09 – Linear Operators

Topics: Linear differential operators/equations

Summary: Linear operators are defined, and students must determine which of five operators are linear. The second part addresses how the components of a complex solution are themselves solutions to a linear differential equation.

Written by: Michael Dubson, Charles Baily and Steven Pollock.

Contact: Steven.Pollock@Colorado.EDU

Comments: Students should be able to finish these tasks in less than 15 minutes. They should be sure to check their answers to the first part, since many will mistakenly believe that option *IV* is linear if they don't think too hard about it. The final task is designed to help students not get confused about how to translate between these complex exponential representations and physical solutions.
(In Fall 2014 we converted this Tutorial into clicker question format, and did not use this paper version)

A. A *function* f takes a number x as input and outputs another number $y = f(x)$.

An *operator* L takes a function $x(t)$ as input and outputs another function $g(t) = L[x(t)]$. A *linear* operator is one where $L[A \cdot x_1(t) + B \cdot x_2(t)] = A \cdot L[x_1(t)] + B \cdot L[x_2(t)]$

A & B are numbers, and x_1 & x_2 are functions.

Which of the following are linear operators? Check each of them to be sure.

I. $L[x] = x^2$

II. $L[x] = A\dfrac{d^2x}{dt^2} + B\dfrac{dx}{dt} + Cx$

III. $L[x] = \sin(x)$

IV. $L[x] = Ax + B$

IV. $L[x] = \exp(x)$

B. Consider the differential equation: $V(t) = A \cdot \frac{d^2 I}{dt^2} + B \cdot \frac{dI}{dt} + C \cdot I$
where *A*, *B* & *C* are constants.

Suppose we find complex solutions $\tilde{V}(t)$ and $\tilde{I}(t)$:

$$\tilde{V}(t) = A \cdot \frac{d^2 \tilde{I}}{dt^2} + B \cdot \frac{d\tilde{I}}{dt} + C \cdot \tilde{I}$$

$\tilde{V}(t)$ and $\tilde{I}(t)$ each have real and imaginary parts:

$$\tilde{V}(t) = V_{\mathrm{R}} + iV_{\mathrm{Im}} \qquad \& \qquad \tilde{I}(t) = I_{\mathrm{R}} + iI_{\mathrm{Im}}$$

Given the complex solutions, is it always true that V_{R} and I_{R} are also solutions? In other words, is the following always true?

$$V_{\mathrm{R}}(t) = A \cdot \frac{d^2 I_{\mathrm{R}}}{dt^2} + B \cdot \frac{dI_{\mathrm{R}}}{dt} + C \cdot I_{\mathrm{R}}$$

Briefly explain your reasoning.