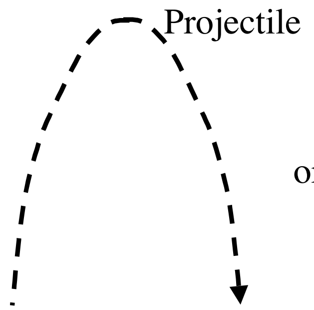
CH. 3: Vectors

In the real world, objects don't just move back and forth in 1-D!

In principle, the world is really 3-dimensional (3-D), but in practice, *lots* of realistic motion is 2-D (like the examples shown here)



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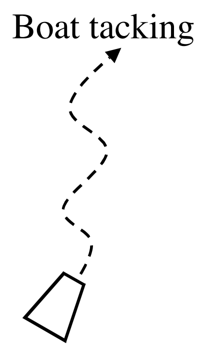
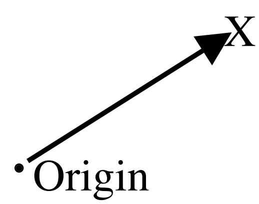


To describe the *position* of an object in 2- (or 3-) D, with respect to an origin requires that you tell not just how FAR from the origin it is, but WHICH WAY.



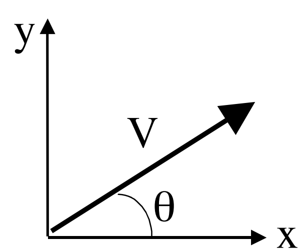
You need a **vector** to describe this. I think of it as a kind of arrow, and indicate it either just with bold letters, like **V**, or with a little arrow on top: $\vec{V}$.

We will often label a vector in a diagram by its magnitude, and its angle with respect to some axis. Mathematically, we write

V = $|\vec{\mathbf{V}}|$ = the magnitude of the vector **V.**

(Note, the magnitude isn't written bold)

The magnitude of any vector is a positive number.

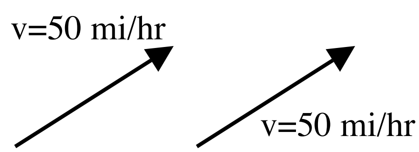


Examples of vectors in nature: velocity (has a magnitude AND direction!), force, acceleration,... I find it easy to visualize that *displacement* (change in position) is a vector, but I find it a little weird to think of position itself as a vector. But it is. Think of it as the vector which points from the origin to where the object is. We usually label position as **r**. (Unlike the displacement vector , the position vector DOES depend on your choice of axes - your coordinate system. Maybe that's why I find it a little unusual!?)

There are also lots of quantities in nature that are *not* vectors. E.g. mass, speed, and time have no direction associated with them. They are **scalars**.

The location (on the page) of a vector is *irrelevant*! (Only magnitude and direction count.)

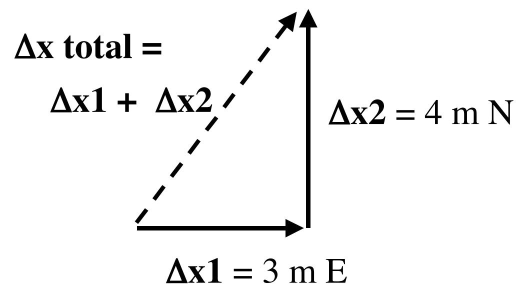
That is, the following two velocity vectors are IDENTICAL! They both represent the *same* velocity.



(Perhaps one is a car in Honolulu, the other in Boulder, but both are going 50 mi/hr heading northeast. That's the exact SAME velocity!)

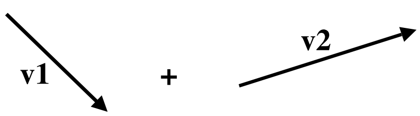
<u>Vectors can be combined!</u>

Suppose I move 3 meters East, first. (That's a displacement, a vector). Then later I move an additional 4 m North. We could ask "what is the total, or net displacement?" Answer: It's the SUM of the two displacements. The *vector sum*.



