



11 – Complex Plane Waves

Topics: Plane waves, complex exponentials

Summary: The initial tasks have students identifying the various quantities that go into a plane wave represented by a complex exponential; the first involves constructing an expression from the quantities given, while the second analyzes the quantities for an expression that's given to them. The remaining tasks involve deriving explicit expressions for the divergence and curl of a plane wave, and relating them to Maxwell's equations in vacuum to determine the orientations of the electric and magnetic fields relative to the direction of propagation.

Written by: Charles Baily, Steven Pollock and Michael Dubson.

Contact: Steven.Pollock@Colorado.EDU

Comments: Most students will be able to finish this activity in less than 40 minutes. We've noticed that students can have trouble parsing out the various vectors and scalar quantities that go into a plane wave expressed in complex exponential notation; the first two tasks give them practice with this. Some students get confused about the difference between *frequency* and *angular frequency*; this provides an opportunity to point out how the argument must advance by 2π for every cycle. Many students didn't take into consideration how multiplying by the complex number $1 + i$ affects the *amplitude* of the wave, instead of it just adding a phase angle. A few were confused about the difference between amplitude (maximum displacement) and the displacement at each moment in time (they wrote the amplitude as a cosine function). When taking partial derivatives of the complex exponential, many students had difficulty correctly applying the chain rule; in particular, they often didn't see that the dot-product $\vec{k} \cdot \vec{r}$ is compact notation for $k_x x + k_y y + k_z z$ - sometimes because they were associating the \vec{r} -vector only with spherical coordinates. This issue could be explicitly addressed in the task, but we feel it is instructive for students to make this error on their own, and hopefully learn from their mistake. The final page asks students to make a *convincing argument* for how the electric field is related to the magnetic field in terms of a cross-product with the wave vector - several students were initially trying to calculate the actual cross-product using determinants, without recognizing that the divergence and curl operations just replace the spatial derivatives with the corresponding components of \vec{k} .

Complex Plane Waves

A. Using complex exponential notation, construct an expression for a plane wave with the following properties:

Amplitude = A_0

Traveling in the $+\hat{z}$ -direction

Linearly polarized in the x -direction

Wavelength = 2 meters

Frequency = 10 Hz

Phase angle = $+\pi/2$



You may continue, but be sure to check your answers with an instructor.


Complex Plane Waves

B. A plane wave is given by the expression $\tilde{\mathbf{A}} \exp[i(ax - by - \omega t)]$, where $\tilde{\mathbf{A}} = A_0(1 + i) \hat{z}$ [a, b, ω & A_0 are all real numbers with appropriate units.]

What is the amplitude of this plane wave?

Re-write the argument of the complex exponential in the form $(\vec{k} \cdot \vec{r} - \omega t)$ to determine the direction of propagation for this plane wave.

What is the complex phase angle associated with this plane wave?

 You may continue, but be sure to check your answers with an instructor.

Complex Plane Waves

C. Suppose the E- and B-fields of an electromagnetic plane wave traveling in free space are given by the following expressions, where the vectors \vec{E}_0 and \vec{B}_0 are real constants.

$$(1) \quad \vec{E}(\vec{r}, t) = \vec{E}_0 \exp \left[i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right]$$

$$(2) \quad \vec{B}(\vec{r}, t) = \vec{B}_0 \exp \left[i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right]$$

Using expression (1) for the electric field, what is the partial derivative with respect to x of the x-component of $\vec{E}(\vec{r}, t)$?

$$\frac{\partial}{\partial x} E_x(\vec{r}, t) = ?$$

Use the above result to find an expression for the divergence of \vec{E} for this plane wave, **written in terms of a dot-product of the vectors \vec{k} and \vec{E} .**

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = ?$$

According to Maxwell's equations $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$. What is the divergence of the electric field equal to for a plane wave traveling in vacuum?

$$\vec{\nabla} \cdot \vec{E} = ?$$

According to Maxwell's equations, what is the divergence of \vec{B} for a plane wave traveling in vacuum?

$$\vec{\nabla} \cdot \vec{B} = ?$$

Using the information on this page, what can you conclude about the direction of \vec{E} & \vec{B} relative to the vector \vec{k} ?

Complex Plane Waves

$$(1) \quad \vec{\mathbf{E}}(\vec{r}, t) = \vec{\mathbf{E}}_0 \exp \left[i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right] \quad (2) \quad \vec{\mathbf{B}}(\vec{r}, t) = \vec{\mathbf{B}}_0 \exp \left[i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right]$$

D. Use Faraday's Law $\vec{\nabla} \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}} / \partial t$ to make a convincing argument that $\vec{\mathbf{E}}$ & $\vec{\mathbf{B}}$ for an electromagnetic plane wave in vacuum are related by:

$$\vec{k} \times \vec{\mathbf{E}} = \omega \vec{\mathbf{B}}$$

[Hint: Do not spend your time trying to explicitly calculate the curl of $\vec{\mathbf{E}}$. Instead, think about what happens any time you take a partial derivative of the complex exponential.]

E. Use your conclusions from this and the previous page to determine the magnitude and direction of the Poynting vector $\vec{\mathbf{S}}$ for this electromagnetic plane wave.

$$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}} = ?$$

Summarize in words how the different spatial orientations of $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$, \vec{k} & $\vec{\mathbf{S}}$ are all related to each other for an electromagnetic plane wave in free space