Linear Momentum

Definition: *Linear momentum* of a mass *m* moving with velocity $\vec{v}$:

$$\boxed{\vec{p} \equiv m\,\vec{v}}$$

Momentum is a vector. Direction of **p** = direction of velocity **v.**

units [p] = kg·m/s (no special name)

(No one seems to know why we use the symbol *p* for momentum, except that we couldn't use "*m*" because that was already used for mass.)

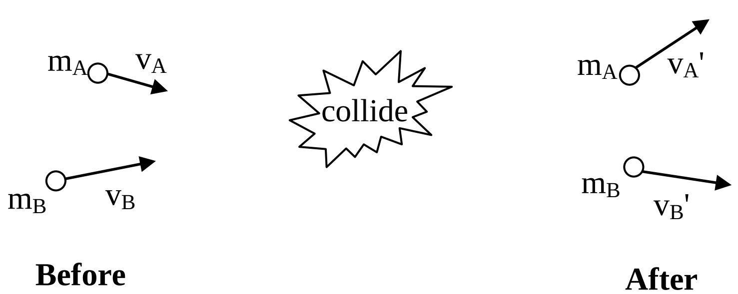
Definition: Total momentum of several masses: m_1 with velocity $\mathbf{v}_1$, m_2 with velocity $\mathbf{v}_2$, etc..

$$\vec{p}_{tot} \equiv \sum_i \vec{p}_i = \vec{p}_1 + \vec{p}_2 + \ldots = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots$$

Momentum is an extremely *useful* concept because total momentum is *conserved* in a system isolated from outside forces. Momentum is especially useful for analyzing collisions between particles.

Conservation of Momentum: You can never create or destroy momentum; all we can do is transfer momentum from one object to another. Therefore, the total momentum of a system of masses isolated from external forces (forces from outside the system) is constant in time. Similar to Conservation of Energy – always true, no exceptions. We will give a proof that momentum is conserved later.

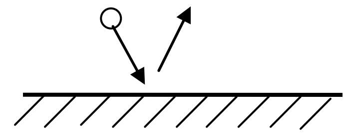
Two objects, labeled A and B, collide. **v** = velocity before collision, **v'** (v-prime) = velocity after collision.



Conservation of momentum guarantees that $\vec{p}_{tot} = m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v}'_A + m_B\vec{v}'_B$. The velocities of all the particles changes in the collision, but the total momentum does not change.

Types of collisions

elastic collision : total KE is conserved (KE before = KE after)



superball on concrete: KE just before collision = KE just after (almost!) The Initial KE just before collision is converted to elastic PE as the ball compresses during the first half of its collision with the floor. But then the elastic PE is converted back into KE as the ball un-compresses during the second half of its collision with the floor.

inelastic collision : some KE is lost to thermal energy, sound, etc

perfectly inelastic collision (or totally inelastic collision) : 2 objects collide and stick together

All collisions between macroscopic (large) objects are inelastic – you always dissipate some KE in a collision. However, you can have an elastic collision between atoms: air molecules are always colliding with each other, but do not lose their KE.

1D Collisions

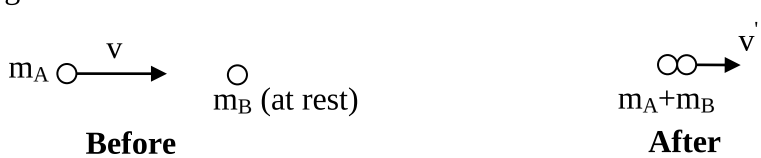
In 1D, we represent direction of vectors **p** and **v** with a sign. (+) = right (–) = left

$v_A = +2$ m/s $\Rightarrow$ moving right

$v_B = -3$ m/s $\Rightarrow$ moving left

Notation Danger!! Sometimes $v = |\vec{v}| = \text{speed}$ (always positive). But in 1D collision problems, symbol "v" represents *velocity* : v can (+) or (–).

