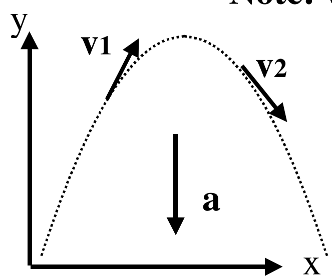
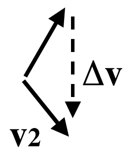
<u>Motion in 2-D (and 3-D)</u>

A "freely falling" objects in 1-D feels *only* gravity, and accel=g, down. In 2-D, a freely falling object is called a **projectile**: with $\left.\begin{array}{l} a_x = 0 \\ a_y = -g \end{array}\right\}$ or in vector notation, $\vec{\mathbf{a}} = 0\,\hat{\mathbf{i}} - g\,\hat{\mathbf{j}}$



Note: **v1**



Gravity pulls straight DOWN, there is no sideways acceleration at all. (ax=0 says that vx is not changing. vx is constant.)

The key physics idea here (discovered by Galileo!) is that for projectile motion, *x and y motions are independent*.

Formally, constant acceleration means **r** = **r**0 + **v0** t + (1/2) **a** t^2.

If you look at the x and y components of this equation, you'll *see* that x and y motion are totally separate:

x motion: $x = x_0 + v_{0x}\, t$ [no (1/2) a t^2 term, because ax=0]

$v_x = v_{0x}$ [again, no a*t term, because ax=0]

y motion: $y = y_0 + v_{0y}\, t + (1/2)\, a_y\, t^2$

$= v_{0y}\, t - (1/2)\, g\, t^2$ (with the coordinate system shown above.

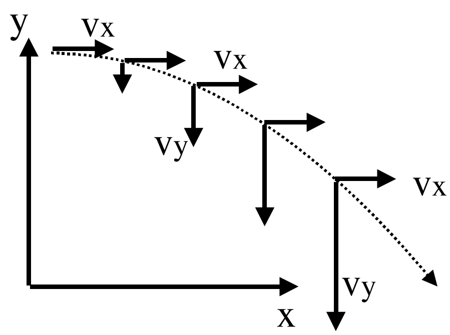
$v_y = v_{0y} + a_y\, t$ If we had chosen a coordinate system with

$= v_{0y} - g\, t$ +y pointing down, we'd instead use

$v_y^{\,2} = v_{0y}^{\,2} + 2a_y(y - y_0)$ ay=+g in these equations, of course.)

$= v_{0y}^{\,2} - 2g(y - y_0)$

All very familiar equations: just the *same old* 1-D constant accel eq'ns of Ch. 2!!



Here's a sketch of an object thrown horizontally.

There is no acceleration in the x direction:

vx is constant.

Gravity causes the downward component of velocity to increase (at 9.8 m/s^2)

All projectile problems amount to doing Ch.2-type problems simultaneously for the x and y motions. Sometimes a little trickier, but not *fundamentally* different.

Example: Thelma and Louise drive off a 1000 m tall cliff with initial (horizontal) velocity v0=30 m/s (about 65 mi/hr) (The figure above sketches their path.)

a) How long (time) are they in the air before they hit the ground?

b) How far do they travel horizontally?

Answer: a) They hit ground when y=0. Apparently, this first question can be answered by looking only at y motion. The x motion can be ignored, it's irrelevant here! So let's look at the equation for y(t) on the previous page:

$$y = y_0 + v_{0y} t + (1/2)\, a_y t^2.$$

What quantities do I know? We have y=0, y0=+1000 m, a_y= -g = -9.8 m/s^2, and v0y =0 (That last one is tricky and very important! v0y is the *y component* of the initial velocity. It's 0, not 30 m/s. v0 is horizontal, it has no "y" component!)

Plugging in, 0 = 1000 m - (1/2)g*t^2, so (can you convince yourself?)

