Simple Harmonic Motion

A pendulum, a mass on a spring, and many other kinds of oscillators exhibit a special kind of oscillatory motion called Simple Harmonic Motion (SHM).

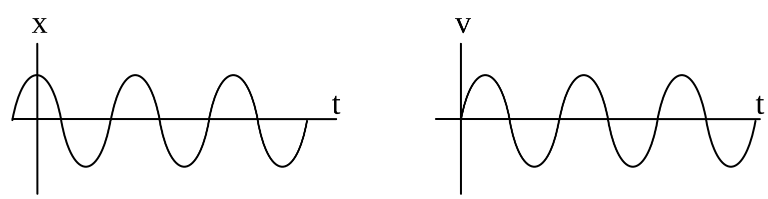
SHM occurs whenever :

i. there is a restoring force proportional to the displacement from equilibrium: $F \propto -x$

ii. the potential energy is proportional to the square of the displacement: $PE \propto x^2$

iii. the period T or frequency f = 1 / T is <u>independent</u> of the <u>amplitud</u>e of the motion.

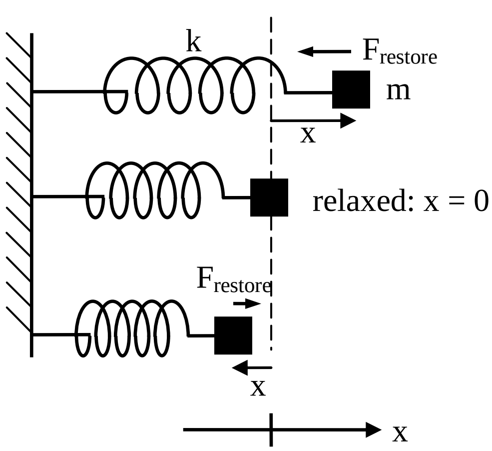
iv. the position *x*, the velocity *v*, and the acceleration *a* are all sinusoidal in time.



(*Sinusoidal* means sine, cosine, or anything in between.)
As we will see, any one of these four properties guarantees the other three. If one of these 4 things is true, then the oscillator is a simple harmonic oscillator and all 4 things must be true.

Not every kind of oscillation is SHM. For instance, a perfectly elastic ball bouncing up and down on a floor: the ball's position (height) is oscillating up and down, but none of the 4 conditions above is satisfied, so this is not an example of SHM.

A mass on a spring is the simplest kind of Simple Harmonic Oscillator.



Hooke's Law: $\mathbf{F}_{spring} = -k\,\mathbf{x}$

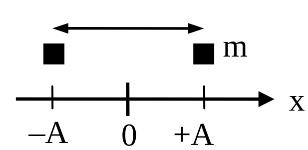
(–) sign because direction of $\mathbf{F}_{spring}$ is opposite to the direction of displacement vector **x** (**bold** font indicates vector)

k = spring constant = stiffness, units [k] = N / m

Big k = stiff spring

Definition: *amplitude* A = $|x_{max}|$ = $|x_{min}|$.

Mass oscillates between extreme positions x = +A and x = –A



Notice that Hooke's Law (F = – kx) is condition i : restoring force proportional to the displacement from equilibrium. We showed previously (Work and Energy Chapter) that for a spring obeying Hooke's Law, the potential energy is $U = (1/2)kx^2$, which is condition ii. Also, in the chapter on Conservation of Energy, we showed that F = –dU/dx, from which it follows that condition ii implies condition i. Thus, Hooke's Law and quadratic PE ($U \propto x^2$) are equivalent.

We now show that Hooke's Law guarantees conditions iii (period independent of amplitude) and iv (sinusoidal motion).

We begin by deriving the *differential equation* for SHM. A differential equation is simply an equation containing a derivative. Since the motion is 1D, we can drop the vector arrows and use sign to indicate direction.

$$F_{net} = ma \quad \text{and} \quad F_{net} = -kx \quad \Rightarrow \quad ma = -kx$$

$$a = dv/dt = d^2x/dt^2 \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

The constants k and m and both positive, so the k/m is always positive, always. For notational convenience, we write $k/m = \omega^2$. (The square on the ω reminds us that ω^2 is always positive.) The differential equation becomes

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x} \qquad \text{(equation of SHM)}$$

This is the *differential equation* for SHM. We seek a solution x = x(t) to this equation, a function x = x(t) whose second time derivative is the function x(t) multiplied by a negative constant ($-\omega^2 = -k/m$). The way you solve differential equations is the same way you solve integrals: you *guess* the solution and then check that the solution works.