1D collision example: 2 objects, A and B, collide and stick together (a perfectly inelastic collision). Object A has initial velocity v, object B is initially at rest. What is the final velocity v' of the stuck-together masses?



$$p_{tot,before} = p_{tot,after}$$

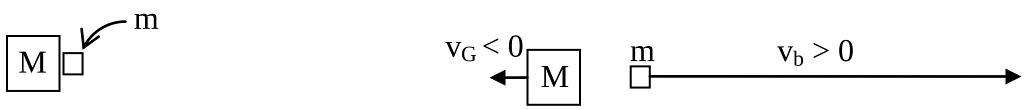
$$m_A \underbrace{v_A}_{v} + m_B \underbrace{v_B}_{0} = (m_A + m_B)\,v'$$

$$m_A v = (m_A + m_B)\,v'$$

$$v' = \left(\frac{m_A}{m_A + m_B}\right) v$$

Notice that $v' < v$, since $m_A/(m_A+m_B) < 1$.

Another 1D collision example (recoil of a gun). A gun of mass *M* fires a bullet of mass *m* with velocity v_b . What is the recoil velocity v_G of the gun?



$p_{tot} = 0$ (both at rest)

Before

After

$$p_{tot(before)} = 0 = p_{tot(after)}$$

$$0 = M\,v_G + m\,v_b$$

$$M\,v_G = -m\,v_b$$

$$v_G = -\left(\frac{m}{M}\right)v_b$$

v_b = 500 m/s, m = 10 gram = 0.01 kg, M = 3 kg $\Rightarrow$ $v_G = -\frac{0.010}{3}\cdot 500 = -1.7$ m/s

Quite a kick! This is how rockets work! Rocket fuel is thrown out the back of the rocket, causing the rocket to recoil forward. There is NO WAY to make a rocket go forward in space except by throwing mass out the back. Any other means of propulsion would violate Conservation of Momentum. (Sorry Star Trek fans, warp drive is impossible.)

Incidentally, why is the barrel of a rifle so long?
Answer: $v = a\cdot t$ $\Rightarrow$ long barrel, more time to accelerate, bigger v

Impulse

To prove that momentum is conserved in collisions, we need the concept of *impulse*, which relates force to changes in momentum.

Newton never wrote $\mathbf{F}_{net} = m\,\mathbf{a}$. He wrote an equivalent relation using momentum:

$$\boxed{\vec{F}_{net} = \frac{d\vec{p}}{dt}}$$

Net force is the rate of change of momentum.

Let's check that this is the same as $\mathbf{F}_{net} = m\,\mathbf{a}$.

$\vec{p} = m\,\vec{v}$, $\Delta\vec{p} = m\,\Delta\vec{v}$ (assuming m = constant) $\Rightarrow$ $\vec{F}_{net} = \frac{\Delta\vec{p}}{\Delta t} = \frac{m\,\Delta\vec{v}}{\Delta t} = m\,\vec{a}$

In the special case that a constant net force is applied during a time interval Δt , we have

$\vec{F}_{net} = \dfrac{\Delta \vec{p}}{\Delta t}$ or $\Delta \vec{p} = \vec{F}_{net} \Delta t$. If the force varies over time, then the correct expression is

$$\Delta \vec{p} = \int_i^f \vec{F}_{net}\, dt = \vec{F}_{net,AVG} \Delta t$$

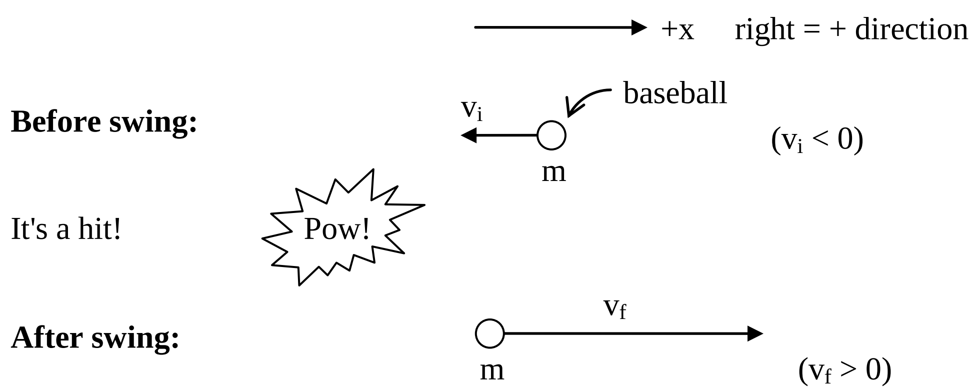
Definition: *impulse* **J** = net force × time $\vec{J} \equiv \vec{F}_{net} \cdot \Delta t$ (F_{net} = constant during time interval Δt)

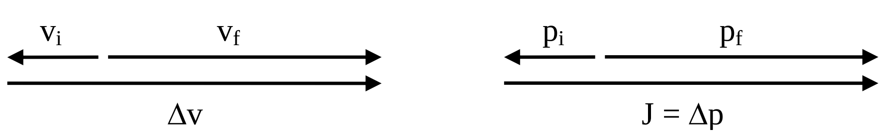
In general, $\vec{J} \equiv \int_i^f \vec{F}_{net}\, dt = \vec{F}_{net,avg} \Delta t$. So we have

$$\boxed{\Delta \vec{p} = \vec{J} = \vec{F}_{net,avg} \Delta t}$$

To change the momentum of an object, you must apply a net force for a time interval.

