Conservation of Angular Momentum = "Spin"
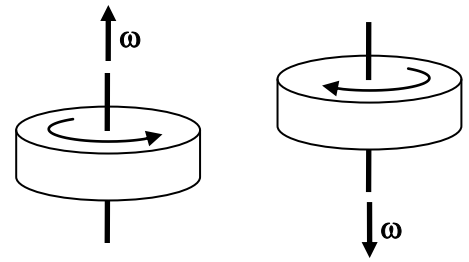
We can assign a direction to the angular velocity:

direction of ω = direction of axis + right hand rule

(with right hand, curl fingers in direction of rotation, thumb points

in direction of ω.) The direction of ω is abstract. There is nothing

actually moving in the direction of the ω vector. We want to choose the direction of ω so that

this gives us information about how the object is spinning. When a wheel is spinning on an axis,

there is no *one* direction of motion (different parts of the wheel are moving in different

directions). But there is always only one direction of the axis of rotation. So by giving both the

direction of the axis (direction of ω) and how fast the object is spinning around that axis
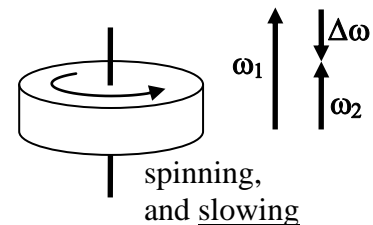
(magnitude of $\vec{\omega}$ $=$ $\left|\vec{\omega}\right|$ $=$ $\left|\dfrac{d\theta}{dt}\right|$) , we are describing its spinning motion exactly.

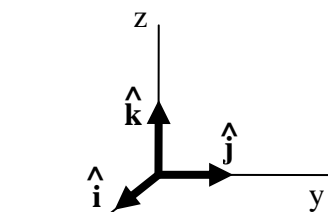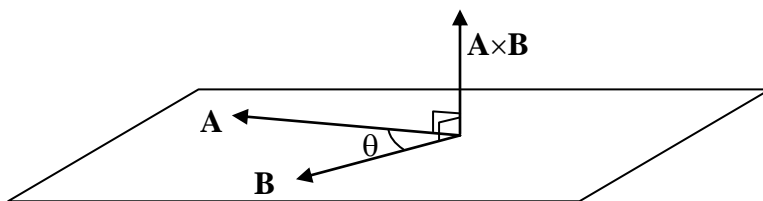Similarly, we can assign a direction to angular acceleration α :

$$\vec{\alpha} \; = \; \frac{d\vec{\omega}}{dt} \; \cong \; \frac{\Delta\vec{\omega}}{\Delta t} \; = \; \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$$

Direction of α is direction of $\Delta\omega$, not the direction of ω (like

direction of **a** is direction of Δ**v**, not the direction of **v**)
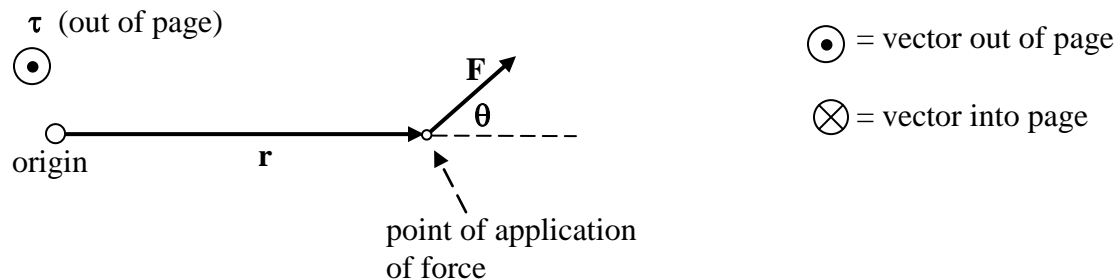
We can also assign a direction to torque with the "cross product".