$t = \sqrt{\dfrac{2 * 1000\ \text{m}}{9.8\ \text{m}/\text{s}^2}} \approx 14\,\text{s}$. Pretty long time. (+ remember, we ignore air friction)

b) Now we know the time they traveled. This question asks about horizontal motion during that time. At this point we can forget about y motion. It is now irrelevant! Focus only on the x motion, use the equation for x(t): x = x0 + vox*t.
In this case, v0x=30 m/s (because v0 is horizontal, purely in the x direction, to start with.)
So x= 30 m/s* t. Plugging in 14 sec gives x=420 m. A spectacular and tragic jump!

Notice that the TIME did not depend in any way on v0. Think about that! A bullet, a car, a dropped coin, all hit the ground simultaneously (if launched with voy=0, i.e. horizontal) However, the bigger v0 is, the farther horizontally they will travel before hitting.

The speed at any time is $\sqrt{v_x^2 + v_y^2} = \sqrt{v_{0x}^2 + (gt)^2}$, which grows with time.
They go faster all the time, even though the horizontal component never changes.

Finally, notice that from the formula x=v0x*t, we can solve for the time when an object passes position x: $t = \frac{x}{v_{0x}}$. Plugging that time back into the equation for y(t), I get

$$y = y_0 + v_{0y} t - (1/2)\, g t^2$$

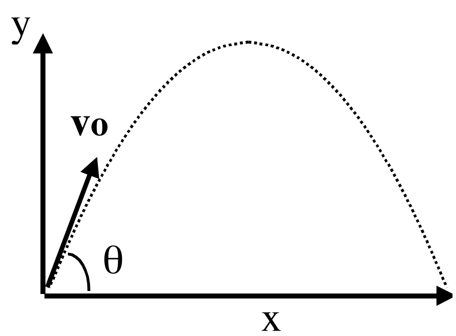
$$= y_0 + \frac{v_{0y}}{v_{0x}} x - \frac{(1/2)g}{v_{0x}^2} x^2$$

This is the mathematical formula for a parabola y(x). It says if you graph y vs x (like the sketch on the previous page, a “time lapse photo”) it will be a parabola. And it tells you how high the object is, given x (rather than given *time*) which is sometimes useful, but I usually just rederive it when I need it, and you probably should too.

Example: A projectile is launched from the ground at some angle theta, speed v0.

Describe its path. E.g. Max height? Time to peak? Range? Impact speed?

Answer: The figure is just a little different now (it doesn't start horizontal.)



The formulas we need to use are the *same old ones* we've been writing down (separate equations for x and y motion, with ax=0, and ay=-g.)

The thing to notice is that v0x = v0*cos(theta), and v0y = v0*sin(theta)

Convince yourself if you don't see that! Look at the components of v0 in the picture

Equations: (refer back to the basic kinematic equations, *nothing* really new here)

x motion: *i)* $x = v_{0x} t \quad = v_0 \cos\theta \, t$ [notice ax=0 in any projectile problem]

ii) $v_x = v_{0x} \quad = v_0 \cos\theta$ [In all equations, I use x0=y0=0, from figure]

y motion: *iii)* $y = v_0 \sin\theta \, t - (1/2) g t^2$

iv) $v_y = v_0 \sin\theta - g t$

v) $v_y^2 = (v_0 \sin\theta)^2 - 2 g y$

Answers: Max height?

When it reaches the top, y=y_max, and vy=0. That last one is important. The projectile is not *stopped* at the top (because v_x never changes) but it is not CLIMBING any more. For an instant in time, at the top, vy = 0. So use equation *v)*

$0^2 = (v_0 \sin\theta)^2 - 2 g y_{max}$ giving y_max = $+(v_0 \sin\theta)^2 / (2 g)$.

Check the units out: (m/s)^2/(m/s^2) = m, it works!

(*Think* about the result physically: if v0 is bigger, it says you'll go higher. Makes good sense. And, if theta = 90, that maximizes the height. Also makes sense)

<u>Time to peak?</u> Just like above, to reach the top, we will want to plug in vy=0 (but now into eq. *iv)* and solve for time:

t=v0 sin(theta)/g. (So far, we haven't even had to look at any x equations!)

