



## **Electrodynamics HW Problems**

### **03 – AC Circuits**

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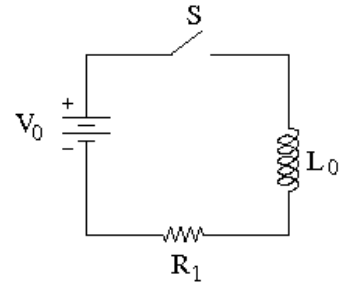
#### 3.01. Self-inductance of a solenoid [Kinney SP11, Rey SP10]

The most common device to store magnetic energy (in the same way that a capacitor stores electric energy) is the solenoid. Consider an “infinite” solenoid with circular cross section of radius  $a$ , and  $N$  coils per length  $l$ , *i.e.*, the turns/length is  $N/l$ .

**(a)** Find the self-inductance  $L$  for a length  $l$  for this infinite solenoid.

**(b)** Qualitatively, how would you expect the self-inductance to change for a finite length solenoid? Estimate the self-inductance  $L_0$  for a solenoid of the same geometry as part (a), but now with finite length  $l_0$ . You do not need to calculate the fringe fields directly but you should use symmetry and scale arguments to provide the (rough) estimate. Does your answer make sense in the limit that  $l_0 \gg R$ ?

**(c)** Now we will study the effect of the solenoid  $L$  when inserted in a simple circuit with a switch  $S$ , a resistor  $R_1$  and an ideal battery which supplies a voltage  $V_0$ , as shown in the schematic. If the switch is open, but then closed at time  $t = 0$ , what is the current in the solenoid as a function of time? Also, what is the voltage across the inductor as a function of time?



**(d)** Now replace the battery with an ideal AC power supply that provides a voltage  $V_0 \cos(\omega t)$ . If the switch is open, but then closed at time  $t = 0$ , what is the current in the solenoid as a function of time? Make a sketch of the magnitude of the current as a function of frequency  $\omega$ , for times long after the transients have died away.

#### 3.02. Inductors and RL circuits [Rogers SP09]

**(a)** Design a coil that has a 10 microhenry inductance. Use a cylindrical solenoid geometry, and 20 gauge copper wire. You will have some flexibility in the choice of coil dimensions and number of turns. Be sure that the coil length is consistent with the wire gauge. Try to design something that uses an integer number of turns. List the coil radius, length, number of turns, and whether the turns are on one layer or multiple layers.

**(b)** Predict the resistance of your coil. Often, you are interested in making a coil with the lowest resistance. Do you expect any decrease in coil resistance if you redesign your inductor to have decreased number of turns, while increasing the solenoid radius so as to keep the solenoid inductance fixed?

**(c)** Draw a closed loop circuit diagram, with an inductor and resistor in series. Use Kirchhoff's law (the sum of the voltage drops around a closed loop is zero) to write an ordinary differential equation for the current in the loop. Solve this equation subject to the initial condition that the current in the loop starts at some initial value,  $I_0$ .

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### 3.03. LC circuits [Rogers SP09]

**(a)** Draw a closed loop circuit diagram with an inductor and a capacitor in series. Use Kirchhoff's law to write an ordinary differential equation for the current in the loop. Solve this equation subject to the initial condition that the current in the loop starts at zero, and that the voltage across the capacitor is some initial value,  $V_0$ .

**(b)** Make a sketch that has the current through the inductor and the associated voltage across the inductor. Make a similar sketch for the current through the capacitor and the voltage across the capacitor. How much electrical power is being dissipated as a function of time?

### 3.04. Phasor notation [Rogers SP09]

Ohm's Law relates current and voltage drop for resistors by a linear relationship,  $V=IR$ . In comparison, capacitors and inductors relate current and voltage through first time derivatives. For example, the capacitance connects the voltage drop across the capacitor to the amount of stored charge by  $Q=CV$ . Therefore, a single time derivative of both sides yields a relationship between the current through a capacitor and the time change of the voltage drop:

$$I = C \frac{dV}{dt}$$

A common electronics technique is to notice that any time dependent quantity can be written as a sum of sine and cosine terms (Fourier components!); it is therefore useful to think about sinusoidal time dependent quantities. In particular, if we assume that the current and voltage are both varying at some angular frequency  $\omega$ :

$$I(t) = I_0 e^{i\omega t} \quad V(t) = V_0 e^{i\omega t}$$

then we find that a capacitor is described by a relationship that is linear between the amplitudes of the two sinusoidally varying quantities,  $I$  and  $V$ :

$$I_0 = i\omega C V_0 \quad \text{or} \quad V_0 = I_0 \frac{1}{i\omega C}$$

This result looks a lot like Ohm's Law,  $V=IR$ , and has converted the differential relationship into an algebraic relationship (under the assumption of sinusoidally varying current and voltage... but HEY! Anything can be written as a sum of such things.) We say that a capacitor has a COMPLEX IMPEDANCE,

$$Z_c = \frac{1}{i\omega C}$$

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#### 3.04 (cont.) Find the complex impedance $Z_L$ of an inductor $L$

**(a)** Redo your work on the series LC circuit from Problem [3.3]. Again, write down a differential equation for the current as a function of time. This time, assume that the solution is of the form:

$$I(t) = I_0 e^{i\omega t}$$

Plug this assumed form into the differential equation. Under what assumptions is this ‘guessed’ solution going to work? In particular, what must the frequency be? How does this result compare to your solution of the differential equation in Problem [3.3]?