The term "impulse" is usually reserved for situations in which a BIG force acts for a short time to cause a rapid change in momentum. Like a bat hitting a baseball:





$$\vec{J} = \Delta \vec{p} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i) = m\Delta \vec{v}$$

Example:
$m_{baseball}$ = 0.30 kg , v_i = – 42 m/s , v_f = +80 m/s , duration of bat/ball collision = Δt = 0.010 s

What is the impulse? And what is the size of the average force exerted by the bat on the ball?

$J = m(v_f - v_i) = (0.30\text{ kg})(80\text{ m/s} - (-42\text{ m/s})) = 0.30(122) \cong +37\text{ kg·m/s}$ (Impulse is to the right.)

$$F = \frac{\Delta p}{\Delta t} = \frac{37\text{ kg}\cdot\text{m/s}}{0.010\text{ s}} = 3700\text{ N} \cong 800\text{ lbs}$$ Bat exerts a BIG force for a short time.

Proof that momentum is conserved

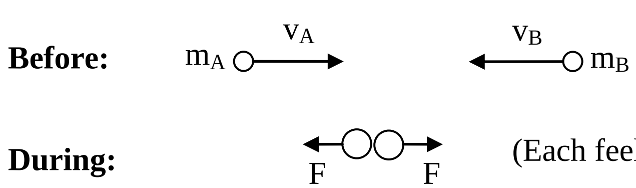
Now finally, we are ready for the proof that momentum is conserved in collisions. We are going to show that Newton's 3rd Law guarantees that

(total momentum before collision) = (total momentum after collision)

We will show that when two objects (A and B) collide, the total momentum $\vec{p}_{tot} \equiv \vec{p}_A + \vec{p}_B$ remains constant because $\Delta\vec{p}_A = -\Delta\vec{p}_B$; that is, the change in momentum of object A is exactly the opposite the change in momentum of object B. Since the change of one is the opposite of the change of the other, the total change is zero: $\Delta\vec{p}_{tot} = \Delta\vec{p}_A + \Delta\vec{p}_B = \Delta\vec{p}_A - \Delta\vec{p}_A = 0$.

Here's the proof: When two objects collide, each exerts a force on the other. NIII says that each feels the same-sized force F, but in opposite directions. Each object experiences the same-sized force for the same duration Δt. So each object receives the same-sized impulse $|J| = |F\cdot\Delta t| = |\Delta p|$ but with opposite directions. Done.

1D collision:



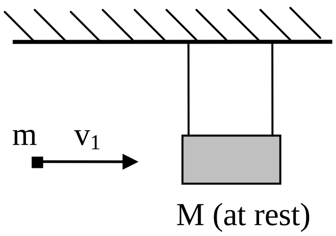
(Each feels same magnitude F, for same duration Δt.)

$\Delta p_A = -F\,\Delta t < 0$ $\Delta p_B = +F\,\Delta t > 0$

$$\Rightarrow \Delta p_A + \Delta p_B = 0 \Rightarrow \Delta(p_A + p_B) = 0 \Rightarrow p_A + p_B = \text{constant}$$

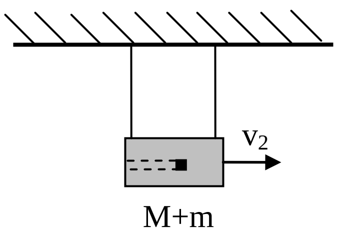
The total momentum is constant, if all forces acting are *internal* to system; that is, if the system is isolated from outside forces. If there are forces from outside the system, then the system's total momentum can change. But any momentum change of the system must be due to transfer of momentum between the system and its surroundings.