<u>Range?</u> This means "how far does it go horizontally before hitting the ground"?

An x question! However, to figure it out from Eq. *i*, we'll need to know the *time* when it hits the ground. We have to solve that first, even though I didn't ask for it explicitly...

<u>Method 1)</u> *(By symmetry)* The time to hit the ground should be twice the time it took to reach the peak, which we just found. So it hits at t_hit = 2 v0 sin(theta)/g.

<u>Method 2)</u> *(Direct)* It hits the ground when y=0, so just plug y=0 into Eq. *iii*.

$$0 = v_0 \sin\theta \, t_{hit} - (1/2) g t_{hit}^2 = t_{hit}(v_0 \sin\theta - (1/2) g t_{hit}).$$

By inspection, solutions are t_hit=0 (uninteresting) or

t_hit = 2 v0 sin(theta)/g. *(Which checks with method 1)*

Now we know the time, we can just plug this into Eq. *i* to get the range:

$$x_{max} = v_0 \cos\theta \, t_{hit} = \frac{v_0^2}{g} 2\cos\theta \sin\theta$$ (*convince yourself* when I do algebra like that.)

There's a little "trig identity" which might make calculator work easier, namely

$\sin(2\theta) = 2\cos\theta \sin\theta$, which is always true. So we can rewrite the formula for the range:

$$\boxed{x_{max} = \frac{v_0^2}{g} \sin(2\theta)}$$ (Only true if the landing spot is the same height as launch spot)

Stare at the formula and think about whether it makes sense. E.g, check units.

Think about it in as many ways as possible: e.g., if v0 is bigger, does it make sense to you that it should go further? (Seems reasonable)

Also, to get the maximum range, this formula tells us you want sin(2 theta) to be as big as possible, i.e. 2*theta = 90 degrees, or theta=45 degrees. Also seems intuitively reasonable, that's what I do naturally when I try to throw a ball far.

(However, it turns out air friction changes the answer noticeably.)

<u>Impact speed?</u> To get *speed*, we need both components of velocity when it hits.

By the Pythagorean theorem, speed = |v| = $\sqrt{v_x^2 + v_y^2}$.

From Eq. *ii)* $v_x = v_0 \cos\theta$. (= constant, this one is very easy)

From Eq. *iv)*, $v_y = v_0 \sin\theta - g t_{\text{hit}}$.

Plugging in our formula for t_hit (found about), I get

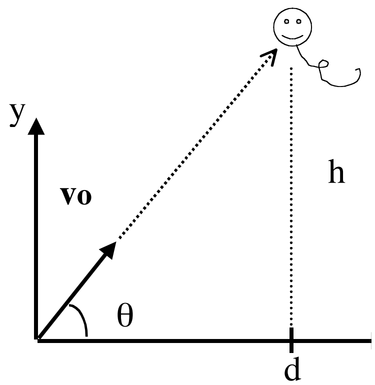
$v_y = v_0 \sin\theta - g(2 v_0 \sin\theta / g)$, which simplifies to vy=-v0 sin(theta). (*Check* this)

This is equal (but opposite) to vy0! So when I square this (as I must, in the Pythagorean formula above) the minus sign squares away, and the final speed is going to be the *exact same* as the launch speed, namely v0. The direction is different, however.

(Just FYI, this is an example of "the conservation of energy", which we'll learn more about later)

The Monkey Hunter: Forgive the politically incorrect nature of this (classic) problem.

Imagine Jane Goodall is firing a tranquilizer dart (or tossing a nice banana)

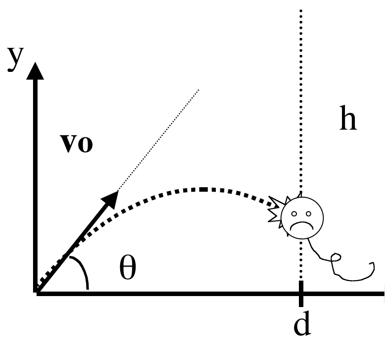


The gun is aimed *straight* at a monkey in a tree.

