Normalizing:

Schrodinger Eq'n, if $V = V(x)$ (no t!)

$$i\hbar \frac{\partial \Psi}{\partial t}(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}(x,t) + V(x)\,\Psi(x,t)$$

we argued $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$ if $|\Psi|^2$ is

to be a "probability density". Need to

check that if it's true at one t, it'll still

be true later!

$$\frac{\partial}{\partial t} |\Psi(x,t)|^2 = \frac{\partial}{\partial t}\left[\Psi^*(x,t)\;\Psi(x,t)\right]$$

$$= \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

Now $\displaystyle \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\cdot\frac{1}{i\hbar}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\cdot\frac{1}{i\hbar}$

so $\displaystyle \left(\frac{\partial \Psi}{\partial t}\right)^* = -\frac{\hbar^2}{2m}\cdot\frac{1}{-i\hbar}\frac{\partial^2 \Psi^*}{\partial x^2} + V^*\Psi^*\cdot\frac{1}{-i\hbar}$

$= V$, for real P.E.!

so $\frac{\partial \Psi^*}{\partial t} = +\frac{\hbar^2}{2m}\cdot\frac{1}{i\hbar}\frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^*$

thus

$$\frac{\partial}{\partial t}|\Psi|^2 = -\frac{\hbar^2}{2m}\cdot\frac{1}{i\hbar}\underbrace{\left[\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right]} + \cancel{V|\Psi|^2\left(\frac{1}{i\hbar}-\frac{1}{i\hbar}\right)}$$

<u>Trick</u>: this is $\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right]$

check for yourself, cross terms cancel!

So $\frac{\partial|\Psi|^2}{\partial t}$ is <u>not</u> zero (!) but $\int_{-\infty}^{\infty}\frac{\partial|\Psi|^2}{\partial t}dx$ <u>is</u>!

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}|\Psi|^2 dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[\text{stuff}\right]dx$$

$$= \frac{i\hbar}{2m}\left[\text{stuff}\Big|_{-\infty}^{\infty}\right] = 0,$$ because <u>(if!)</u>

$\Psi(\infty)$ vanishes.

That means $\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx = 0$, "Prob. is conserved" over time.

Momentum operator:

we've argued $\langle x \rangle = \int x \, |\Psi(x,t)|^2 \, dx$

what about $\langle p \rangle = m \langle v \rangle$ (?)

cannot just try $m \int v \, |\Psi(x,t)|^2 \, dx$,

because I have no quantum formula for $v(x)$ ← velocity

(+ turns out you can't, due to Heisenberg uncertainty)

Griffiths says $\langle v \rangle = \frac{d\langle x \rangle}{dt}$ (seems reasonable!)

This deriv is a lot like what we just did (prev. 2 pages)

but there's an extra "x" inside the integral.

But $\frac{\partial x}{\partial t}$ is not velocity, it's zero! [(x, t) are our two variables!]

so $\langle p \rangle = m \frac{d\langle x \rangle}{dt} = m \int x \frac{\partial}{\partial t} |\Psi(x,t)|^2 \, dx$

using result from prev pages,

$$\langle p\rangle = m\int x\cdot \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right]dx$$

Integrate by parts, toss "surface term"

(assuming $\psi\frac{\partial\psi^*}{\partial x}$ vanishes fast enough as ∞ to kill the "x" that comes along too, now!)

$$\langle p\rangle = \frac{i\hbar}{2}(-)\int\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]dx$$

Now do it again, $\int -\psi\frac{\partial\psi^*}{\partial x}dx = \psi^*\psi\Big|_{-\infty}^{\infty} + \int\psi^*\frac{\partial\psi}{\partial x}dx$

$-u\,dv \qquad -uv + \int v\,du$

so two terms are both same, and

$$\langle p\rangle = -i\hbar\int\psi^*\frac{\partial}{\partial x}\psi\,dx$$

this looks like $\int\psi^*\hat{p}\,\psi\,dx$

with $\hat{p} = -i\hbar\,\partial/\partial x$, the "momentum operator"

OK, check it out:

Classically $\langle p \rangle = \int \overset{\text{(momentum)}}{p(x)} \; \overset{*\text{(prob. density)}}{\rho(x,t)}\, dx$

In QM, you'd expect $\rho \to \Psi^*(x,t)\, \Psi(x,t)$

And that's close, but p is not longer a function of x, it's the operator $-i\hbar \frac{\partial}{\partial x}$, and you "sandwich" it

$$\underline{\underline{\text{QM}}}: \langle p \rangle = \int \Psi^*(x,t)\, \hat{p}\, \Psi(x,t)\, dx$$

$\hookrightarrow$ with $\hat{p} = -i\hbar \frac{\partial}{\partial x}$.

Does this make sense? You know, de Broglie says

$$\Psi(\text{free particle}) = A e^{i(kx - \omega t)}$$

has momentum $p = \hbar k$.

But notice $-i\hbar \frac{\partial \Psi_{free}}{\partial x} = -i\hbar (ik) \Psi_{free}$

$$= +(\hbar k)\Psi_{free}$$

so $\underbrace{\hat{p}}_{\text{operator}} \Psi_{free} = \underbrace{(\hbar k)}_{\text{a number, the momentum according to DeBroglie}} \Psi_{free}$. Nice! (Strange though, $\hat{p}$ is novel...)

- If $\hat{O}[f] = c[f]$ (for some special function(s) f)

we say f is an "eigenfunction of $\hat{O}$" with "eigenvalue c".

Apparently Ψ_{free} is an eigenfunction of $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ with eigenvalue $\hbar k$ (which De Broglie says is the number p.)

In Class. Mech, all dynamical quantities "Q" depend on x and/or p:

e.g position = x

Kin Energy = $\frac{p^2}{2m}$, total Energy = $\frac{p^2}{2m} + V(x)$

Ang momentum = $\vec{r} \times \vec{p}$

In Quantum, we just replace every "p" in these classical eq'ns with $-i\hbar \frac{\partial}{\partial x}$, i.e. with $\hat{p}$, and get QUANTUM OPERATORS.

so $\langle \hat{Q} \rangle = \int \Psi^*_{(x,t)} \hat{Q} \, \Psi_{(x,t)} \, dx$

where $\hat{Q}$ is just $Q(x,p)$, where you let $p \to \frac{\hbar}{i}\frac{\partial}{\partial x}$

e.g. $\langle \text{Energy} \rangle = \int \Psi^* \left[\frac{p^2}{2m} + V(x) \right] \Psi \, dx$

but p is really $\frac{\hbar}{i}\frac{\partial}{\partial x}$, so $p^2 = \underset{\uparrow}{-} \hbar^2 \frac{\partial^2}{\partial x^2}$

right!!

so $\langle \text{Energy} \rangle = \int \Psi^* \cdot \underbrace{\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right]} \Psi \, dx$

this is called $\hat{H}$, the Hamiltonian.

It's also the right side of Schrodinger's Eq'n,

which is $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$. See p. 1 of Griffiths!

So $\langle \text{Energy} \rangle = \langle \hat{H} \rangle$.