Conservation of Energy

The important *conclusions* of this chapter are:

- If a system is isolated and there is no kinetic friction (no *non-conservative* forces), then

  $$KE + PE = \text{constant}$$

  (Some texts use the notation K +U = constant)

- If there is friction, then $KE + PE + E_{therm} = \text{constant}$. ($E_{therm}$ = thermal energy)

- Two examples of PE (potential energy)

  $PE_{grav} = mgh$

  $PE_{elastic} = (1/2)kx^2$

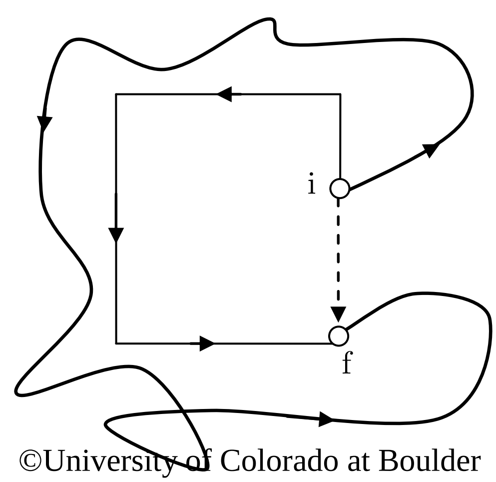
At this point, there are two questions you should be wondering about:
What is the definition of potential energy, PE, and why $PE_{grav} = mgh$, $PE_{elastic} = (1/2)kx^2$?
Why is KE+PE = constant, when system isolated and no friction?
It is not enough to know formulas. You should know where the formulas come from.

The Big Picture

We can define energy as the conserved, scalar quantity which obeys The First Law of Thermodynamics: $W + Q = \Delta U$.
In words, "work done + heat added = the change in energy of a system".
In this course, we will not consider heat exchanges, so $Q = 0$, and $W = \Delta U$. In some special cases, we can derive $W = \Delta U$ from Newton's Laws, but the general form $W + Q = \Delta U$ cannot be derived. We accept it as an experimental fact, and a new law of physics independent of Newton's Laws.

Potential Energy

So, how do we define potential energy, PE, and get $PE_{grav} = mgh$?
If a force involves no dissipation (no friction), then it can be a special type of force called a *conservative* force.



The defining property of a conservative force is that the work done by the force depends only the initial and final positions, not on the path taken. We showed in a previous concept test that gravity is a conservative force. The force of friction is not a conservative force, because the work done depends on the path taken: the longer the path the more work is done by friction.

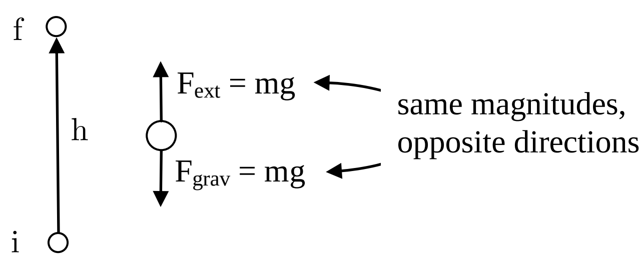
We have only two examples of conservative forces (so far) :

- gravity (F = mg)
- the spring force, or elastic force (F = –kx)

The normal force is not a conservative force, but it is something of a special case. The work done by the normal force when an object slides along a surface is always zero, so the normal force does zero work and we can ignore it, as far as energy problems are concerned.

Associated with every conservative force is a kind of energy called potential energy (PE or U). PE is a kind of stored energy. If a configuration of objects has PE, then there is the potential to change that PE into other kinds of energy (KE, thermal, light, etc). The definition of the PE associated with a conservative force involves the work done by that force. Let's first review the concept of work.

Recall: If I lift a mass m, a distance h, at constant velocity (**v** = constant), with an external force $\mathbf{F}_{ext}$, such as my hand, then the work done by gravity is the negative of the work done by the external force.



So W_{ext} = +mgh and W_{grav} = –mgh . This is true for the special case **v** = constant, but it turns out that it is always true that $W_{ext} = -W_{grav}$, regardless of the motion, so long as the KE at the final position is the same as the KE at the intial position. So, $W_{ext} = -W_{grav}$, if the mass starts and finishes at rest: $v_i = v_f = 0$. With this example in mind, we are ready to define PE.