Based on observation, we guess a sinusoidal solution: $\boxed{x(t) = A\cos(\omega t + \varphi)}$,

where A, φ are <u>any</u> constants and (as we'll show) $\boxed{\omega = \sqrt{\frac{k}{m}}}$.

A = amplitude: x oscillates between +A and –A

φ = phase constant (more on this later)

Danger: ωt and φ have units of radians (not degrees). So set your calculators to radians when using this formula.

Just as with circular motion, the angular frequency ω for SHM is related to the period by

$$\boxed{\omega = 2\pi f = \frac{2\pi}{T}}, \quad T = \text{period.}$$

(What does SHM have to do with circular motion? We'll see later.)

Let's check that $x(t) = A\cos(\omega t + \varphi)$ is a solution of the SHM equation.

Taking the first derivative dx/dt , we get $v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$.

Here, we've used the Chain Rule: $\frac{d}{dt}\cos(\omega t + \varphi) = \frac{d\cos(\theta)}{d\theta}\frac{d\theta}{dt}$, $(\theta = \omega t + \varphi)$

$$= -\sin\theta \cdot \omega = -\omega\sin(\omega t + \varphi)$$

Taking a second derivative, we get

$$a(t) = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}\left(-A\omega\sin(\omega t + \varphi)\right) = -A\omega^2\cos(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -\omega^2\left[A\cos(\omega t + \varphi)\right]$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

This is the SHM equation, with $\omega^2 = \frac{k}{m}$, $\omega = \sqrt{\frac{k}{m}}$

We have shown that our assumed solution is indeed a solution of the SHM equation. (I leave to the mathematicians to show that this solution is unique. Physicists seldom worry about that kind of thing, since we know that nature usually provides only one solution for physical systems, such as masses on springs.)

We have also shown condition iv: x, v, and a are all sinusoidal functions of time:

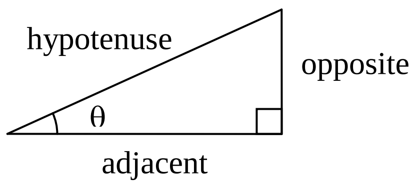
$$x(t) = A\cos(\omega t + \varphi)$$
$$v(t) = -A\omega\sin(\omega t + \varphi)$$
$$a(t) = -A\omega^2\cos(\omega t + \varphi)$$

The period T is given by $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$. We see that T does not depend on the amplitude A (condition iii).

Let's first try to make sense of $\omega = \sqrt{k/m}$: big ω means small T which means rapid oscillations. According to the formula, we get a big ω when k is big and m is small. This makes sense: a big k (stiff spring) and a small mass m will indeed produce very rapid oscillations and a big ω.

A closer look at x(t) = A cos(ωt+φ)

Let's review the sine and cosine functions and their relation to the *unit* circle. We often define the sine and cosine functions this way:

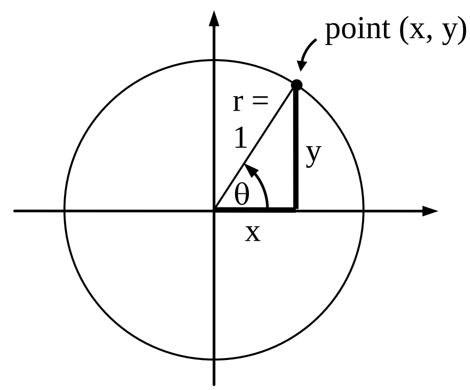


$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

This way of defining sine and cosine is correct but incomplete. It is hard to see from this definition how to get the sine or cosine of an angle greater than 90^o.

A more complete way of defining sine and cosine, a way that gives the value of the sine and cosine for *any* angle, is this: Draw a *unit* circle (a circle of radius r = 1) centered on the origin of the x-y axes as shown:



Define sine and cosine as

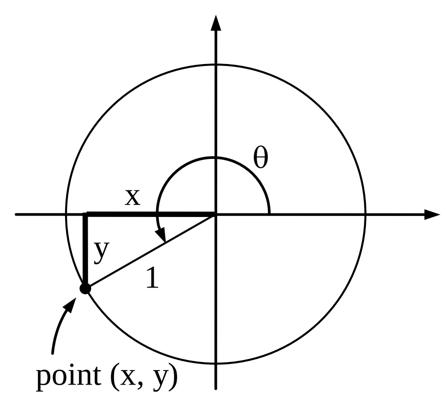
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

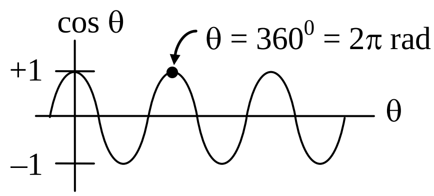
This way of defining sin and cos allows us to compute the sin or cos of *any* angle at all.