As a note: You can finish the solution by asking what value of  $I_0$  (possibly a complex number) is required so that the Real part of the assumed form satisfies the initial conditions. You don’t need to do that here.

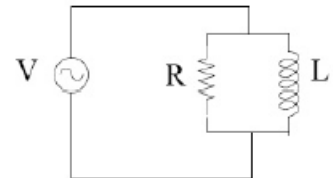
**(b)** Draw a closed loop circuit with a resistor, capacitor, and inductor in series. Again, use Kirchhoff to create an equation for the current flow around this circuit. Using the assumed form from part (a), reduce the differential equation to algebra. What must the ‘frequency’ be if the assumed form is to satisfy the equation? Argue for a simplified form for the frequency that comes from assuming that the resistance is ‘small’ and state what ‘small’ means... Small compared to what? Interpret your result for the ‘frequency’. What type of time dependence does it predict?

#### 3.05. Resistor and inductor in parallel [Pollock FA11, Rey SP10]

In the LR circuit shown in the figure, the power supply provides an EMF of  $V = V_0 \cos(\omega t)$ .

**(a)** Find the current through the power supply as a function of time after all the transients have died away.

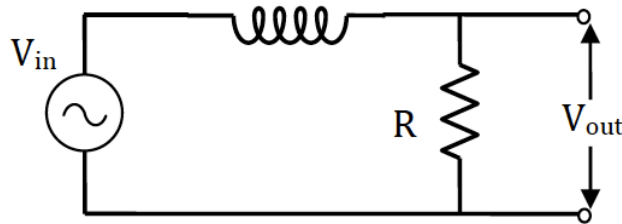
**(b)** How does your answer behave in the limits  $\omega \rightarrow 0$  and  $\omega \rightarrow \infty$ ? Explain if this makes sense given the characteristics of an inductor.



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#### 3.06. LR filter [Dubson SP12]

The following LR circuit is driven by an AC voltage source  $V_{in} = V_0 \cos(\omega t)$ . It can be regarded as a filter that changes  $V_{in}$  into  $V_{out}$ .



**(a)** Using complex analysis (the phasor method) solve for the "true" current  $I_{true}$  in this circuit:  $I_{true} = \text{Re}[I]$ .

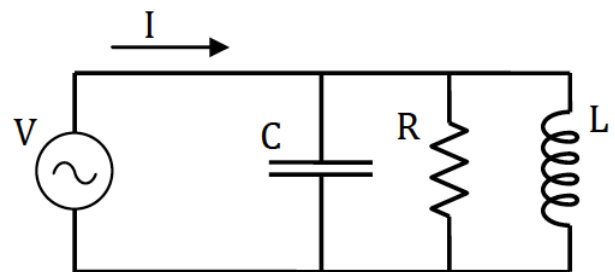
**(b)** Solve for the complex ratio  $V_{out} / V_{in}$  (give the magnitude and phase of this ratio) as a function of frequency. Check that your answer makes sense in the limits  $\omega \rightarrow 0$  and  $\omega \rightarrow \infty$ .

**(c)** Make a sketch of the magnitude of  $V_{out} / V_{in}$  vs. frequency. How would you describe this filter? Is it a high-pass filter? Low-pass? Band-pass?

#### 3.07. Parallel RLC circuit [Dubson SP12]

This parallel RLC circuit is driven by an AC voltage  $V = V_0 \cos(\omega t)$ .

**(a)** What is the total complex impedance that the voltage source sees? Write the answer in the format  $Z = |Z| \exp(i\theta)$ ; i.e., give the magnitude and phase angle of the impedance.