If a force **F** (such as gravity) is a conservative force, then we define the PE associated with that force by

$$\boxed{\Delta PE_F \equiv -W_F = +W_{ext}}$$

In words: the change in potential energy is the negative of the work done by the conservative force and it is therefore the positive of the work done by an external force opposing the conservative force.

Only *changes* in PE are physically meaningful. We are free to set the zero of potential energy wherever we want.

$\Delta PE_{grav} = +W_{ext} = mgh$. In this formula, the height h is the height above (h+) or below (h–) the h=0 level. So h is really $\Delta h = h_f - \underset{0}{h_i} = h_f = h$.

So I should really write the formula as

$$\boxed{\Delta PE_{grav} = mg\Delta h\,.}$$

If I choose to set $PE_i = 0$ at $h_i = 0$, then the formula $\Delta PE_{grav} = mg\Delta h$ becomes $PE - \underset{0}{PE_i} = mg(h - \underset{0}{h_i})$ or simply, $PE_{grav} = mgh$

In the previous chapter, we showed that the work done by an external force to stretch or compress a spring by an amount x is $W_{ext} = \frac{1}{2}kx^2$. We therefore have that the elastic potential energy contained in a spring is $\Delta PE_{elas} = +W_{ext}$ or

$$\boxed{\Delta PE_{elas} = \tfrac{1}{2}kx^2}$$

In writing this formula, we have set $PE_{elas} = 0$ at x = 0 (the unstretched position).

(The normal force never does work, so $\Delta PE_{normal} = -W_{normal} = 0$. We can set the PE associated with the normal force equal to zero and forget about it.)

Where is potential energy located?

I lift a book of mass m a height h and say that the book has PE_{grav} = mgh. But it is not correct to say that the PE "in the book". The gravitational PE is associated with the system of (book + earth + gravitational attraction between book and earth). The PE is not "in the book" or "in the earth"; it is in the book-earth system which includes the "gravitational field" surrounding the book and the earth.

For the case of elastic potential energy, the PE_{elas} actually is inside the spring. It is located in the increased electrostatic potential energy in the chemical bonds joining the atoms of the spring.

Conservation of mechanical energy.

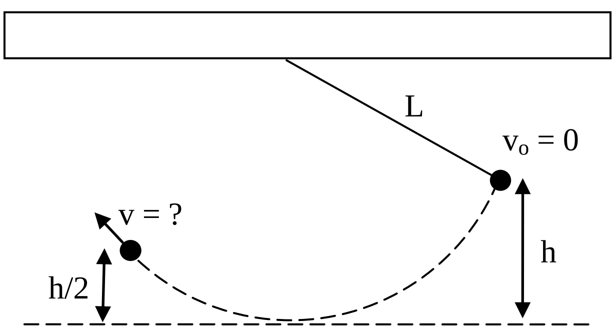
Definition: *mechanical energy* E_{mech} = KE + PE. We are now in a position to show that

$$E_{mech} = KE + PE = \text{constant (if no friction and system isolated).}$$

Recall the Work-KE Theorem: $W_{net} = \Delta KE$. Now if there is no friction, the net force involves conservative forces only, and $W_{net} = W_c$ (c for conservative force). But we just *defined* $\Delta PE \equiv -W_c$, so we have $W_{net} = W_c = \Delta KE = -\Delta PE$ or

$$\Delta KE + \Delta PE = 0 \quad \Leftrightarrow \quad KE + PE = \text{constant} \;\; \text{(if no friction)}$$

Example of Conservation of Energy (no friction). A pendulum consists of a mass *m* attached to a massless string of length *L*. The pendulum is released from rest a height *h* above its lowest point. What is the speed of the pendulum mass when it is at height *h*/2 from the lowest point? Assume no dissipation (no friction).



