Electrodynamics HW Problems
06 – EM Waves

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06 – EM Waves

6.01. Energy in a wave on a string [Dubson SP12]

In class we discussed a wave on a string, and showed that the wave speed v is given by $\mathrm{v} = \sqrt{\mathrm{T}/\mu}$, where T is the tension in the string and μ is the mass density (mass per length) of the string.

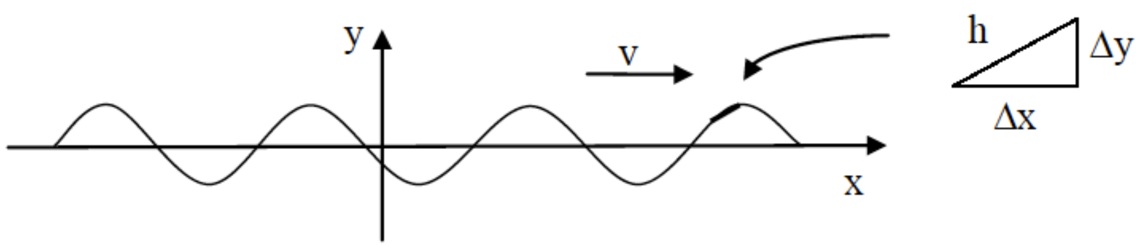
In this problem, you will show that the energy in a sinusoidal string wave is proportional to the amplitude-squared (A^2) of the wave. Let's consider a (real) sinusoidal traveling wave given by $y(x,t) = A\cos(kx - \omega t)$. Let's say x is the horizontal direction and y is the vertical direction. This traveling wave contains kinetic energy because each atom in the string is moving up and down. The wave contains potential energy because the string is stretched when the wave is present, compared to the straight horizontal string when no wave is present. We will assume "small amplitude" motion so that:

- each atom in the string moves vertically only,
- each portion of the string experience a very small fractional change in the length as the wave goes by,
- the tension T in the string remains constant, independent of position and time,
- the slope of the string is always very small, $\partial y/\partial x \ll 1$. (Note that the figure below is greatly exaggerated.)

Only in this small amplitude limit is the wave speed independent of amplitude. If the amplitude is large, the string is stretched so much that the tension changes locally, which affects the wave speed.

(a) Derive a formula for the time-averaged kinetic energy per length in the string when the wave is present. Your answer should be in terms of A, ω, and μ.

(b) Show that, in this small amplitude limit, the amount by which a small segment Δx of the string is stretched is $\Delta s = h - \Delta x \simeq \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \Delta x$, where $(\partial y/\partial x)$ is the slope of the string.



(c) Derive an expression for the time-averaged potential energy per length in the string. Your answer should be in terms of A, T, and k.

(d) Show that the average potential energy in the string is equal to the average kinetic energy in the wave.

6.02. Traveling wave on a string [Dubson SP12]

Two points on a string are observed as a traveling wave passes. The points are at $x_1 = 0$ and $x_2 = 1$ m. The two points are known to be less than one wavelength apart. The transverse motions of the two points are observed to be

$$y_1 = 0.2\cos(3\pi t)$$
$$y_2 = 0.2\cos(3\pi t + \pi/8)$$

where all numbers are in SI units.

(a) What is the frequency of this wave in hertz?

(b) If we write the wave in the form $y = A\cos(kx \pm \omega t + \delta)$, what is the smallest possible value of δ? [Since the cosine function is periodic with period 2π, the phase constant δ is only determined modulo 2π.]

(c) What is the wavelength?

(d) What is the wave speed?

(e) Can you tell if this wave is moving to the right or to the left? If so, which way is it moving?

6.03. Standing wave [Dubson SP12]

(a) Starting with the complex relation $\exp(i\alpha)\cdot\exp(i\beta) = \exp(i[\alpha+\beta])$, complete the trig identities: $\cos(\alpha+\beta) = ?$ and $\sin(\alpha+\beta) = ?$

(b) If you add two sinusoidal traveling waves, with the same frequency and wavelength, but traveling in opposite directions, you get a standing wave. Use the results of part (a) to show that the sum $A\cos(kx - \omega t) + A\cos(kx + \omega t)$ is a standing wave. Make a sketch of this standing wave, showing its shape at two different times, and indicate on the sketch where the nodes and anti-nodes are.

