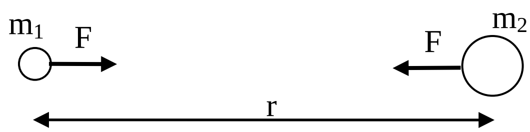
Gravity

Newton's Universal Law of Gravitation (first stated by Newton): any two masses m_1 and m_2 exert an attractive gravitational force on each other according to

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$



This applies to all masses, not just big ones.

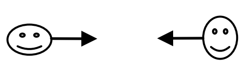
G = universal constant of gravitation = 6.67×10^{-11} N m^2 / kg^2 (G is very small, so it is very difficult to measure!)

Don't confuse G with g: "Big G" and "little g" are totally different things.

Newton showed that the force of gravity must act according to this rule in order to produce the observed motions of the planets around the sun, of the moon around the earth, and of projectiles near the earth. He then had the great insight to realize that this same force acts between *all* masses. [That gravity acts between all masses, even small ones, was experimentally verified in 1798 by Cavendish.]

Newton couldn't say *why* gravity acted this way, only *how*. Einstein’s (1915) General Theory of Relativity explained why gravity acted like this.

Example: Force of attraction between two humans. 2 people with masses $m_1 \cong m_2 \cong 70$ kg, distance r = 1 m apart.

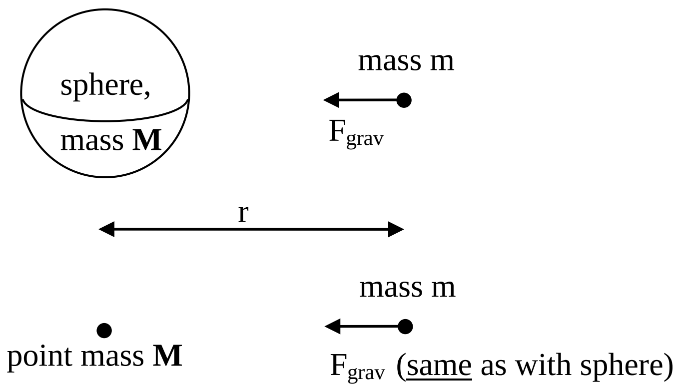


$$F = G\frac{m_1 m_2}{r^2} = \frac{(6.67\times 10^{-11})(70)^2}{1^2} = 3.3\times 10^{-7}\ \text{N}$$

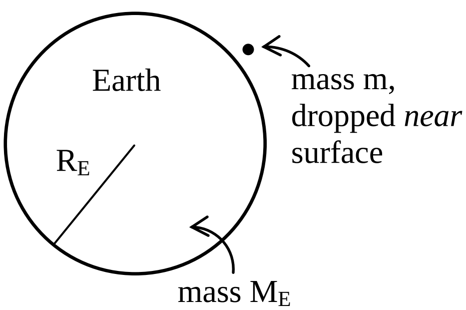
This is a very tiny force! It is the weight of a mass of 3.4×10^{-5} gram. A hair weighs 2×10^{-3} grams – the force of gravity between two people talking is about 1/60 the weight of a single hair.

Computation of g

Important fact about the gravitational force from spherical masses: a spherical body exerts a gravitational force on surrounding bodies that is the same as if all of the sphere's mass were concentrated at its center. This is difficult to prove (Newton worried about this for 20 years.)



We can now *compute* the acceleration of gravity g ! (Before this chapter, g was experimentally determined, and it was a mystery why g was the same for all masses.)



$$F_{grav} = m\,a = m\,g$$

$$G\,\frac{M_E m}{R_E^{\,2}} = m\,g$$

(since $r = R_E$ is distance from m to center of Earth)

m's cancel ! $\Rightarrow$ $\boxed{g = \dfrac{G\,M_E}{R_E^{\,2}}}$

If you plug in the numbers for G, M_E, and R_E, you get $g = 9.8\ \mathrm{m/s^2}$.