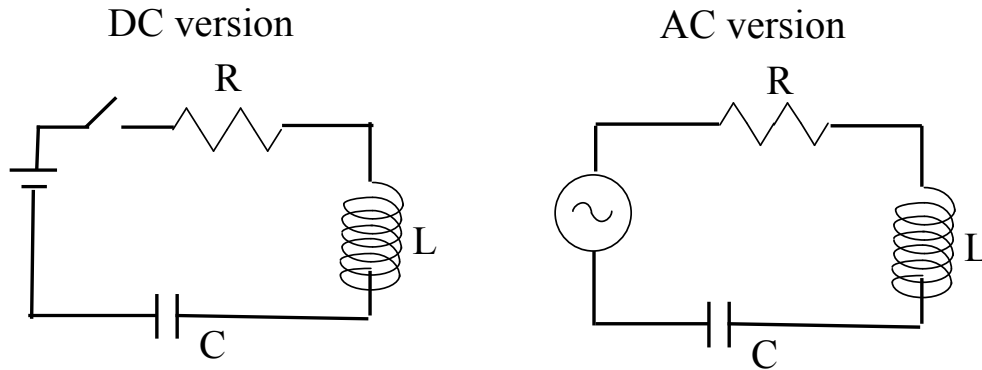
(Hint: It is simplest to begin by computing the magnitude and phase of  $1/Z$ .)

**(b)** Using complex analysis (the phasor method) solve for the "true" current  $I_{true}$  through the voltage source  $I_{true} = \text{Re}[I]$ .

**(c)** Make a sketch of the magnitude of the current  $I$  vs. frequency. At what frequency is the current at an extremum?

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#### 3.08. Series RLC circuit [Pollock FA11]



Consider the RLC series circuit shown in the figures above. First we'll use a battery to supply EMF, then an AC power supply. In both cases, assume the circuit is *underdamped*. (i.e.,  $R$  is small; as you solve the problem, you should be able to say what's small compared to what.)

**(a)** Consider the DC circuit on the left. If the switch is open, but then closed at time  $t = 0$ , describe qualitatively what happens just after  $t = 0$ , and then as  $t$  gets very large. Then compute a formula for the current through the capacitor as a function of time.

**(b)** Now replace the battery with an ideal AC power supply that provides a voltage  $V_0 \sin(\omega t)$ . After letting all the transients from turning on the power die away, what is the current through the capacitor as a function of time? Make a rough sketch of the magnitude of the current as a function of frequency, showing (and briefly explaining) the main features of your sketch (e.g., what are its limiting behaviors, and what are any “interesting features”).

**(c)** Let's use some realistic values appropriate to an inexpensive (homebrew) oscillator, say a 1 Ohm resistor, a 30 pF (picoFarad) capacitor, and a 100 nH (nanoHenry) inductor. This circuit has a natural “resonant” frequency – what is it in this case? (Does that number give you a clue as to what this circuit might be useful for?)

Use Mathematica to plot the magnitude of current as a function of frequency. (Be careful to set the scale of frequency to run past the resonance) Does the graph match the expectations of your “sketch” in the previous part? Briefly comment.

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#### 3.09. Simple LC circuit

[Dubson SP12, Pollock FA11]

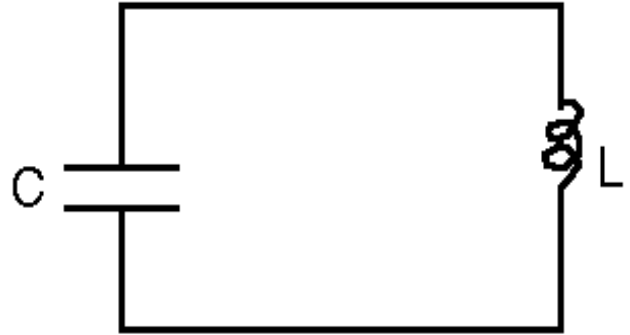
This LC circuit starts at time  $t = 0$  with charge  $Q_0$  on the capacitor and zero current.

(a) Write down the differential equation for charge on the capacitor as a function of time and solve for  $Q(t)$ .

(b) Find the voltage across the capacitor, the voltage across the inductor, the energy stored in the capacitor, and the energy stored in the inductor as functions of time.

(c) Describe in words the energy flow in the circuit. Show that the total electromagnetic energy in the circuit is conserved.

(d) Describe an analogous mechanical system (i.e. one that obeys the same differential equation) and provide “translations” between any important variables.



#### 3.10. Underdamped RLC circuit [Pollock FA11]

This underdamped RLC circuit starts at time  $t = 0$  with charge  $Q_0$  on the capacitor and zero current.

(a) Write down the differential equation for charge on the capacitor as a function of time and solve for  $Q(t)$ .

Describe what *underdamped* means in this context.

Find the voltage across the capacitor, the voltage across the inductor, and the current in the loop as a function of time.

(b) Find the energy stored in the capacitor, and the energy stored in the inductor as functions of time. Describe in words the energy flow in the circuit.

(c) Describe an analogous mechanical system (i.e. one that obeys the same differential equation) and provide “translations” between any important variables. How is the overdamped system different and how is it similar?

**Extra credit:** Show that the stored energy is **not** conserved, and analyze how the rate of change of the total electromagnetic energy is related to the Joule heating in the resistor.