In all Conservation of Energy problems, begin by writing **(initial energy) = (final energy)** :

$E_i = E_f \;\Rightarrow\; KE_i + PE_i = KE_f + PE_f$

$\Rightarrow \;\; 0 + mgh = (1/2)\, mv^2 + mg(h/2)$

(cancel m's and multiply through by 2) $\Rightarrow$

$2gh = v^2 + gh \;\Rightarrow\; v^2 = gh \;\Rightarrow\; \boxed{v = \sqrt{g\,h}}$

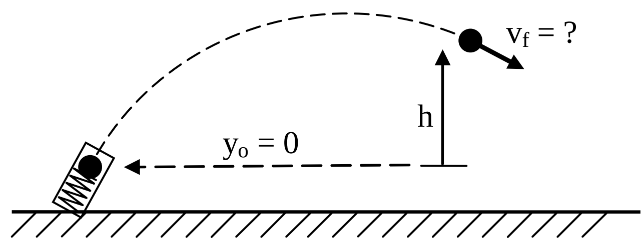
Notice: Using Conservation of Energy, we didn't need to know anything about the details of the forces involved and we didn't need to use $\mathbf{F}_{net} = m\mathbf{a}$. The Conservation of Energy strategy allows us to relate conditions at the beginning to conditions at the end; we don't need to know anything about the details of what goes on in between.

Suppose there are two conservative forces acting on a system, and no non-conservative forces. Then we have $\vec{F}_{net} = \vec{F}_{c1} + \vec{F}_{c2}$ (For instance, there may be gravity and a spring force, but no friction.) Then we have $W_{net} = \int \vec{F}_{net} \cdot d\vec{r} = \int \vec{F}_{c1} + \vec{F}_{c2} \cdot d\vec{r} = \int \vec{F}_{c1} \cdot d\vec{r} + \int \vec{F}_{c2} \cdot d\vec{r} = W_{c1} + W_{c2}$

The Work-KE Theorem then gives $W_{net} = W_{c1} + W_{c2} = -\Delta PE_1 - \Delta PE_2 = \Delta KE$ or

$$\Delta KE + \Delta PE_1 + \Delta PE_2 = 0 \quad \Leftrightarrow \quad KE + PE_1 + PE_2 = \text{constant (if no friction)}$$

Another example of Conservation of Energy: A spring-loaded gun fires a dart at an angle θ from the horizontal. The dart gun has a spring with spring constant *k* that compresses a distance *x*. Assume no air resistance. What is the speed of the dart when it is at a height *h* above the initial position?



$$E_i = E_f \quad \Rightarrow \quad KE_i + PE_{grav,i} + PE_{elas,i} = KE_f + PE_{grav,f} + PE_{elas,f}$$

$$0 \;+\; 0 \;+\; (1/2)kx^2 \;=\; (1/2)mv^2 \;+\; mgh \;+\; 0$$

$$x = \sqrt{\frac{m\,v^2 + 2\,m\,g\,h}{k}} = \sqrt{\left(\frac{m}{k}\right)\left(v^2 + 2\,g\,h\right)}$$

Notice that the angle θ never entered into the solution.

What if there is friction?

Up till now, we have assumed that there is no sliding friction in any of these problems. (Having static friction in a problem causes no difficulties, because static friction does not generate thermal energy.) How do we handle sliding friction and the thermal energy generated?

If a system is isolated from external forces so that no external work is done, and if no heat is transferred, and if there is no sliding friction so that no thermal energy is generated (that's a lot of "if's"), then we can assert that

KE + PE = constant (isolated system, no thermal energy involved)

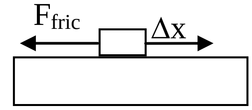
If, however, there is sliding friction inside the system, then some of the mechanical energy (KE+PE) can be transformed into thermal energy (E_{therm}). In this case, we have

$$KE + PE + E_{therm} = \text{constant (isolated system)}$$

We now show that the amount of thermal energy generated is the negative of the work done by friction:

$$\Delta E_{therm} = -W_{fric}$$

Notice that the work done by sliding friction is always negative, since sliding friction always exerts a force in the direction opposite the motion. Consequently, $-W_{fric}$ is a positive quantity.