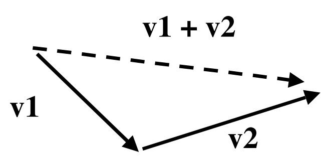
In this case, it's graphically clear what's going on: (Note that the sum is not 3+4=7 m long , it's 5 m, by the Pythagorean theorem.

Graphical addition of vectors always works just like this example, no matter what physical vector you're talking about.

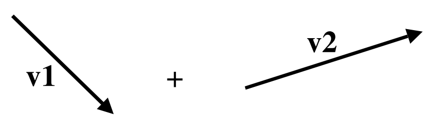


To add vectors, line them up "tip to tail", and then the sum goes from the start of the first to the end of the last!

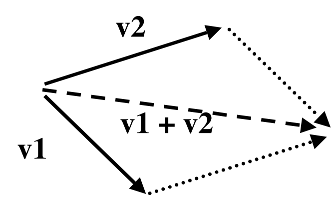
Moved together, tip to tail... :

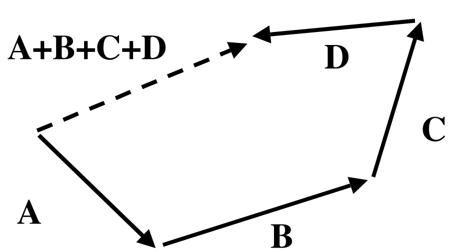


There's another method that's geometrically equivalent, called the "parallelogram method". Look at the picture and convince yourself it's the same thing!



The sum is the diagonal of this parallelogram... (tails together)



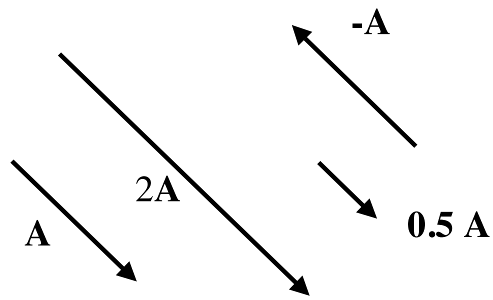


You can always add more than two vectors. Just put them all "tip to tail" in a chain. (Think of making a series of displacements!)

You can also MULTIPLY a vector times a scalar.

E.g. 2**V** doubles the length of **V**, but doesn't change its direction.

In general, c**V** is "c" times longer than **V** but in the same direction. The only exception is if c is negative, which *flips* the direction!

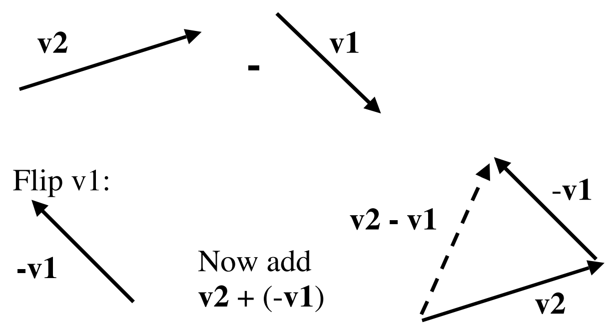


By the tip to tail rule, **A** + (-**A**) = **0**, (convince yourself!) which is what you would hope! (**0** is the "zero vector", which has no length)

You can subtract vectors The most straightforward way is this:

A-**B** is the same thing as **A**+(-**B**). So to subtract a vector, you add its negative.

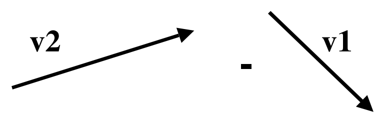
(Flip and add) An example:



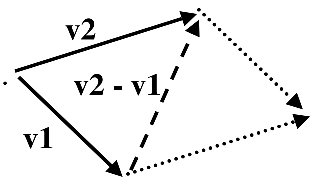
There's another method , the "parallelogram method of subtraction" .