6.04. Spherical traveling wave [Dubson SP12]

Here we will derive the expression for a spherical traveling wave, originating at the origin and moving outward. The wave equation in 1D is:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

and has solutions of the form $f(x,t) = g(x \pm vt)$. In 3D, the wave equation is:

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

We hunt for a spherically symmetric solution by assuming that $f = f(r)$, where r is the radial distance from the origin.

(a) The Laplacian for a spherically symmetric function $f = f(r)$ is $\nabla^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right)$.
Show that an equivalent way to write this equation is $\nabla^2 f = \frac{1}{r}\frac{d^2}{dr^2}(r \cdot f)$.

(b) Show that the spherically symmetric solution to the 3D wave equation is of the form $f(r) = \frac{g(r - vt)}{r}$, where g in an arbitrary function. **Hint**: Using the result from part (a), you can re-write the 3D wave equation so that it has a form like the 1D wave equation. Then you can use the 1D wave equation solution.

(c) Show that the form $f(r) = \frac{g(r - vt)}{r}$ is equivalent to the form $f(r) = \frac{h(t - r/v)}{r}$ where h is an arbitrary function.

6.05. Traveling EM wave [Pollock FA11]

Consider the electric field given by $\mathbf{E}(x,y,z,t) = E_0 \cos\left[k(x-ct)\right]\hat{y}$ (where "*k*" and "*c*" are known constants).

(a) What is the charge density $\rho(x,y,z,t)$ associated with this E-field? What are the units of *k* and *c*?

(b) Come up with the *simplest possible* B-field that satisfies Faraday's law, given this E-field. Then, CHECK that the remaining free-space Maxwell equations are satisfied, *if* you make the correct choice for the constant "*c*". What is the required magnitude of the B-field? What is the current density $\mathbf{J}(x,y,z,t)$ associated with this E- and B-field?

(c) What is the Poynting vector $\vec{S}$ associated with these fields? Describe in words what this E- and B-field look like. Can you "interpret" them physically? What does the constant "*k*" tell you? Does this set of **E** and **B** provide a valid, self-consistent, physically possible solution to Maxwell's equations? What's the physics here?

6.06. 3-D electromagnetic plane wave [Dubson SP12, Pollock FA11, Kinney SP11]

Consider a 3D electromagnetic plane wave in vacuum, described in usual complex form by $\tilde{\vec{\mathbf{E}}}(\mathbf{r},t) = \tilde{\vec{\mathbf{E}}}_0 \exp\left[i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right]$, in which $\tilde{\vec{\mathbf{E}}}_o$ is a constant vector equal to $\tilde{E}_0\,\hat{\mathbf{x}}$, with $\tilde{E}_0 = E_0 \exp(i\pi/2)$. Assume **k** is the wave vector k $\hat{\mathbf{y}}$, ω is the angular frequency. As usual, the real field is $\vec{\mathbf{E}} = \mathrm{Re}[\tilde{\vec{\mathbf{E}}}]$

(a) Describe in words what this mathematical expression represents physically. You may use sketches, but if you do, they should be well described. In which direction the wave is moving? What is the speed, wavelength, and period of the wave? (What does that phase of π/2 in $\tilde{E}_0$ do?) Sketch the *real* field **E**(*x*=0,*y*,*z*=0,*t*=0) (a 2D plot with y as the horizontal axis) and **E**(*x*=0,*y*=0,*z*=0,*t*) (a 2D plot with t as the horizontal axis). Clearly indicate the *direction* of the field and the *scale* of both your axes. How is the field at *x*=*a*, i.e. **E**(*x*=a,*y*,*z*=0,*t*=0), different from the case at *x*=0?

(b) Why is this called a plane wave? (where is (are) the plane(s)?) <u>Sketch or represent this in 3D</u>. Describe how the direction of the electric field changes in time. If **E** always points in the same direction, the wave is said to be *linearly* polarized. Is this wave linearly polarized?

(c) Find the associated magnetic field **B**(**r**,*t*) for this plane electric wave. Sketch the magnetic fields, **B**(*x*=0,*y*,*z*=0,*t*=0) and **B**(*x*=0,*y*=0,*z*=0,t) indicating field direction. (As above, be clear about your axes) A 3D sketch of B would be helpful here too, what's the simplest way to draw it? Describe in words how **B** compares/contrasts with **E**.