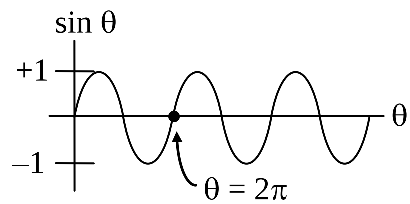
For instance, suppose the angle is $\theta = 210^o$. Then the diagram looks like this:

The point on the unit circle is in the third quadrant, where both x and y are negative. So both cosθ = x and sinθ = y are negative

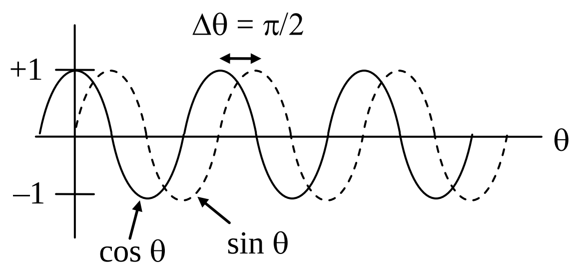


For any angle θ, even angles bigger than 360^o (more than once around the circle), we can always compute sin and cos. When we plot sin and cos vs angle θ, we get functions that oscillate between +1 and –1 like so:





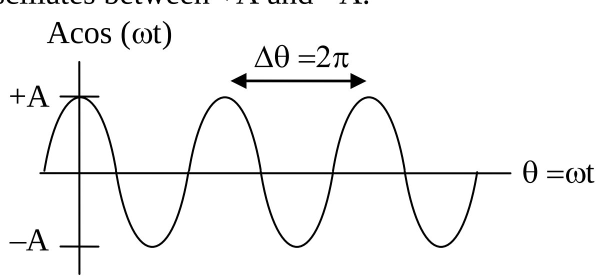
We will almost always measure angle θ in radians. Once around the circle is 2π radians, so sine and cosine functions are periodic and repeat every time θ increases by 2π rad. The sine and cosine functions have exactly the same shape, except that sin is shifted to the right compared to cos by $\Delta\theta = \pi/2$. Both these functions are called *sinusoidal* functions.



The function cos(θ + φ) can be made to be anything in between cos(θ) and sin(θ) by adjusting the size of the *phase* φ between 0 and −2π.

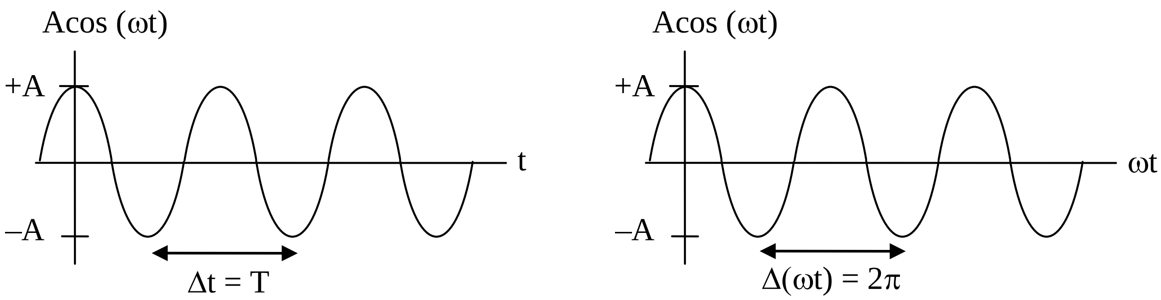
$$\cos\theta,\ (\varphi = 0) \ \rightarrow \ \sin\theta = \cos\left(\theta - \frac{\pi}{2}\right), \quad (\varphi = -\pi/2)$$

The function cos(ωt + φ) oscillates between +1 and −1, so the function Acos(ωt + φ) oscillates between +A and −A.



Why $\omega = \dfrac{2\pi}{T}$? The function f(θ) = cosθ is periodic with period Δθ = 2π. Since θ =ωt+φ, and φ is some constant, we have Δθ =ω Δt. One complete cycle of the cosine function corresponds to Δθ = 2π and Δt = T, (T is the period). So we have 2π = ω T or $\omega = \dfrac{2\pi}{T}$. Here is another way to see it: $\cos(\omega t) = \cos\left(2\pi\dfrac{t}{T}\right)$ is periodic with period Δt

= T. To see this, notice that when t increases by T, the fraction t/T increases by 1 and the fraction 2πt/T increases by 2π.



Now back to simple harmonic motion. Instead of a circle of radius 1, we have a circle of radius A (where A is the amplitude of the Simple Harmonic Motion).