(It's different from adding, we want the OTHER diagonal this time!)

There's only one hard thing about this method. *Which way* does the arrow point on **v2**-**v1**? The rule is just as shown in this picture, but WHY?



Form the parallogram with v1 and v2 as sides. (i.e. tails together)



Here's how I think of it:

Can you see that mathematically, you *must* have **v1 + (v2-v1) = v2.**

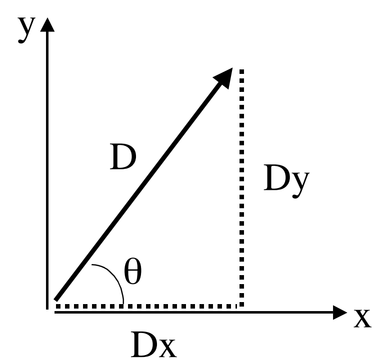
(See, I just "cancelled out" the **v**1 terms!) Now look back at the picture above. Can you see that **v1** and **(v2-v1)** are perfectly lined up "tip to tail" in the picture? So the picture looks like I'm ADDING **v1** and **(v2-v1)**, and the sum is **v2**, just as I said it should be. (If I had put the arrow on **(v2-v1)** backwards, this little addition wouldn't make any sense!!! If the arrow on the difference was drawn the wrong way, the picture would then look like **v2**+**(v2-v1)** = **v1**, which is clearly nonsense. The v2's don't cancel out!!)

So with addition and subtraction and scalar multiplication, you can manipulate vectors just like you do numbers, algebraically. For example, if you define $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, then you can add **r1** to both sides to get $\mathbf{r}_1 + \Delta\mathbf{r} = \mathbf{r}_2$,

initial position + change = final position. (True for vectors, just like it was in 1-D)

Components of vectors

These graphical methods are conceptually important, but in practice, you don't want to have to draw careful pictures to be able to add vectors. Components save the day. Any vector **D** has components in a coordinate system.



Here, D_x is the "x component of vector **D**".

If **D** is a displacement, think of it as "how much of the motion was towards the right".

To find components, you must define a coordinate system. Here, I picked a regular rectangular or "Cartesian" x-y coordinate system.

Trigonometry relates the components to the magnitude and angle:

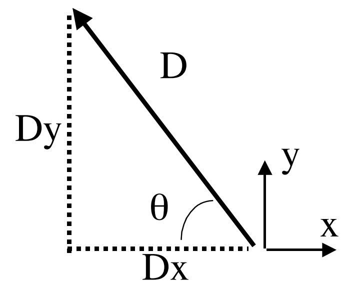
$$D_x = D\cos\theta$$
$$D_y = D\sin\theta$$

and similarly, you can go the other way. If you know the components, you can figure out the magnitude and angle by

$$D = \sqrt{D_x^2 + D_y^2}$$
$$\theta = \tan^{-1}(D_y / D_x)$$

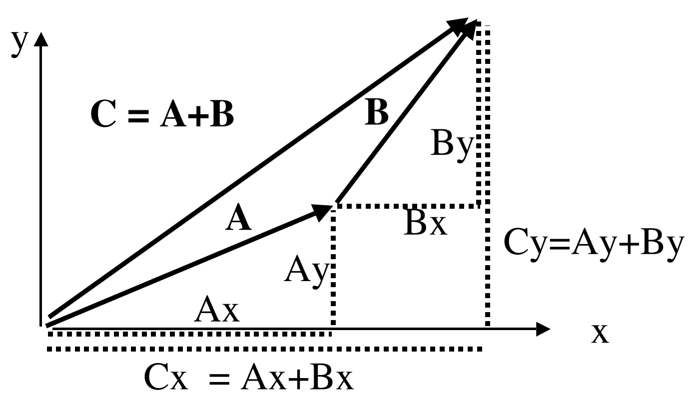
(Can you convince yourself that these formulas are correct, just by looking at the picture above?)