The *cross-product* of two vectors **A** and **B** is a third vector $\vec{A}\times\vec{B} = \vec{C}$ defined like this: The magnitude is $|\vec{A}\times\vec{B}| = A\,B\sin\theta$. The direction of $\vec{C} = \vec{A}\times\vec{B}$ is the direction perpendicular to the plane defined by the vectors **A** and **B** , plus right-hand-rule. (Curl fingers from first vector **A** to second vector **B**, thumb points in direction of $\vec{A}\times\vec{B}$

$$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{i}\times\hat{i} = 0$$
$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{j}\times\hat{j} = 0 \quad \text{etc.}$$

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Vector torque defined as $\boxed{\hat{\tau} = \vec{r} \times \vec{F}}$

τ (out of page)

⊙

origin   **r**

F

θ

point of application
of force

⊙ = vector out of page

⊗ = vector into page

The vector torque τ is defined with respect to an origin (which is usually, but not always, the axis of rotation). So, if you change the origin, you change the torque (since changing the origin changes the position vector **r**).

With these definitions of vector angular acceleration and vector torque, the fixed-axis equation $\tau = I\alpha$ becomes

$$\vec{\tau} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} \qquad (\text{like } \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt})$$

Now, a new concept: angular momentum **L** = "spin". Angular momentum, a vector, is the rotational analogue of linear momentum. So, based on our analogy between translation and rotation, we expect $\vec{L} = I\vec{\omega}$ (like $\vec{p} = m\vec{v}$). Note that this equation implies that the direction of **L** is the direction of ω.

L

Definition of angular momentum of a particle with momentum **p** = m**v** at position **r**

relative to an origin is $\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$

Like torque τ, the angular momentum **L** is defined w.r.t. an origin, often the axis of rotation.

L (out of page)

⊙

origin   **r**   **p** θ

particle

We now show that the <u>total</u> angular momentum of a object spinning about a fixed axis is $\vec{L} = I\vec{\omega}$. Consider a object spinning about an axis pointing along the +z direction. We place the origin at the axis.
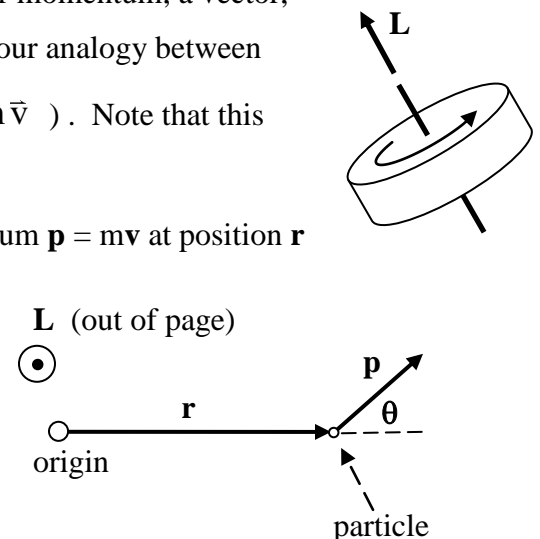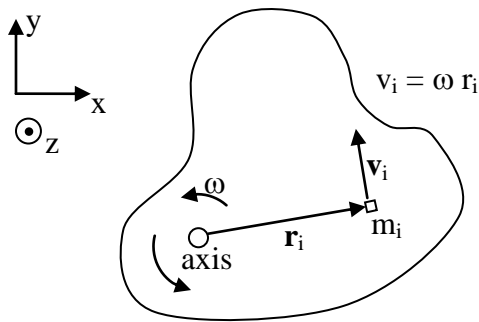
$$\vec{L}_{tot} \ = \ \sum_i \vec{L}_i \ = \ \sum_i \vec{r}_i \times \vec{p}_i \ = \ \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

$$= \ \hat{z} \sum_i r_i \, m_i \, v_i \ = \ \hat{z} \sum_i m_i \, r_i^2 \, \omega \ = \ \hat{z} \, I \omega$$

$v_i = \omega \, r_i$

$$\vec{L}_{tot} \ = \ I \vec{\omega}$$

If something has a big moment of inertia I and is spinning fast (big ω), then it has a big "spin", big angular momentum. Angular momentum is a very useful concept, because angular momentum is conserved.