The instant the dart is released, however, the monkey drops out of the tree.

Question: will the dart overshoot, undershoot, or hit?

(Does the answer depend on the initial dart speed?)

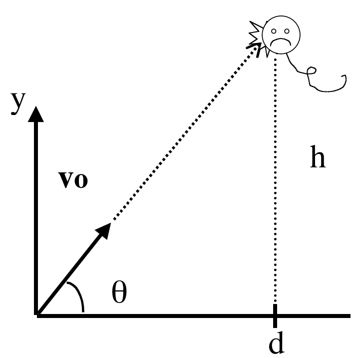


The remarkable answer is that it will always hit, no matter what the dart speed v0 is.

How can we understand this?

It's basic 2-D kinematics with constant acceleration.

I find it easiest to FIRST image what would happen if gravity were turned off...



If g=0, the monkey doesn't fall, it just sits there.

Similarly, the bullet just follows a straight line path.

The path of the bullet is $\mathbf{r}=\mathbf{r0}+\mathbf{v0}*t$, or, in components:

$y = v0y*t$ (no acceleration term if g=0)

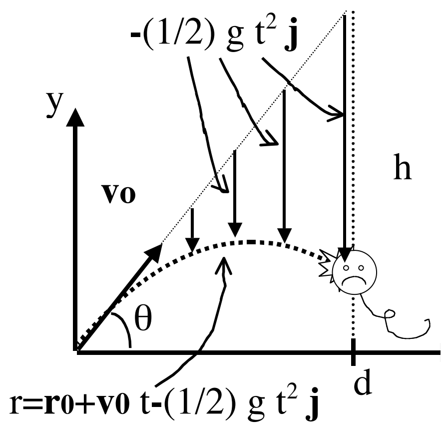
$= v0 * \sin(\theta) * t$

$x = v0 * \cos(\theta) * t$

From that last equation, $t_{hit} = d/(v0*\cos(\theta))$, if you were interested...)

In any case it should be clear that you WILL hit the monkey regardless of v0.

Now, let's turn g back on. With gravity turned on, the path of the dart is



r=r0+v0*t-(1/2)g **j** t^2, or in components:

x = v0x*t

y=v0y*t - (1/2) g t^2.

The second term in the "y equation" is the reduction in height due to gravity pulling the dart down.

The dart is always BELOW the straight path, by an amount (1/2) g t^2.

Meanwhile, the monkey is also falling, and its height is given simply by

y_monkey = h - (1/2) g t^2. (Just 1-D kinematics, with initial velocity=0, starting at h)

So the monkey is ALSO falling BELOW the straight line path, by an amount (1/2) g t^2.

The bullet and monkey will, sooner or later, be at the same x position. When that happens, (that's t_hit), BOTH will be below the straight line path by the *same amount*, (1/2) g t^2: they'll be at the same height. That means the dart hits!

If v0 is large, this will happen sooner. (The collision will be near the original height h.)

If v0 is small, it will happen later. (The collision is lower, e.g. as shown in the figure)

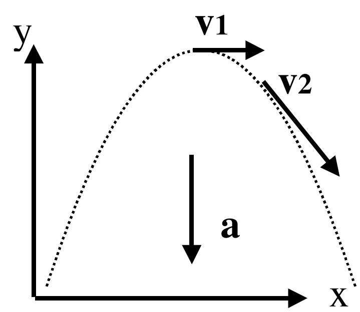
Admission: if v0 is *really* small, the dart hits the ground before reaching the horizontal distance d, and misses. But then it's not in "free-fall" any more, so that's not really fair. As long as the bullet makes it over as far horizontally as the monkey's position, they'll hit. (If there's a cliff right in front of the gun, so the bullet has no ground to hit, it will STILL hit the monkey, no matter how slow v0 is, maybe at some height far below the starting position)

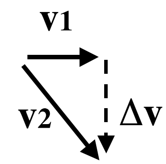
Review to here:

In 2- (or 3-) D, the equations are just like in 1-D, except written as vectors. E.g.