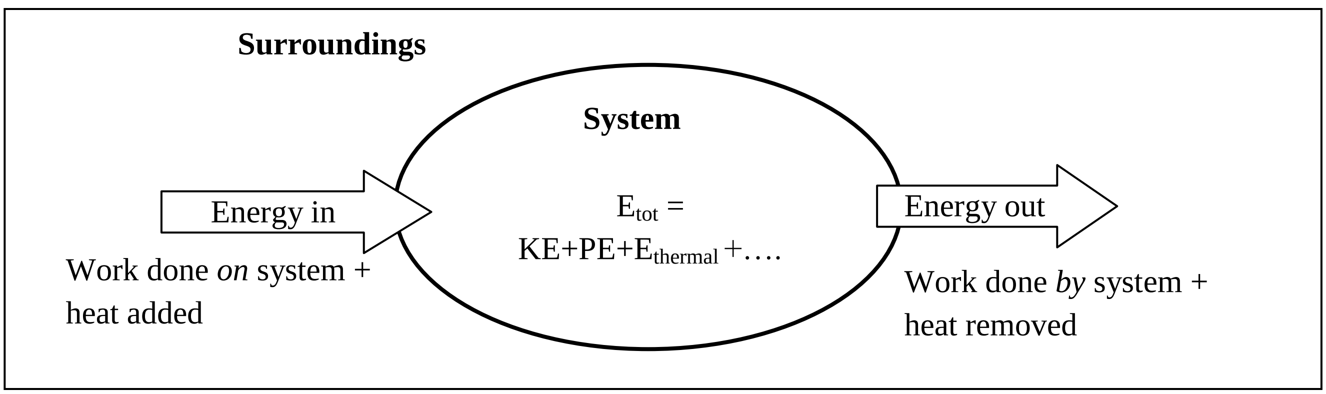
When there is sliding friction, the net force usually consists of both conservative forces (like gravity) and non-conservative forces (the friction), so $W_{net} = W_c + W_{nc}$. But $W_{net} = \Delta KE$ (by Work-KE theorem) and $W_c = -\Delta PE$, so we have

$$\Delta KE + \Delta PE - W_{nc} = 0 \quad \text{(isolated system, with friction)}$$

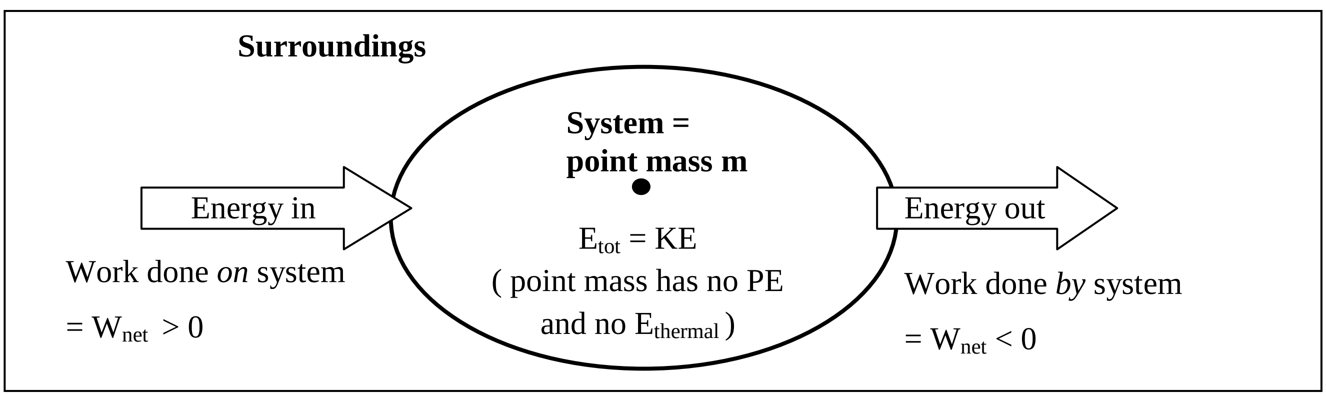
Since the total energy must remain constant in an isolated system, we identify the term $-W_{nc}$ as $\Delta E_{therm} = -W_{fric}$, since we want $\Delta KE + \Delta PE + \Delta E_{therm} = 0$ (which is the same as $KE + PE + E_{therm} =$ constant).