(cont…)

6.06. (cont.)

(d) Calculate the energy density u_{EM}, Poynting vector **S**, and momentum density for these fields. Interpret the answers physically (Make sense of them, including units, signs, directions, etc!)

(e) Calculate the angular momentum density ℓ_{EM} about the origin (0,0,0). If you integrate this density over a cube of centered at the origin at one instant in time, would the angular momentum in that cube be zero or non-zero? Briefly discuss.

(f) Suppose now that we add two plane waves, $\mathbf{E_1}$ and $\mathbf{E_2}$, (superposition still works!) to find the total electric field. Let $\mathbf{E_1}(\mathbf{r},t) = \mathbf{E_1}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_1)$ and $\mathbf{E_2}(\mathbf{r},t) = \mathbf{E_2}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_2)$ so in this simple case the waves propagate in the same direction. Let's say the amplitudes are $\mathbf{E_1} = E_1\mathbf{z}$ and $\mathbf{E_2} = E_2\mathbf{z}$. <u>Use complex notation</u> (taking the real part only at the very end) to find $\mathbf{E_T}(\mathbf{r},t) = \mathbf{E_1}(\mathbf{r},t)+\mathbf{E_2}(\mathbf{r},t)$ in the form $\mathbf{E_T}(\mathbf{r},t) = \mathbf{E_T}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_T)$, giving expressions for the total amplitude and phase shift in terms of those from $\mathbf{E_1}(\mathbf{r},t)$ and $\mathbf{E_2}(\mathbf{r},t)$. Explicitly check your answer in the special case $d_1 = d_2$.

(g) Let's examine one more situation, this time

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E_1}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + \mathbf{E_2}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \pi/2),$$

in which $\mathbf{E_1}$ is a constant vector equal to $E_0\hat{z}$, $\mathbf{E_2}$ is a constant vector equal to $E_0\,\hat{\mathbf{x}}$, **k** is the wave vector k $\hat{\mathbf{y}}$ (as before,) and ω is the angular frequency. Find the total $\mathbf{E}(\mathbf{r},t)$. Describe how the direction and magnitude of **E** changes in time. Is this wave linearly polarized? Consider all points in space where $\mathbf{k.r} = 0$ (in this case, how would you describe such a set of points in words?), and describe in words or pictures what your E field looks like. Does this help you describe the polarization state? (If you look down the axis with the wave approaching you, is the **E** vector circling CW? CCW? Or, something else?)

6.07. Wave equation in vacuum [Dubson SP12]

(a) Prove that, for any vector field **E**:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{E}}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{E}}\right) - \nabla^2\vec{\mathbf{E}}$$

(b) Prove that $\vec{\nabla}\times\vec{\mathbf{E}} = i\vec{\mathbf{k}}\times\vec{\mathbf{E}}$ for a complex plane wave $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\exp\left[i\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t\right)\right]$, where $\vec{\mathbf{E}}_0$ is a constant (independent of position and time).

(c) Prove that in free space the magnetic field obeys the wave equation:

$$\nabla^2\vec{\mathbf{B}} = \mu_0\varepsilon_0\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2}$$

6.08. Radiation pressure of a plane wave

[Pollock FA11, Kinney SP11, Munsat FA10]

Consider a plane electromagnetic wave of the form:

$$\vec{E} = E_0 \cos(kz - \omega t)\ \hat{x} \qquad \text{and} \qquad \vec{B} = B_0 \cos(kz - \omega t)\ \hat{y}$$

which strikes a perfectly absorbing black screen.

(a) Starting from the law of conservation of linear momentum, show that the time-averaged pressure exerted on the screen is equal to the field energy per unit volume in the wave. This is called "radiation pressure".

Hint: Set up your sample volume behind the screen, so that there are only fields on the front surface, and not inside the volume.

(b) Near the earth, the time-averaged flux of electromagnetic energy (Poynting vector) from the sun is about 0.14 watt/cm^2. How does the radiation pressure from this light compare to atmospheric air pressure?

(c) If I made a 100 kg spacecraft with a 10,000 m^2 large black sail to absorb the sunlight and propel me away from the sun, what would its acceleration be?