Important fact: the angular momentum of a object spinning about an axis that passes through the center of mass is given by $\mathbf{L} = I_{cm} \, \boldsymbol{\omega}$, <u>independent</u> of the location of the origin; that is, even if the origin is chosen to be outside the spinning object, the angular momentum has the same value as if the origin was chosen to be at the axis. (Proof not given here).

Conservation of Angular Momentum: If a system is isolated from external torques, then its

total angular momentum L is constant.

$$\boxed{\tau_{ext} = 0 \ \Rightarrow \ L_{tot} = I \, \omega \ = \ constant}$$ (like $F_{ext} = 0 \ \Rightarrow \ p_{tot} = constant$)

Here is a proof of conservation of angular momentum:

First, we argue that $\vec{\tau}_{net} \ = \ \dfrac{d\vec{L}}{dt}$ (this is like $\vec{F}_{net} \ = \ \dfrac{d\vec{p}}{dt}$) :

$$\frac{d\vec{L}}{dt} \ = \ \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) \ = \ \sum_i \left(\frac{d\vec{r}}{dt}_i \times \vec{p}_i \ + \ \vec{r}_i \times \frac{d\vec{p}_i}{dt}\right).$$

Now, the first term in the last expression is zero:

$$\sum_i \left(\frac{d\vec{r}}{dt}_i \times \vec{p}_i\right) \ = \ \sum_i \left(\vec{v}_i \times m_i \vec{v}_i\right) = \sum_i m_i\left(\vec{v}_i \times \vec{v}_i\right) \ = \ 0 \ , \text{ since any vector crossed into itself is}$$

zero. So, we have $\dfrac{d\vec{L}}{dt} \ = \ \sum_i \left(\vec{r}_i \times \dfrac{d\vec{p}_i}{dt}\right) \ = \ \sum_i \left(\vec{r}_i \times \vec{F}_i\right)$ (since $\vec{F}_{net} \ = \ \dfrac{d\vec{p}}{dt}$). Finally,

$$\sum_i \left(\vec{r}_i \times \vec{F}_i\right) \ = \ \sum_i \vec{\tau}_i \ = \ \vec{\tau}_{net} \ , \text{ so we have } \ \frac{d\vec{L}}{dt} \ = \ \vec{\tau}_{net} \, .$$

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So now we have, $\vec{\tau}_{net} = \dfrac{\Delta\vec{L}}{\Delta t} \Rightarrow$ if $\tau_{net} = 0$, then $\dfrac{\Delta\vec{L}}{\Delta t} = 0 \Rightarrow \vec{L} = \text{constant}$. Done.

It turns out that only 4 things are conserved:
- Energy
- Linear momentum p
- Angular momentum L
- Charge q

Conservation of Angular Momentum is very useful for analyzing the motion of spinning objects isolated from external torques — like a skater or a spinning star.

If $\tau_{ext} = 0$, then $L = I\,\omega = \text{constant}$. If I decreases, ω must increase, to keep L = constant.

Example: spinning skater.

$I_i \ \omega_i \qquad = \qquad I_f \ \omega_f$

(I big, ω small)   (I small, ω big)
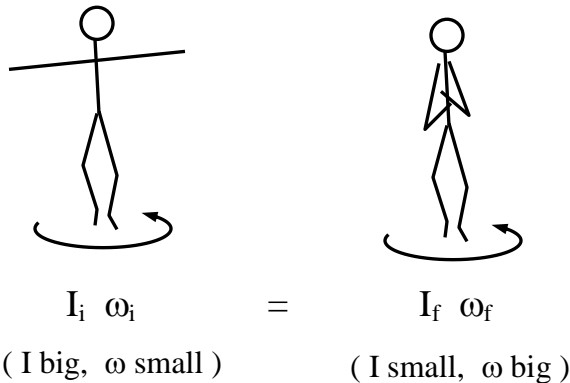
Example: rotation of collapsing star. A star shines by converting hydrogen (H) into helium (He) in a nuclear reaction. When the H is used up, the nuclear fire stops, and gravity causes the star to collapse inward.