SHM and Conservation of Energy:

Recall $PE_{elastic} = (1/2)\,k\,x^2$ = work done to compress or stretch a spring by distance x.

If there is no friction, then the total energy E_{tot} = KE + PE = constant during oscillation. The value of E_{tot} depends on initial conditions – where the mass is and how fast it is moving initially. But once the mass is set in motion, E_{tot} stays constant (assuming no dissipation.)

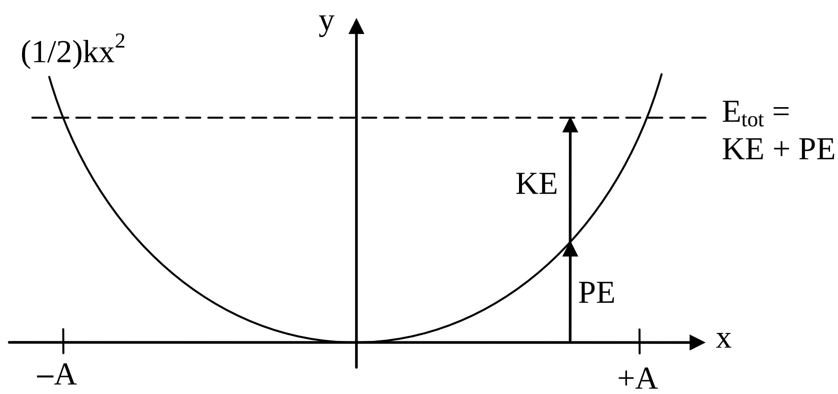
At any position x, speed v is such that $\boxed{\tfrac{1}{2}m\,v^2 + \tfrac{1}{2}k\,x^2 = E_{tot}}$.

When |x| = A, then v = 0, and all the energy is PE: $\underset{0}{KE} + \underset{(1/2)kA^2}{PE} = E_{tot}$

So total energy $E_{tot} = \tfrac{1}{2}k\,A^2$

When x = 0, v = v_{max}, and all the energy is KE: $\underset{(1/2)m{v_{max}}^2}{KE} + \underset{0}{PE} = E_{tot}$

So, total energy $E_{tot} = \tfrac{1}{2}m\,{v_{max}}^2$.



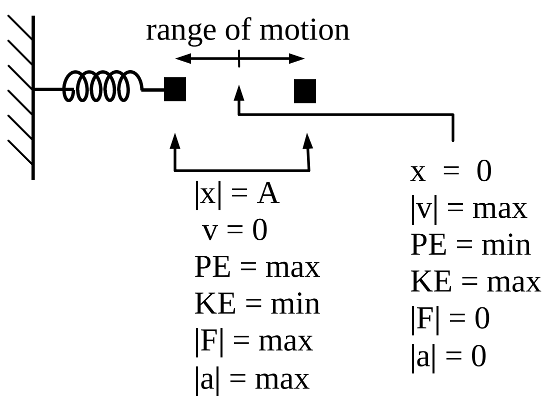
So, we can relate v_{max} to amplitude A : $PE_{max} = KE_{max} = E_{tot} \Rightarrow \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}{}^2 \Rightarrow$

$$v_{max} = \sqrt{\frac{k}{m}}\,A$$

Example Problem: A mass m on a spring with spring constant k is oscillating with amplitude A. Derive a general formula for the speed v of the mass when its position is x.

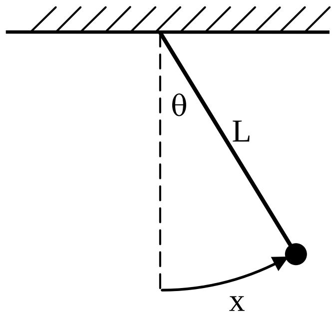
Answer: $v(x) = A\sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{x}{A}\right)^2}$

Be sure you understand these things:

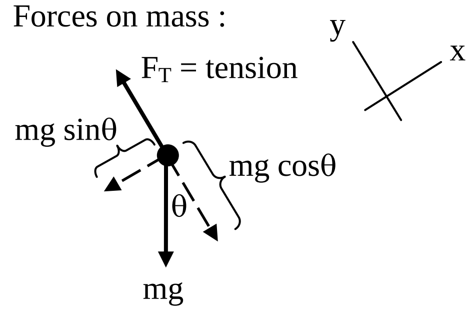


Pendulum Motion

A simple pendulum consists of a small mass m suspended at the end of a massless string of length L. A pendulum executes SHM, if the amplitude is not too large.