It's perfectly possible for a component to be NEGATIVE. For example, this vector has a negative x component, Dx < 0 (but it has a positive y component).



Suppose theta=60 degrees in this figure, and D=4.
If I just blindly plug into the formula I had above, Dx=Dcos(theta) = 4 cos(60)= +2, I don't see that minus sign. Why not?
Because "theta" in the formula earlier is the angle with the +x axis. In this figure, even though I *called* it theta, it's a different angle! (angle with the -x axis).
Don't use that formula without thinking, look at the picture!
Here, clearly Dx is "leftwards", it's got to be -2. (If you *really* want to use the formula, notice that the real theta is 180-60 = 120 degrees, and 4 cos(120) = -2...)

Why are components useful? Because they make adding and subtracting so easy!
Suppose **C** = **A**+**B**. Then, $C_x = A_x + B_x$ (similarly for y components).
I can add NUMBERS now, on a calculator, rather than trying to add "pictures".
Components are just plain old numbers!
I can prove $C_x = A_x + B_x$ with a simple picture:



(Similarly, if **D**=**A**-**B**, then $D_x = A_x - B_x$.)

<u>Unit vectors</u>

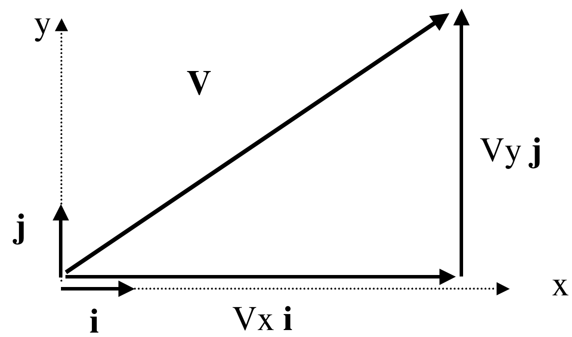
We define a little "unit vector" called $\hat{\mathbf{i}}$ (that's "i-hat"), which has LENGTH 1, and points in the +x direction. Similarly, $\hat{\mathbf{j}}$ ("j-hat") has $|\hat{\mathbf{j}}| = 1$ and points in the +y direction. (In 3-D, we'll add "k-hat" in the +z direction)

I claim that ANY old vector **V** can be (uniquely!) written like this:

$\vec{\mathbf{V}} = V_x\,\hat{\mathbf{i}} + V_y\,\hat{\mathbf{j}}$

(where Vx and Vy are the usual x and y components)

This can be seen by staring at this picture:



$V_x\,\hat{\mathbf{i}}$ is the product of a number (Vx) times a vector ($\hat{\mathbf{i}}$), so it's a vector. Its length is Vx times the length of $\hat{\mathbf{i}}$, which is one, so its length is Vx.

Look at the picture: $V_x\,\hat{\mathbf{i}}$ is the "right pointing vector", that makes up the bottom leg of **V**. (It's just a convenient notation, useful when doing "vector arithmetic".)

In this notation, if you want to add **A+B**, with $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}$ you just have $\vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}}$

(note that the "x component" of **A+B**, i.e. the coefficient of $\hat{\mathbf{i}}$ above, is $(A_x + B_x)$, just as it should be!

We can now go back to the kinematics equations we had in 1-D, and write them down immediately (and correctly) in 2-D or 3-D, just by adding vector signs.

Example: average velocity (vector) is defined by $\vec{\bar{\mathbf{v}}} \equiv \dfrac{\Delta \vec{\mathbf{r}}}{\Delta t}$, where **r** is the position.

(Just like we had $\bar{\mathbf{v}} \equiv \dfrac{\Delta x}{\Delta t}$ in 1-D) (Note: Delta t is NOT a vector, it's a number!)

You can manipulate the equation in usual algebraic ways, e.g.