Newton's Theory explains why all objects near the Earth's surface fall with the same acceleration (because the m's cancel in $F_{grav} = \dfrac{G\,M\,m}{R^2} = m\,a$.) Newton's theory also makes a quantitative prediction for the value of g, which is correct.

Example: g on Planet X. Planet X has the same mass as earth ($M_X = M_E$) but has ½ the radius ($R_X = 0.5\ R_E$). What is g_x , the acceleration of gravity on planet X?

Planet X is denser than earth, so expect g_x larger than g.

Method I:

$$g_x = \frac{G\,M_X}{R_X^{\,2}} = \frac{G\,M_E}{\left(R_E/2\right)^2} = \frac{1}{(1/2)^2}\,\underbrace{\frac{G\,M_E}{R_E^{\,2}}}_{\text{g of earth}} = 4\,g\,.$$

Don't need values of G, M_E, and R_E!

Method II, set up a ratio:

$$\frac{g_x}{g_E} = \frac{\left(\dfrac{\cancel{G} M_X}{R_X{}^2}\right)}{\left(\dfrac{\cancel{G} M_E}{R_E{}^2}\right)} = \frac{M_X}{M_E}\left(\frac{R_E}{R_X}\right)^2 = 1 \cdot 2^2 = 4 \ , \quad g_X = 4 g_E$$

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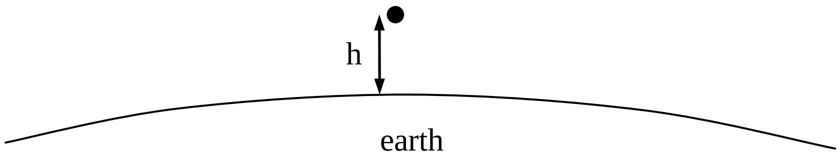
At height h above the surface of the earth, g is less, since we are further from the surface, further from the earth's center.



$r = R_E + h \Rightarrow$

$$g = \frac{G M_E}{r^2} = \frac{G M_E}{(R_E + h)^2}$$

The space shuttle orbits earth at an altitude of about 200 mi × 1.6 km/mi ≅ 320 km. Earth's radius is R_E = 6380 km. So the space shuttle is only about 5% further from the earth's center than we are. If r is 5% larger, then r^2 is about 10% larger, and

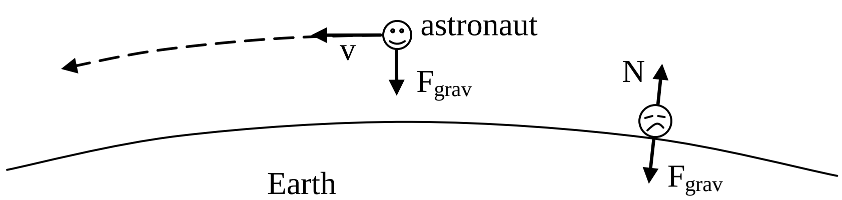
$$F_{grav}(\text{on mass m in shuttle}) = G\frac{M_E\, m}{(R_E + h)^2} \cong \text{ about 10\% less than on earth's surface}$$

Astronauts on the shuttle experience almost the same F_{grav} as when on earth. So why do we say the astronauts are *weightless*??

"Weightless" does not mean "no weight".

"Weightless" means "freefall" means "the <u>only</u> force acting is gravity".