Example of Conservation of Energy and Momentum: The Ballistic Pendulum. The ballistic pendulum is a simple device which can accurately measure the speed of a bullet. It consists of a block of wood hanging from some strings. When a bullet is fired into the block, the kick from the bullet causes the block to swing upward. From the height of the swing, the speed of the bullet can be determined.



The Situation: bullet of mass m, with unknown initial velocity v_1 , is fired into a large wooden block of mass M, hanging at rest from strings.

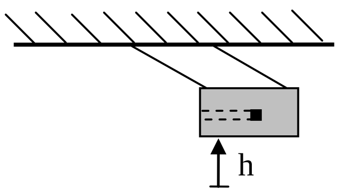
$p_{ot} = m\, v_1$



Immediately after collision, bullet is buried in block, but block has not yet had time to move. The impulse from bullet gives block+bullet a velocity v_2.

Momentum conservation $\Rightarrow$ $m \cdot v_1 = (M + m) \cdot v_2$ (1)

Momentum is conserved, but KE is not. Most of the bullet's initial KE has been converted to thermal energy: bullet and block get hot. Some KE is left over: $KE = \frac{1}{2}(m+M)\,v_2^{\,2}$



Block+bullet rise to max height h, which is measured.

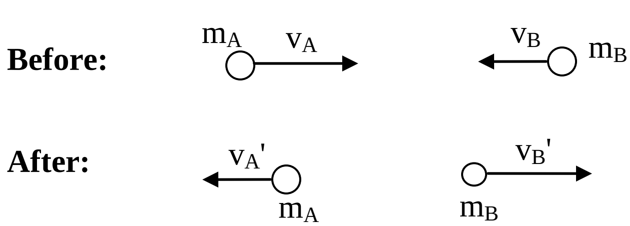
Conservation of energy $\Rightarrow$

$$\underbrace{KE_i}_{} + \underbrace{PE_i}_{0} = \underbrace{KE_f}_{0} + PE_f$$

$$\tfrac{1}{2}\,(M+m)\,v_2^{\,2} = (M+m)\,g\,h \qquad (2)$$

Now have 2 equations [(1) and (2)] in two unknowns (v_1 and v_2). So you can solve for the velocity of the bullet v_1 terms of the knowns (m, M, g, and h). I'll let you do the algebra.

Elastic Collisions



In a collision between two masses, momentum is ALWAYS conserved (when there are no outside forces). So, for an isolated system, we can always write:

$$\vec{p}_{tot} = m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v}'_A + m_B\vec{v}'_B$$

IF the collision is elastic, then KE is *also* conserved, so we can also write:

$$KE_{Before} = KE_{After} \quad , \quad \tfrac{1}{2}m_A v_A{}^2 + \tfrac{1}{2}m_B v_B{}^2 = \tfrac{1}{2}m_A v'_A{}^2 + \tfrac{1}{2}m_B v'_B{}^2$$

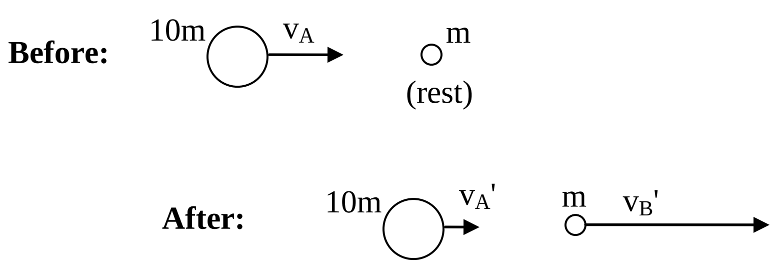
If the initial conditions (masses and initial velocities) are known, and we seek the final velocities, then we have two equations (Conserv of p, Conserv of KE) in two unknowns (v_A' and v_B'), and it is possible to solve. But the algebra gets very messy, because of the squared terms in the KE equation.

It turns out that when the collision is elastic, the ***relative*** velocity of the two objects (velocity of one relative to the other) is reversed, according to the equation:

$$\left(\vec{v}_B - \vec{v}_A\right) = -\left(\vec{v}'_B - \vec{v}'_A\right) \quad \text{(elastic collision)}$$

Because this equation has no squared terms, it is much easier to use than the KE conservation equation. This equation says that the relative velocity of approach before the collision is the negative of the relative velocity after the collision. The proof of this equation is in the Appendix.

Example of elastic collision in 1D: A mass m_A = 10m with initial velocity v_A collides head-on with a mass m_B = m that is at rest. What are the final velocities, v_A' and v_B', of the two masses?