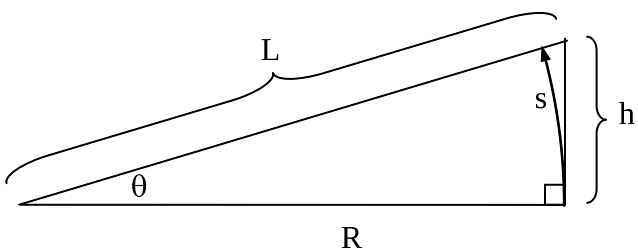
$\theta = x / L$ (rads)



The restoring force is the component of the force along the direction of motion:

$$\text{restoring force} = -mg\sin\theta \cong -mg\,\theta = -mg\frac{x}{L}$$

Claim: $\sin\theta \cong \theta\,(\text{rads})$ when θ is small. $\sin\theta = \dfrac{h}{L}$



$$\theta = \frac{s}{R}$$

If θ small, then $h \approx s$, and $L \approx R$, so $\sin\theta \approx \theta$.

Try it on your calculator: $\theta = 5^o = 0.087266..$ rad, $\sin\theta = 0.087156..$

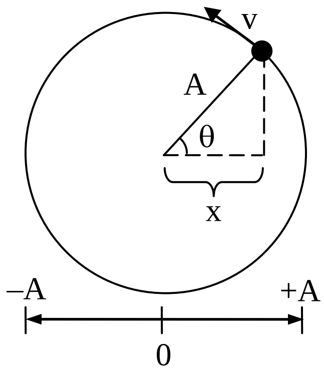
$F_{restore} = -\left(\dfrac{mg}{L}\right)x$ is exactly like Hooke's Law $F_{restore} = -k\,x$, except we have replaced the constant k with another constant (mg / L). The math is exactly the same as with a mass on a spring; all results are the same, except we replace k with (mg/L).

$$T_{spring} = 2\pi\sqrt{\frac{m}{k}} \;\Rightarrow\; T_{pend} = 2\pi\sqrt{\frac{m}{(mg/L)}} = 2\pi\sqrt{\frac{L}{g}}$$

Notice that the period is independent of the amplitude; the period depends only on length L and acceleration of gravity. (But this is true only if θ is not too large.)

Appendix: **SHM and circular motion**

There is an exact analogy between SHM and *circular motion*. Consider a particle moving with constant speed v around the rim of a circle of radius A.
The x-component of the position of the particle has *exactly* the same mathematical form as the motion of a mass on a spring executing SHM with amplitude A.



angular velocity $\omega = \dfrac{d\theta}{dt} = \text{const} \quad \Rightarrow$

$\theta = \omega t$ so

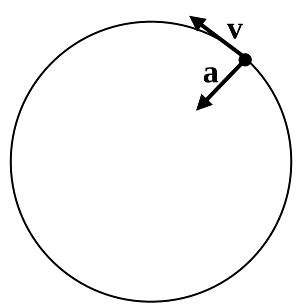
$x = A\cos\theta = A\cos\omega t$

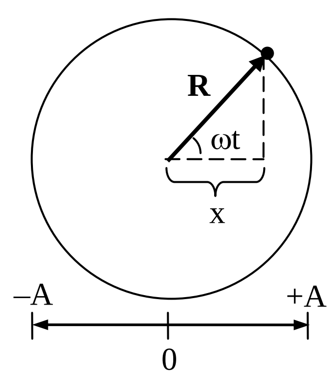
This same formula also describes the *sinusoidal* motion of a mass on a spring.

That the same formula applies for two different situations (mass on a spring & circular motion) is no accident. The two situations have the same solution because they both obey the same equation. As Feynman said, "The same equations have the same solutions". The equation of SHM is $\dfrac{d^2x}{dt^2} = -\omega^2 x$. We now show that a particle in circular motion obeys this same SHM equation.

Recall that for circular motion with angular speed ω, the acceleration of a the particle is toward the center and has magnitude $|\vec{a}| = \dfrac{v^2}{R}$. Since v = ω R , we can rewrite this as

$$|\vec{a}| = \frac{(\omega R)^2}{R} = \omega^2 R$$





Let's set the origin at the center of the circle so the position vector **R** is along the radius. Notice that the acceleration vector **a** is always in the direction opposite the position vector **R** . Since $|\vec{a}| = \omega^2 |\vec{R}|$, the vectors **a** and **R** are related by $\vec{a} = -\omega^2 \vec{R}$. The x-component of this vector equation is: $a_x = -\omega^2 R_x$. If we write $R_x = x$, then we have $\dfrac{d^2x}{dt^2} = -\omega^2 x$, which is the SHM equation. Done.