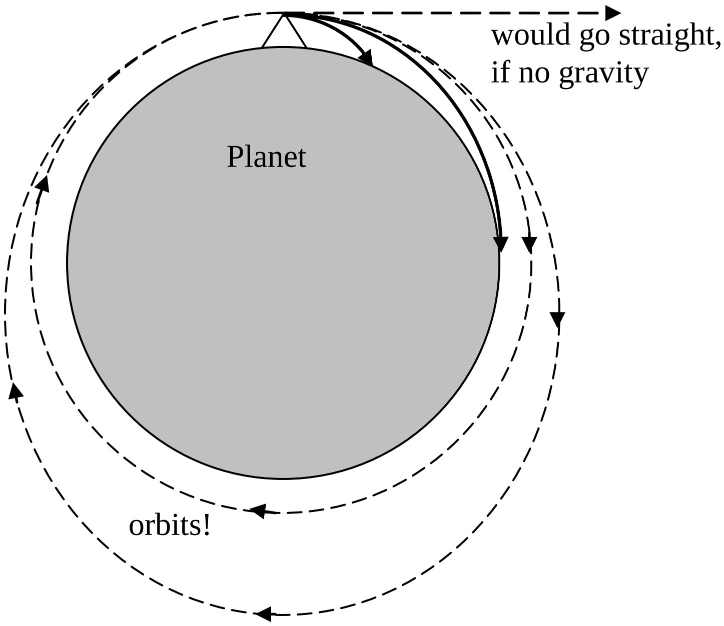
If you fall down an airless elevator shaft, you will feel exactly like the astronauts. You will be weightless, you will be in free-fall.



An astronaut falls toward the earth, as she moves forward, just as a bullet fired horizontally from a gun falls toward earth.

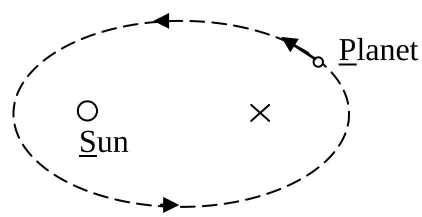
Orbits

Consider a planet like Earth, but with no air. Fire projectiles horizontally from a mountain top, with faster and faster initial speeds.



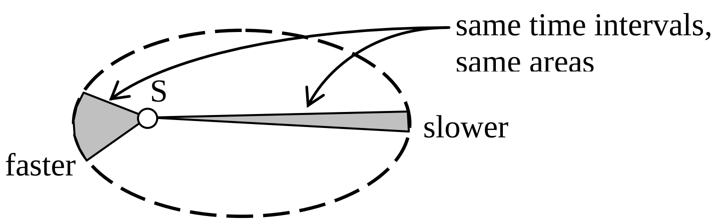
The orbit of a satellite around the earth, or of a planet around the sun obeys *Kepler's 3 Laws*.

Kepler, German (1571-1630). Before Newton. Using observational data from Danish astronomer Tycho Brahe ("Bra-hay"), Kepler discovered that the orbits of the planets obey 3 rules.



KI : A planet's orbit is an ellipse with the Sun at one focus.

KII : A line drawn from planet P to sun S sweeps out equal areas in equal times.



KIII: For planets around the sun, the period T and the mean distance r from the sun are related by $\frac{T^2}{r^3} = \text{constant}$. That is for any two planets A and B, $\frac{T_A^{\,2}}{r_A^{\,3}} = \frac{T_B^{\,2}}{r_B^{\,3}}$. This means that planets further from the sun (larger r) have longer orbital periods (longer T).

Kepler's Laws were empirical rules, based on observations of the motions of the planets in the sky. Kepler had no theory to explain these rules.

Newton (1642-1727) started with Kepler's Laws and NII ($\mathbf{F}_{net} = m\mathbf{a}$) and <u>deduced</u> that

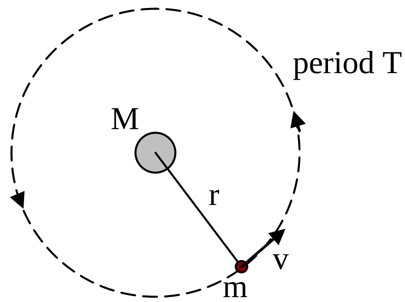
$$\underset{(\text{Sun}-\text{planet})}{F_{grav}} = G\frac{M_S\, m_P}{r_{SP}{}^2}.$$

Newton applied similar reasoning to the motion of the Earth-Moon system (and to an Earth-apple system) and deduced that $\underset{(\text{Earth-mass}\, m)}{F_{grav}} = G\frac{M_E\, m}{r_{Em}{}^2}$.

Newton then made a mental leap, and realized that this law applied to <u>any</u> 2 masses, not just to the Sun-planet, the Earth-moon, and Earth-projectile systems.