Some Pictorial Representations

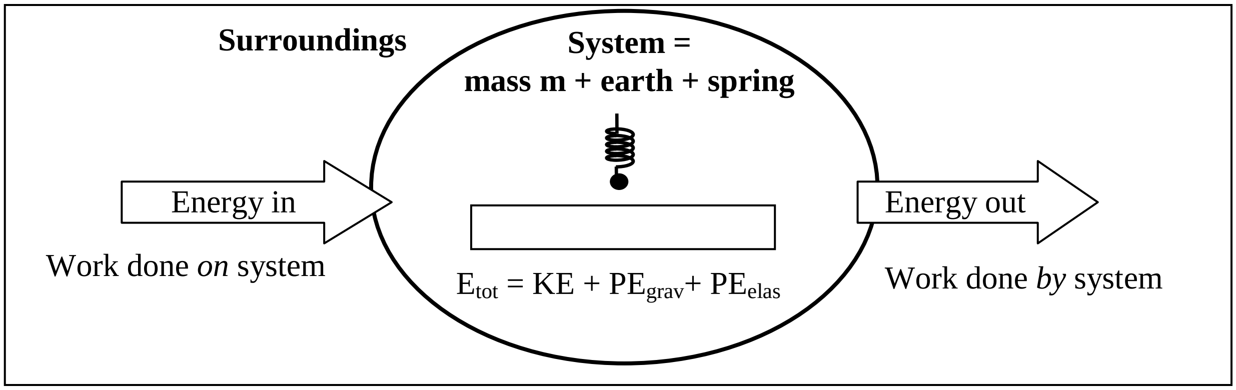
The First Law of Thermodynamics says:



The 1st Law applied to a simple point mass: the Work-KE theorem.

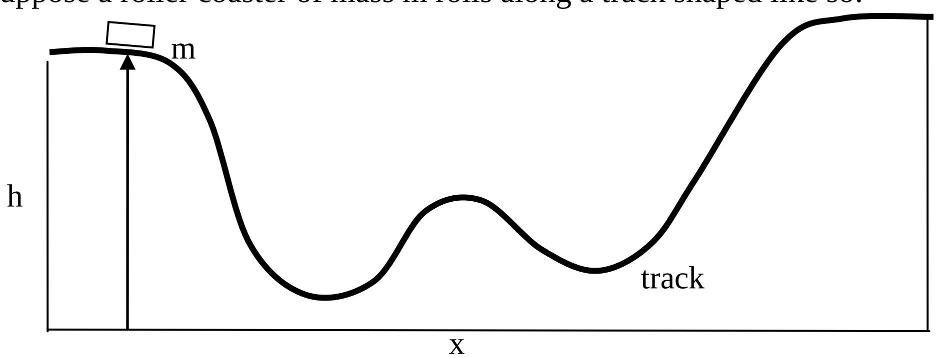


The 1st Law applied to a simple mechanical system.

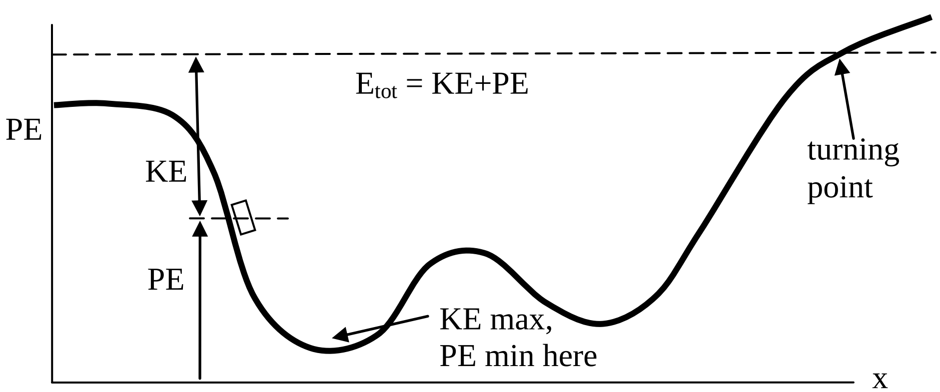


KE+PE=total energy graphs

Suppose a roller coaster of mass m rolls along a track shaped like so:



The shape of the track is a graph of height h vs. horizontal position x. Since the gravitational potential energy of the coaster is PE = mgh, where mg is a constant, a graph of PE vs. x looks the same as the graph of h vs. x, but with the vertical axis measuring energy (joules) rather than height (meters). Assuming no friction, the total mechanical energy E_{tot} = KE+PE of the roller coaster remains constant as it rolls along the track. We can represent this constant energy with a horizontal line on our graph of energy vs. x. From this "energy graph", we can read the KE and the PE of the coaster at any point.



Relation between force and PE.