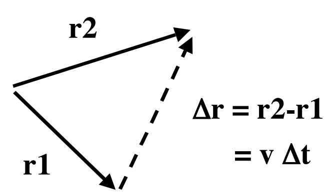
$\Delta \vec{\mathbf{r}} = \vec{\bar{\mathbf{v}}}\, \Delta t$ (just multiplied the definition through by the scalar number Delta t)

Since $\Delta \vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$ (by definition), we have $\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = \vec{\bar{\mathbf{v}}}\, \Delta t$, so $\vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1 + \vec{\bar{\mathbf{v}}}\, \Delta t$.

The picture shows you this equation graphically:

final position = initial position + change.

The CHANGE in position is given by $\Delta \vec{\mathbf{r}} = \vec{\bar{\mathbf{v}}}\, \Delta t$



(Study the picture to see that the arrow on Delta r is in the right direction!)

Similarly, $\vec{\bar{\mathbf{a}}} \equiv \dfrac{\Delta \vec{\mathbf{v}}}{\Delta t}$, which leads to $\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1 + \vec{\bar{\mathbf{a}}}\, \Delta t$ (a familiar 1-D eqn, only now with vector signs) It means **v** final = **v** initial + change in **v**,

and change in v is given by $\Delta \vec{\mathbf{v}} = \vec{\bar{\mathbf{a}}}\ \Delta t$. (Same picture, just replace **r**'s with **v**'s)

Instantaneous quantities are also defined just as before:

$$\vec{\mathbf{v}} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt}, \qquad \text{and similarly} \qquad \vec{\mathbf{a}} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt}.$$

What do those nasty looking formulas mean?

Whenever you see a vector equation, you can always think about it in terms of *components*! If **A**=**B**, that simply means Ax = Bx, *and* Ay=By.

E.g., the x component of that a" equation above (right) tells you $a_x = \frac{dv_x}{dt}$

Suppose you know what x(t) and y(t) are for a particle. That means you know where it IS at all times, you know its position vector $\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$.

(Can you see that the "x component" of position is simply x?!)

So we can immediately find $\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}$

(or, in other words, looking at the "x components" of that equation, $v_x = \frac{dx}{dt}$)

And we also know $\vec{\mathbf{a}} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}}$.

(or, in other words, looking at the "x components" of that equation, $a_x = \frac{dv_x}{dt}$)

If you know position, $\vec{\mathbf{r}}(t)$, these formulas tell you what the velocity and acceleration are. If you feel intimidated, just work separately on the x equations (like $v_x = \frac{dx}{dt}$, or $a_x = \frac{dv_x}{dt}$, by themselves, just like you had in 1-D)

Example: A particle moves in 2-D.

Its x-coordinate is steadily increasing with time, x=20 t (it's moving to the right) but at the SAME time, its y-coordinate is y=-5 t^2.

(it's moving down, and indeed going farther and farther down each second because of that "squared" on the time)

How do you *picture* this? Think of a ball tossed in the air - moving sideways AND up and down simultaneously! This will be the main focus of Ch. 4, by the way!

The formulas say $\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = 20\hat{\mathbf{i}} - 10t\ \hat{\mathbf{j}}$ (Do you see why?)

Which means vx = 20, a constant (makes sense, that's what I said. In the x direction, it moves rightwards in a steady fashion, x=20t, uniform x-motion)

It also says vy=-10 t is changing with time. (This also makes sense - it's going faster and faster, DOWN (in the negative y direction) because of the t^2 term...)

Finally $\vec{\mathbf{a}} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} = 0\hat{\mathbf{i}} - 10\ \hat{\mathbf{j}}$.

There is NO acceleration in the x direction (sideways), which is what we said in words - the motion is STEADY in that dimension. (vx is a constant, no *change*.)

But it is not zero in the y-direction - it is CONSTANT =-10. It is accelerating DOWN in the y direction. (Just like a ball in gravity does!)