Starting with $\mathbf{F}_{net} = m\mathbf{a}$ and $F_{grav} = G\, Mm / r^2$, Newton was able to <u>derive</u> Kepler's Laws (and much more!). Newton could explain the motion of everything!

Derivation of KIII (for special case of circular orbits). Consider a small mass m in circular orbit about a large mass M, with orbital radius r and period T. We aim to show that $T^2 / r^3 = \text{const}$.



Start with NII: $F_{net} = m\, a$

The only force acting is gravity, and for circular motion $a = v^2 / r \quad \Rightarrow$

$$G\frac{M\cancel{m}}{r^{\cancel{2}}} = \cancel{m}\frac{v^2}{\cancel{r}} \Rightarrow G\frac{M}{r} = v^2 = \left(\frac{2\pi r}{T}\right)^2$$

[recall the v = dist / time = 2πr / T]

$$\Rightarrow \quad G\frac{M}{r} = \frac{4\pi^2 r^2}{T^2} \quad \Rightarrow \quad \frac{T^2}{r^3} = \frac{4\pi^2}{G\,M} = \text{constant, independent of m}$$

So, not only did Newton derive the experimentally observed fact the T^2/r^3 = constant, but his theory also explained the value of the constant. (Deriving this result for elliptical orbits is much harder, but Newton did it.)

An extra result of this calculation is a formula for the speed v of a satellite in circular orbit:

$$v = \sqrt{\frac{GM}{r}}.$$

For low-earth orbit (few hundred miles up), this orbital speed is about 7.8 km/s ≅ 4.7 miles/second. The Space Shuttle must attain a speed of 4.7 mi/s when it reaches the top of the atmosphere (and it fuel has run out) or else it will fall back to Earth.

Measurement of Big G

The value of G ("big G") was not known until 1798. In that year, Henry Cavendish (English) measured the very tiny F_{grav} between 2 lead spheres, using a device called a torsion balance.

$$F_{grav} = G\frac{m_1\, m_2}{r^2} \quad \Rightarrow \quad G = \frac{F_{grav}\, r^2}{m_1 m_2}$$ (If F_{grav}, r, and m's known, can compute G.)

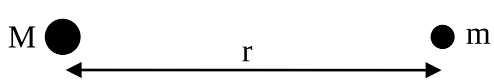
Before Cavendish's experiment, g and R_E were known, so using $g = \dfrac{G M_E}{R_E{}^2}$, one could compute the product $G \cdot M_E$, but G and M_E could not be determined separately.

With Cavendish's measurement of G, one could then compute M_E. Hence, Cavendish "weighed the earth".

Gravitational Potential Energy

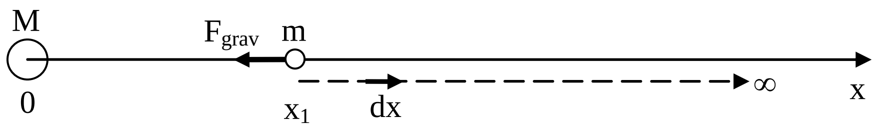
Previously, we showed that PE_{grav} = mgh. But to derive PE = mgh, we assumed that F_{grav} = mg = constant, which is only true near the surface of the Earth. In general, $F_{grav} = G\dfrac{M m}{r^2} \neq$ constant (it depends on r). We now show that for the general case,

$$PE_{grav} = \boxed{U(r) = -\frac{G M m}{r}} \quad , \qquad [U(r = \infty) = 0]$$



This is the gravitational potential for two masses, M and m, separated by a distance r. By convention, the zero of gravitational potential energy is set at r = ∞. [I will use the common notation U(r) , instead of PE.]

Recall the definition of PE: $\Delta PE_F \equiv -W_F = -\int_{x1}^{x2} F(x)\,dx$. Here, we have used the definition of work for the case of 1D motion: $W_F \equiv \int_i^r \vec{F} \cdot d\vec{r} \underset{(1D)}{=} \int_{x1}^{x2} F(x)\,dx$.