Here v_B (initial velocity of object B) is zero, so Conservation of Momentum gives:

$$10\,m\,v_A = 10\,m\,v'_A + m\,v'_B \quad \text{(m's cancel)} \Rightarrow \quad 10\,v_A = 10\,v'_A + v'_B \quad (*)$$

Because the collision is elastic (meaning KE is conserved), we can write

$$\left(v_B - v_A\right) = -\left(v'_B - v'_A\right), \quad (v_B = 0) \Rightarrow \quad v_A = v'_B - v'_A \ , \quad v'_B = v_A + v'_A$$

Substitution into (*) gives

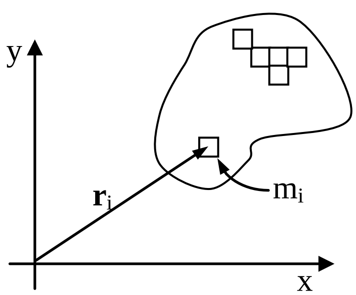
$$10\,v_A = 10\,v'_A + v_A + v'_A \ , \quad 9\,v_A = 11\,v'_A \ , \quad v'_A = \frac{9}{11}v_A$$

$$v'_B = v_A + v'_A = \frac{11}{11}v_A + \frac{9}{11}v_A = \frac{20}{11}v_A$$

Notice that the big mass is slowed by the collision (makes sense) and the little mass is shot forward with a velocity that is larger than the initial velocity of the big mass.

Center of mass

The formula $\vec{F}_{net} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$ applies to a <u>point</u> particle. What about an extended object, made of many particles? We can regard any object as a collection of N particles.



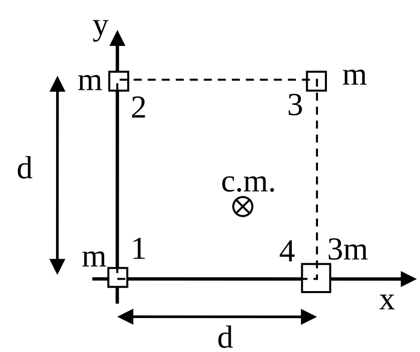
Total mass = $M = \sum_{i=1}^{N} m_i$

Definition: *center of mass* $\vec{R} \equiv \frac{\sum_i m_i \vec{r}_i}{M}$

(Wolfson uses notation r_{cm}, but I will use capital letters for center-of-mass)

This is easier remember if you think of the definition like this: $M\vec{R} \equiv \sum_i m_i \vec{r}_i$

Example: Where is c.m. of this 4 mass system? The masses, labeled 1, 2, 3, 4, form a square of edge length d. The four masses are m, m, m, and 3m.



$$X = \frac{1}{M}\left(m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4\right)$$
$$= \frac{1}{6m}\left(m\cdot 0 + m\cdot 0 + m\cdot d + 3m\cdot d\right) = \frac{4}{6}d \cong 0.67\,d$$

$$Y = \frac{1}{M}\left(m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4\right)$$
$$= \frac{1}{6m}\left(m\cdot 0 + m\cdot d + m\,d + 3m\cdot 0\right) = \frac{1}{3}d \cong 0.33\,d$$

Notice that the c.m. is closer to the heavy corner in the lower right. Roughly speaking, the c.m. is the "balance point".

We will now show that, for an extended object:

$$\vec{F}_{net} = M\frac{d^2\vec{R}}{dt^2} = m\vec{A}\text{ , where } \vec{A} = \text{acceleration of c.m.}$$

We define the velocity of the c.m. **V** (capital V for center-of-mass velocity) and the acceleration of the c.m. **A** like so:

$$\vec{V} \equiv \frac{d\vec{R}}{dt} = \frac{\sum_i m_i \frac{d\vec{r}_i}{dt}}{M} = \frac{\sum_i m_i \vec{v}_i}{M} \quad , \quad \vec{A} \equiv \frac{d\vec{V}}{dt} = \frac{\sum_i m_i \frac{d\vec{v}_i}{dt}}{M} = \frac{\sum_i m_i \vec{a}_i}{M}$$

The center-of-mass has x-, y-, and z-components:

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}\text{ , where } X = \frac{\sum_i m_i x_i}{M}, \quad Y = \frac{\sum_i m_i y_i}{M}\text{ , etc}$$

and likewise for the velocity and acceleration

$$\vec{V} = V_x\,\hat{i} + V_y\,\hat{j} + V_z\,\hat{k}\text{ , where } V_x = \frac{\sum_i m_i v_{xi}}{M}\text{ , etc.}$$

The total force or net force on an *extended* object is the vector sum of <u>all</u> the forces on <u>all</u> the particles. Some of the forces are *external* forces, from outside the object (for example, gravity), and some of the forces are *internal* forces, acting between particles in the object. The internal forces all cancel in pairs, because of NIII.