$\vec{a} = \dfrac{\Delta \vec{v}}{\Delta t}$,

$\vec{v}_f = \vec{v}_i + \Delta \vec{v}$





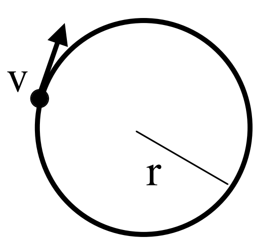
For projectiles, **a**= -g **j**, the acceleration is CONSTANT and in the negative y direction. Constant acceleration is nice, it means we can use our kinematics equations we derived back in Chapter 2 for the x and y components separately. Of course **a** does not always have to be constant in real life, there are other forces in the world besides simple gravity!

Uniform Circular Motion: Uniform means steady, constant speed.

An object moving in a circle is a rather different situation than projectile motion. Consider the moon (or a satellite) in orbit around the earth. Or just a ball on a string, swinging around my head. Since the objects SPEED is constant, is it accelerating?

The answer is YES: The velocity vector is changing *direction* (all the time) and changing velocity *always* means acceleration.

For uniform circular motion, $\vec{a} = \dfrac{\Delta \vec{v}}{\Delta t}$ is definitely not zero. (Is this semantics, due to our arbitrary definition of acceleration? *No*, it's a very real acceleration; you can *feel* it.)



Some lingo:

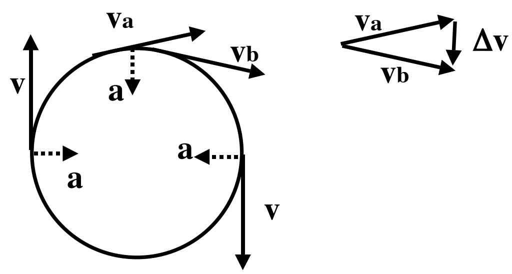
T = period = time for one revolution (rev, cycle, time around)

f = frequency = # rotations/sec = 1/T (convince yourself of that.)

speed = |v| = (distance/time) = (2 Pi r)/T = 2 Pi r f.

When an object runs around in a circle with constant speed, it is accelerating.

The acceleration is a vector, it points towards the CENTER of the circle at all times.



The figure here should at least show you that this is believable. (See my Appendix for a more careful geometric demonstration)

The acceleration is called **centripetal**, meaning "center-seeking". The acceleration is ALWAYS exactly perpendicular to the velocity. (If it wasn't, the size of **v** would be changing, this wouldn't be "*uniform* circular motion")

The magnitude of acceleration is given by a simple formula (also in my Appendix)

$|a| = \frac{v^2}{r}$. (Check the units: (m/s)^2 / m = m/s^2, they work out)

The formula is plausible: the tighter a circle you try to turn, the bigger a is. That's sure how it feels in a bike or car. Similarly, the faster you go, the bigger acceleration is...

Question: If you accelerate towards the center all the time, why don't you ever GET to the center? The answer is Douglas Adams like - you really are accelerating towards the center, but you constantly MISS because of your sideways motion! Think of a bullet, which is traveling sideways and accelerating down. Does it move STRAIGHT down? No, it always keeps its sideways motion, that never goes away. So it never hits the ground directly below where it is, even though that's the direction that its accelerating.

Same with circular motion. (The main difference is that the acceleration of gravity is always in the *same* direction, which yields parabolic paths. For uniform circular motion, the acceleration is constantly changing direction to stay perpendicular to velocity, and that yields *circular* paths.)

Example: A giant merry go round spins, with radius 5 m, and period T=3 s.

What is the acceleration of a small child near the edge?