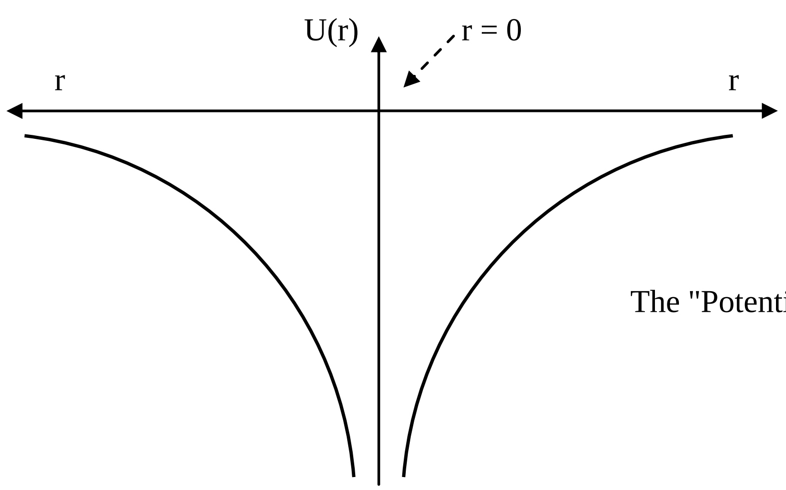


Consider a mass M at the origin and a mass m at position x_1, as shown in the diagram. We compute the work done by the force of gravity as the mass m moves from $x = x_1$ to $x = \infty$. The force F(x) on mass m is in the negative direction, so, indicating direction with a sign, we have $F(x) = -\frac{G\,M\,m}{x^2}$. Here, the work done by gravity is negative, since force and displacement are in opposite directions:

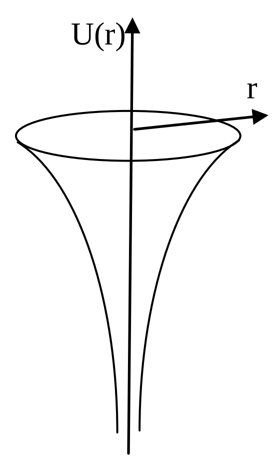
$$W_{grav} = \int_{x1}^{\infty} F(x)\,dx = -\int_{x1}^{\infty} \frac{GMm}{x^2}\,dx = +\left.\frac{GMm}{x}\right|_{x1}^{\infty} = -\frac{GMm}{x_1}$$

From the definition of PE, $\Delta PE = \Delta U = \underbrace{U(x{=}\infty)}_{0} - U(x_1) = -W_{grav} = +\frac{GMm}{x_1}$. Calling the initial position r (instead of x_1), we have $U(r) = -\frac{G\,M\,m}{r}$.

A slight notation change now: r is the radial *distance* from the origin, so r is always positive (unlike x which can be positive or negative.) Plotting U(r) vs. r, we see a "gravitational potential well".

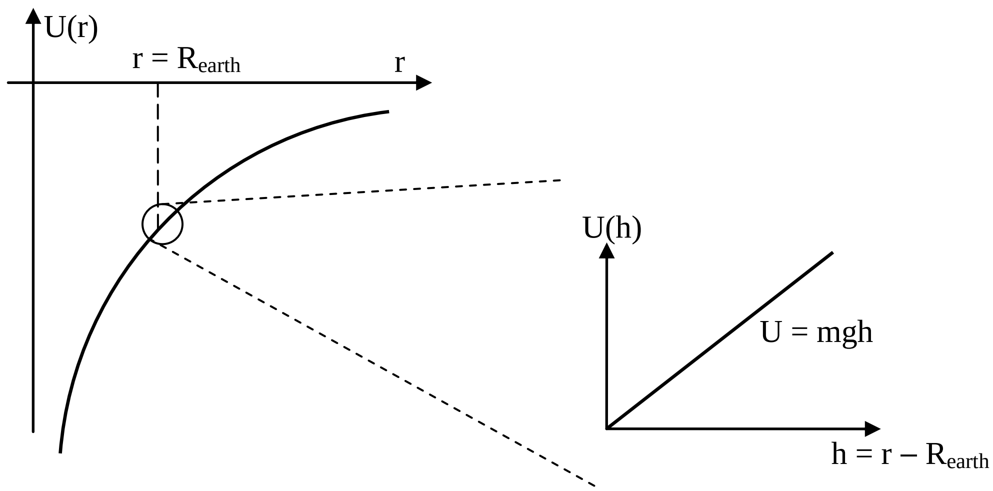


The "Potential Well"