6.09. EM plane wave surfing [Kinney SP11, Munsat FA10]

(a) A particle of charge q and mass m is held at the origin in the field of an electromagnetic wave $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$ in which $\mathbf{E_0}$ is a constant vector equal to $E_0\,\hat{\mathbf{y}}$, $\mathbf{k}$ is the wave vector $k\hat{\mathbf{z}}$, and ω is the angular frequency. Find the magnetic field $\mathbf{B}(\mathbf{r},t)$ associated with this electric field.

(b) At t = 0 the particle is released. What is the acceleration of the particle the instant after release? Give both magnitude and direction. As the particle now begins to move, it has a velocity $\mathbf{u}$ in the direction of the initial acceleration. What is the acceleration of the particle now that it has some velocity after some short time Δt? Assume that the particle is still essentially at the origin, and the time Δt is much less than $T/4$, where T is the period of the EM wave.

(c) Without attempting to solve the equations of motion, qualitatively describe and sketch the motion of the charge during this first quarter of the period. Take into consideration the relative magnitudes of the electric and magnetic forces. Does the motion of the charge lie in a line, a plane or in 3-dimensions (*e.g.,* helical motion)?

(d) After $t > T/4$ but before $3T/4$, the fields reverse direction. Again assuming that the particle is at the origin (this just means we are assuming that q/m is very small), what is the acceleration of the particle with velocity $\mathbf{u}$? As in part (c), using all the information you can, but without actually solving the equations of motion, qualitatively describe and sketch the motion of the charge. Now that you have considered the forces when the fields are in opposite directions, what do you expect the long term (time-averaged) motion of the charge to be?

6.10. Traveling wave packet [Pollock FA11]

Consider a localized wave packet that satisfies the one-dimensional wave equation from a sum of sinusoidal waves using Fourier's integral method:

$$f(x,t) = \int_{-\infty}^{+\infty} A(k)\exp\left[ik(x-ct)\right]\,dk$$

First, show that $f(x,t)$ satisfies the wave equation with wave speed *c*. Then, assume $A(k)$ is given by a Gaussian distribution centered at some positive wavevector k_0.

$$A(k) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(k-k_0)^2\sigma^2}{2}\right)$$

Sketch this. What range of wave vectors Δk contribute significantly to the wave packet? Calculate $f(x,t)$ from the above $A(k)$.

$$\text{Hint: } \int_{-\infty}^{+\infty} \exp\left[-\frac{q^2y^2}{2}+zy\right]dy = \sqrt{\frac{2\pi}{q^2}}\exp\left(\frac{z^2}{2q^2}\right).$$

Show that $f(x,t)$ is a localized wave packet with wavevector k_0 and a Gaussian envelope that moves to the right with speed *c*. How is the width Δx of the wave packet related to the width Δk?

6.11. Quantum wave velocities [Kinney SP11, Munsat FA10]

The quantum mechanical wave function that describes a particle of mass *m* moving in the *z* direction with momentum *p* and energy kinetic energy *E* is

$$\Psi(z,t) = Ae^{i(pz-Et)/\hbar}.$$

The motion is non-relativistic, *i.e.,* $E = p^2/2m$. Calculate both the wave (phase) velocity and the group velocity for this wave function and compare them with the classical speed *p/m.*

6.12. Complex wave notation [Munsat FA10]

Consider a plane wave in free space with oscillating magnetic field described by

$$\vec{B} = B_0\exp\left[i(kz-\omega t)\right]\,\hat{y}$$

Calculate the induced electromotive force through a loop with area *A*, located at the origin and oriented in the direction, in two different ways:

(a) First take the real part of $\vec{B}$, then *without* using complex notation find $E(t)$.

(b) Use complex notation until the very end, then take the real part of $E(t)$.

6.13. Polarization of waves [Munsat FA10]

Consider a superposition of two waves with a relative phase difference θ, defined by

$$E_{\mathrm{x}} = E_{0\mathrm{x}} \exp\left[i\left(kz - \omega t\right)\right] \hat{x} \qquad \& \qquad E_{\mathrm{y}} = E_{0\mathrm{y}} \exp\left[i\left(kz - \omega t\right)\right] \hat{y}$$

Use a computer plotting routine to plot E_{y} vs. E_{x} at a fixed location z, for θ ranging from 0 to π, in increments of π/6 (i.e. 7 plots).