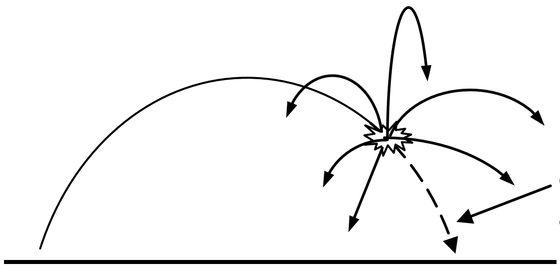
$$\vec{F}_{net} = \sum_j \vec{F}_j = \underbrace{\sum \vec{F}_{ext}}_{\text{external forces}} + \underbrace{\sum \vec{F}_{int}}_{\substack{=0 \\ \text{internal forces} \\ \text{cancel in pairs}}} = \sum \vec{F}_{ext}$$

We can also write the net force on an object as the vector sum of the net forces on each particle:

$$\vec{F}_{net} = \sum_i \vec{F}_{net,i} = \sum_i m_i\,\vec{a}_i \quad .$$

Now using our definition of acceleration of c.m., $M\vec{A} = \sum_i m_i\,\vec{a}_i$ and the fact that $\vec{F}_{net} = \sum \vec{F}_{ext}$, we have $\vec{F}_{net} = \sum \vec{F}_{ext} = M\vec{A}$.

The center-of-mass moves like a point particle even if the particles are not glued together. Example: a projectile bomb is launched, and explodes in flight.



Now have alternative way of showing that total momentum of many-particle system is conserved, if the system is isolated from external forces.

Recall $\vec{P}_{tot} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i$. Can show that $\vec{P}_{tot} = M\vec{V}_{c.m.}$:

$$\vec{P}_{tot} = \sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt}\underbrace{\left(\sum_i m_i \vec{r}_i\right)}_{M\cdot\vec{R}_{cm}} = M\frac{d\vec{R}}{dt} = M\vec{V}_{cm}$$

Now $\sum \vec{F}_{ext} = M\vec{A} = M\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(M\vec{V}\right) = \frac{d\vec{P}}{dt}$

So, if no external forces are acting, $\frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} = \text{constant} \Rightarrow \sum_i \vec{p}_i = \text{constant}$

Appendix. Proof of $(\vec{v}_B - \vec{v}_A) = -(\vec{v}'_B - \vec{v}'_A)$ for elastic collisions.

Working in 1D, so we can drop the "vector arrow" notation.
Conservation of momentum gives

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \qquad (1)$$

Conservation of KE gives

$$m_A v_A^2 + m_B v_B^2 = m_A v'^2_A + m_B v'^2_B \qquad (2)$$ [We've cancelled out all the (1/2) factors.]

We can rearrange these equations to put all the m_A terms on one side and all the m_B terms on the other:

(1) $\Rightarrow$ $$m_A(v_A - v'_A) = m_B(v'_B - v_B) \qquad (3)$$

(2) $\Rightarrow$ $$\begin{cases} m_A(v_A^2 - v'^2_A) = m_B(v'^2_B - v_B^2) \\ m_A(v_A + v'_A)(v_A - v'_A) = m_B(v'_B + v_B)(v'_B - v_B) \qquad (4) \end{cases}$$

[We have used the identity $(x^2 - y^2) = (x + y)(x - y)$.]

If we divide equation (4) by equation (3), we get:

$$(4) \div (3) \Rightarrow \frac{m_A(v_A + v'_A)(v_A - v'_A)}{m_A(v_A - v'_A)} = \frac{m_B(v'_B + v_B)(v'_B - v_B)}{m_B(v'_B - v_B)}$$

Notice that almost everything cancels out in this equation, leaving only

$(v_A + v'_A) = (v'_B + v_B)$, which is the same as $(v_B - v_A) = -(v'_B - v'_A)$. Done.