For this discussion, I'll use the standard notation for potential energy: U = PE. I want to show that (in 1D)

$$\boxed{F = -\frac{dU}{dx}}$$

(Force is the negative "gradient" of the potential energy.)

Consider a small change in U, ΔU, that occurs when a small change in x, Δx, is made. By the definition of potential energy

$$\Delta PE_F = \Delta U = -W_F = -F \cdot \Delta x$$

(I can write $W_F = F \cdot \Delta x$ even if F varies with x, because Δx is small and so F ≈ constant.) If I divide through by Δx, we get,

$$F = -\frac{\Delta U}{\Delta x} \xrightarrow[\lim_{\Delta x \to 0}]{} -\frac{dU}{dx}$$

Here F is the conservative force associated with the potential energy U. This F is not, in general, the net force in the problem.

Power

Power = *rate* at which work is done = rate at which energy is converted from one form to another:

$$\boxed{P \equiv \frac{W}{\Delta t} = \frac{\Delta E}{\Delta t}}$$

units of power = [P] = joules/second = J/s = watts (W)

Every second, a 100 W light bulb converts 100 joules of electric potential energy into heat and light.

The power company sells potential energy in units of *kilowatt–hours*.
1 kW·hr = 1000 J/s × 3600 s = 3.6×10^6 J

Another popular unit of energy is the Calorie (spelled with a capital C). A typical candy bar has 300 Calories of stored chemical energy. There are two kinds of calories, spelled with a little "c" or a big "C":

1 calorie (cal) = "little calorie" = 4.186 J
1 Calorie = 1000 cal = "big Calorie" = "food calorie" = 1 kcal = 4186 J

The "food Calorie" is the "big Calorie" and it should be spelled with a big C. (Chemists like to use the little calorie, which is defined as the amount of heat required to raise the temp of a gram of water by 1^o centigrade.)
Calorie example: Dr. D, who has mass m = 68 kg, eats a 300 Cal candy bar and then climbs 10 stories (Δh = 35 meters) to his office on the 10^{th} floor of Gamow Tower. How many Calories has he burned?

Work done = $\Delta PE = mg\,\Delta h = (68\text{ kg})\,(9.8\text{ m/s}^2)\,(35\text{ m}) = 23300\text{ J} \times (1\text{Cal}/\ 4186\text{ J}) = 5.6\text{ Cal}$

A measly 5.6 Cal !?!? Well, it's not quite that bad. He was also doing a lot of ineffective work turning around in the stairwell, flailing his limbs, etc as he climbed, so the total mechanical work was more, maybe 10 Cal total. Also, the human body is not a very efficient machine: only about 25% of the food Calories burned come out of the body as mechanical work; the rest goes into heat. (Dr. D was flushed and panting after his 10-story climb.) So to produce 10 Cal of work, his body burned about 40 Cal — still not very much.

Moral: You can't burn many Calories instantly by exercising. However, by exercising regularly, you build muscles which increases your resting metabolic rate (RMR). A typical out–of-shape male has a RMR of about 70 watts, meaning 70 joules per second burned by just breathing, digesting, thinking. (70 W is about 1400 Cal/day). By exercising regularly, that RMR can be raised to 90 watts (1860 Cal/day). So by exercising regularly, you burn about an extra 500 Cal per day just from your increased resting metabolic rate. "Lose weight while you sleep!" With the increased RMB, you can eat about 1 candy bar per day more than normal and still not gain weight.

Power example: The same Dr. D can climb to the 10^{th} floor in 60 seconds (if he pushes!). What is the mechanical power he generates (due to increased PE only, not including heat generated)?

$P = \Delta PE\ /\ \Delta t = mg\,\Delta h/\ \Delta t = (68\text{ kg})\,(9.8\text{ m/s}^2)\,(35\text{ m})\ /\ (60\text{ s}) = 390\text{ W}$

— almost enough to light four 100W light bulbs for that 1 minute.

A horse can generate a power of 1 horsepower for several hours. 1 hp = 746 W. So Dr. D can generate about ½ hp for 1 minute (and then he is needs to take a nap). Horses are pretty powerful!

Currently (2013) the power company sells energy at a rate of $0.10 per kilowatt·hour. One kW·hr is enough to light ten 100 W bulbs for 1 hour — and you get that for 10 cents!