gravity
nuclear
slow
fast!
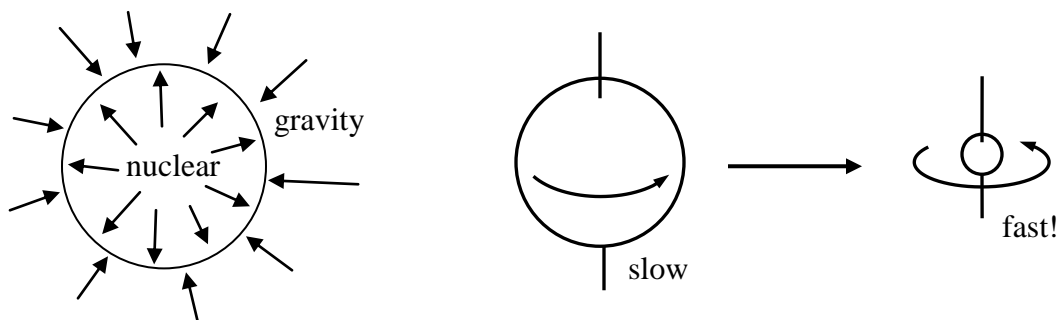
As the star collapses (pulls its arms in), the star rotates faster and faster. Star's radius can get **much** smaller: $R_i \approx 1$ million miles $\rightarrow R_f \approx 30$ miles

$$I_i\,\omega_i \;=\; I_f\,\omega_f \qquad \text{(Sphere } I = \tfrac{2}{5}M\,R^2\text{)}$$

$$\left(\tfrac{2}{5}M\,R_i^{\,2}\right)\omega_i \;=\; \left(\tfrac{2}{5}M\,R_f^{\,2}\right)\omega_f$$

$$R_i^{\,2}\,\omega_i \;=\; R_f^{\,2}\,\omega_f$$

$$\frac{R_i^{\,2}}{R_f^{\,2}} \;=\; \frac{\omega_f}{\omega_i} \;=\; \frac{T_i}{T_f} \qquad (\text{ using } \omega = 2\pi f = \frac{2\pi}{T})$$

If $R_i \gg R_f$, then $T_i \ggg T_f$.

The sun rotates once every 27 days. "Neutron stars" with diameter of about 30 miles typically rotate 100 time per second.

Let's review the correspondence between translational and rotational motion

Translation	$\leftrightarrow$	**Rotation**
x	$\leftrightarrow$	θ
$v = \dfrac{\Delta x}{\Delta t}$	$\leftrightarrow$	$\omega = \dfrac{\Delta \theta}{\Delta t}$
$a = \dfrac{\Delta v}{\Delta t}$	$\leftrightarrow$	$\alpha = \dfrac{\Delta \omega}{\Delta t}$
F	$\leftrightarrow$	$\tau = r\,F_{\perp}$
M	$\leftrightarrow$	$I = \sum m\,r^2$
$F_{net} = M\,a$	$\leftrightarrow$	$\tau_{net} = I\,\alpha$
$KE_{trans} = (1/2)M\,v^2$	$\leftrightarrow$	$KE_{rot} = (1/2)\,I\,\omega^2$
$p = m\,v$	$\leftrightarrow$	$L = I\,\omega$
$F_{net} = \Delta p / \Delta t$	$\leftrightarrow$	$\tau_{net} = \Delta L / \Delta t$
If $F_{ext} = 0$, $p_{tot} = $ constant	$\leftrightarrow$	If $\tau_{ext} = 0$, $L_{tot} = $ constant