10 – EM Wave Equation

Topics: Wave equation, Maxwell's equations.

Summary: This is a mostly mathematical exercise, to have students derive the wave equation for the electric field in a vacuum (where there are no charges or currents).

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Comments: The initial task of deriving the wave equation should be completed within 5-10 minutes, though some may need help in getting started. The final part asking about static fields has been added since the implementation in our class, but we expect that there will be some students who are confused by whether this statement about fields in a vacuum is completely general. As a *challenge question* for students who complete this quickly, students could be asked to make the same derivation for the magnetic field, though this is slightly redundant. The biggest confusion we've seen for students is with the wave equation for EM fields usually being written as a compact vector equation, where it can look as though the Laplacian is operating on the entire vector, instead of each of its components separately.

Using Maxwell's equations:

(1) $\vec{\nabla}\cdot\vec{\mathbf{E}}=\rho/\varepsilon_0$ (2) $\vec{\nabla}\cdot\vec{\mathbf{B}}=0$

(3) $\vec{\nabla}\times\vec{\mathbf{E}}=-\partial\vec{\mathbf{B}}/\partial t$ (4) $\vec{\nabla}\times\vec{\mathbf{B}}=\mu_0\vec{\mathbf{J}}+\mu_0\varepsilon_0\,\partial\vec{\mathbf{E}}/\partial t$

and the following vector identity:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{f}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{f}\right)-\nabla^2\vec{f}$$

show that in vacuum (where there are no charges or currents) each of the three spatial components of the *electric* field satisfy the three-dimensional wave equation.

$$\nabla^2\vec{f}\left(\vec{r},t\right)=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\vec{f}\left(\vec{r},t\right)$$

What is the speed v of this wave?

Do *static* electric fields satisfy the wave equation? Explain briefly why or why not.