Recall that negative potential energy simply means less energy than the zero of energy.

Question: How is PE = mgh a special case of U(r) = –GMm/r ?



Escape Speed v_{escape}

Throw a rock away from an (airless) planet with a speed v. If $v < v_{escape}$, the rock will rise to a maximum height and then fall back down. If $v > v_{escape}$, the rock will go to r = ∞ , and will still have some speed left over and be moving away from the planet. If $v = v_{escape}$, the rock will have just enough initial KE to escape the planet: its distance goes to r = ∞ at the same time its speed approaches zero: $v \to 0$ as $r \to \infty$.

We can use conservation of energy to compute the escape speed v_{esc} (often called , incorrectly, the "escape velocity").
Initial configuration: r = R (surface of planet), $v = v_{esc}$.
Final configuration: r = ∞ , v = 0.

$$KE_i + PE_i = KE_f + PE_f \quad \Rightarrow \quad \tfrac{1}{2}\cancel{m}\, v_{esc}^2 - \frac{G M \cancel{m}}{R} = 0 + 0$$

$$\Rightarrow v_{esc} = \sqrt{\frac{2GM}{R}}$$

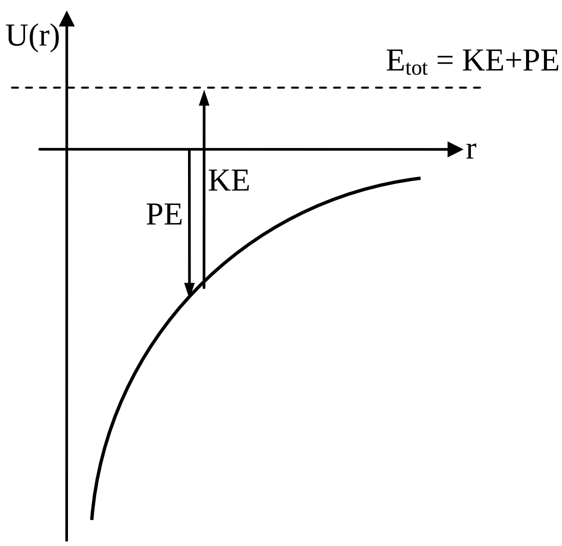
Notice that $v_{esc} = \sqrt{2}\, v_{orbit}$

If the rock is thrown with speed $v > v_{esc}$, it will go to r = ∞, and will have some KE left over, $v_{final} > 0$.

$$KE_i + PE_i = KE_f + PE_f \quad \Rightarrow \quad \tfrac{1}{2}\cancel{m}\,v_i^2 - \frac{G\,M\,\cancel{m}}{R} = \tfrac{1}{2}\cancel{m}\,v_f^2 + 0$$

$$\Rightarrow v_f = \sqrt{v_i^2 - \frac{2GM}{R}}$$

$$\underset{(+)}{KE} + \underset{(-)}{PE} = \underset{(+)\text{ or }(-)}{E_{tot}} = \text{constant}$$



Note that the KE is always positive and, in this case, the PE is always negative. The total energy E_{tot} = KE + PE can be either positive or negative. If $E_{tot} < 0$, then we have *bound* system; the KE of the mass m is not large enough for m to escape to infinity, and m remains in elliptical orbit about M . If $E_{tot} > 0$, then we have an *unbound* system, and the mass m will escape to infinity along a hyperbolic orbit. If $E_{tot} = 0$, then the mass m will escape to infinity along a parabolic orbit.

