quantum field theory where anti-matter particles follow a time dependence that is mathematically equivalent to moving backward in time.

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8.02. Temporal gauge [Dubson SP12, Pollock FA11]

Consider a point charge at rest at the origin. You know $V(\mathbf{r}, t) = q/4\pi\epsilon_0 r$, and $\mathbf{A}(\mathbf{r}, t) = 0$. [Right? Convince yourself!] Explicitly check to see if we are in the Coulomb gauge, the Lorentz gauge, or perhaps both, or maybe neither!

Now, introduce a gauge transformation $\mathbf{A}_{\text{new}} = \mathbf{A} + \nabla f$, $V_{\text{new}} = V - \partial f / \partial t$, using the particular choice $f(\mathbf{r}, t) = qt/4\pi\epsilon_0 r$. Find the new V and \mathbf{A} . Briefly discuss your results: are the potentials “static” or time-dependent? Does this represent the same physical situation, or is something different here? Are we in the Coulomb gauge, or the Lorentz gauge now? (Do we *have* to be in one of those to solve physics problems?) What has changed, and what is the same?

8.03. Vector potential for EM plane waves [Pollock FA11, Kinney SP11]

Say the potentials throughout space and time are $V(\mathbf{r}, t) = 0$ and $\mathbf{A}(\mathbf{r}, t) = A_0 \cos[k(z - ct)] \hat{x}$, where A_0 and k are given constants, and c is the speed of light.

(a) Find the \mathbf{E} and \mathbf{B} fields everywhere in space and time, and comment on the *physics* here – what have we got going on?

(b) Are we in the Coulomb gauge, the Lorentz gauge, both, or neither?

(c) Is it possible, in principle, to find a different gauge for this problem (following the usual general procedure of gauge transformations) in which $\mathbf{A}(\mathbf{r}, t) = 0$? [If your answer is yes, you don’t necessarily need to find the particular gauge transformation here – I’m just asking if it’s possible *in principle*, and how you know.]

8.04. Vector potential for an infinite wire [Dubson SP12]

Let’s get some familiarity with the vector potential \mathbf{A} by using it to compute something we already know: the B-field of a long straight wire carrying a steady current I . If you try to compute the vector potential at a distance r from an *infinitely-long* straight wire, you will get a nasty surprise: the integral for \mathbf{A} diverges logarithmically. There is a nice trick for getting around the infinity in such cases.

(a) Compute the vector potential $A(r)$ at a distance r from the center of a *very long* wire, so the limits of integration are $+L$ to $-L$ (where $L \gg r$), instead of $-\infty$ to $+\infty$. As a reality check, use Mathematica or Mathcad to plot $A(r)$ vs. r over some reasonable range of r , but with some $r \ll L$ and a big value of L (say $L = 1000$). $A(r)$ blows up at $r = 0$, so be careful with your plot.

(b) Now compute \mathbf{B} by taking the curl of \mathbf{A} and then taking the limit as $L \rightarrow \infty$. (When computing the curl, chose coordinates intelligently. Should you work in Cartesian coordinates, spherical coordinates,...?)

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8.05. Time-dependent fields from an infinite wire [Kinney SP11]

Consider an infinite straight wire in which the current $I = kt$ (k is a constant) begins to flow at $t = 0$; there is no current for $t < 0$. The wire remains neutral for all time, and we will magically assume that the current begins to flow at $t = 0$ everywhere along the entire length of the wire(!).

(a) Using the retarded potentials in the Lorentz gauge:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}', t_R)}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}', t_R)}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

find the scalar and vector potentials (V, \mathbf{A}) as a function of time in all space.

(b) Using your solutions to part (a), find the \mathbf{E} and \mathbf{B} fields in all space. Compare your results to what you would have found in the quasi-static approximation.

8.06. Infinite current sheet [Dubson SP12, Pollock FA11]

Consider a neutral infinite sheet that lies in the xy -plane. The surface current $K(t) = 0$ for $t \leq 0$, but at $t = 0$, suddenly the surface current turns on, so it is a constant K_0 in the $+x$ -direction, instantly, and everywhere in this plane. (There is no charge density anywhere.)

(a) Find the (retarded) scalar and vector potentials everywhere in space. In computing your answer, note that $\sqrt{z^2} = |z|$. This will be essential in the parts below.

(b) Find the \mathbf{E} and \mathbf{B} fields at a distance z above (or below) the xy -plane. Sketch these fields, and describe them in words. Briefly, comment on them. (Does your answer make sense, is it what you would have expected from a static situation? I find *part* of the answer a little surprising!)

(c) I claim that your answer to part (b) is really an “outgoing plane wave”. Show/convince us that this is the case. What direction is the wave propagating? At what speed?

(d) Show explicitly that \mathbf{B} is *not* continuous at the ($z = 0$) xy -plane, but that it satisfies the appropriate *boundary condition* there. Also, show explicitly that \mathbf{E} *also* satisfies the appropriate boundary condition at the ($z = 0$) xy -plane.

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8.07. Poynting vector of a point charge [Munsat FA10]

Calculate the Poynting vector and energy density of the electromagnetic field of a particle of charge q moving with constant velocity. Show that the field energy is carried along with the particle.

8.08. Scalar and vector potential of a rotating point charge [Munsat FA10]

A particle of charge q moves in a circle of radius a in the xy -plane at constant angular velocity ω . Assume the particle passes through the Cartesian coordinates $(a,0,0)$ at $t = 0$. Find the vector and scalar potentials for points on the z -axis.

8.09. Electric and magnetic fields of a moving point charge [Munsat FA10]

The electric and magnetic fields from a point charge q moving with constant velocity can be given by the following:

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{3/2}} \frac{\hat{R}}{R^2} \quad \text{and} \quad \vec{B}(\vec{r}, t) = \frac{\vec{v} \times \vec{E}}{c^2}$$

where $\vec{R} \equiv \vec{r} - \vec{v}t$ and θ is the angle between \vec{R} and \vec{r} . (See Griffiths, fig. 10.9.)

Use these formulas to calculate the electric and magnetic fields measured at a distance d away from an infinite straight wire carrying a uniform line charge density λ moving at a constant speed v down the wire.