Answer: First we need the speed, v = (2 Pi r)/T.

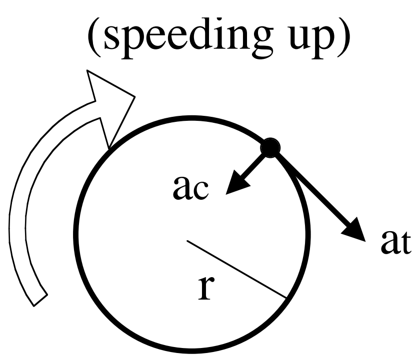
Then we use a = v^2/r = (2 Pi r)^2 / T^2 / r.

One of the r's cancels, and plugging in #'s gives 4 Pi^2 (5 m) / (3 s)^2 = 21.3 m/s^2.

Since g=9.8 m/s^2, this answer is about 2.2g. That's a pretty hefty acceleration.

(5 g's will knock you out after a few minutes. 10 g's will knock you out in a few secs!)

What if the circular motion isn't *uniform*? What if the object moves in a circle but, say, speeds up?



You still have centripetal acceleration,

a_centrip = a_c = v^2/r,

(v is the instantaneous velocity at the point considered)

You ALSO have an additional component of acceleration parallel to velocity (antiparallel if you are slowing down), called a_tangential, or just a_t.

a_t = d |v| / dt, it's the derivative of the SPEED.

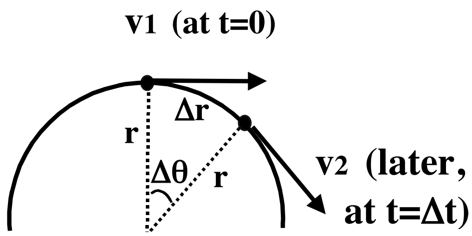
It tells you how much faster (or slower) you're going every second.

The total vector acceleration is given by the sum of these, $\vec{a} = \vec{a}_{centrip} + \vec{a}_{tan}$

The magnitude of total acceleration is thus $|a| = \sqrt{a_c^2 + a_{tan}^2}$. (because $a_c \perp a_{tan}$)

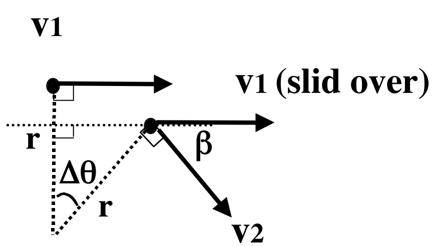
Any arbitrary motion in 2-D can be thought of as (instantaneously) being "non-uniform circular motion." (See Fig 4-7 in your text to try to picture this...) We may come back to this more when we talk more about rotational motion later.

APPENDIX: Geometric argument for centripetal acceleration.



An object moves around the circle with constant (uniform) speed. The magnitude of v1 = v2 = v. In time Delta t, it moves through angle Delta theta, and travels a distance Delta r. So $v = \dfrac{\Delta r}{\Delta t}$.

Let's look graphically at Delta v, which we will need (to figure out acceleration.)



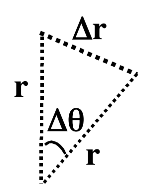
The easiest way to find Δv is to slide the two velocity vectors together, as shown. Stare at the geometry. Remember: the sum of angles in a triangle is 180, and the sum of angles along a straight line is 180.

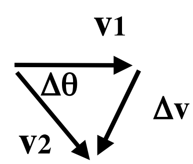
This should convince you that $\beta = \Delta\theta$

Now look at the following two isosceles triangles.

(The "v" triangle is isosceles because |v1| = |v2| = v.)

Since they have the same angle, and they're isosceles, they are similar triangles, which means $\dfrac{\Delta r}{r} = \dfrac{\Delta v}{v}$.





We're all set: $a = \Delta v / \Delta t = v\dfrac{\Delta r}{r} / \Delta t$ (I just used the formula above to eliminate Δv),

and now I reorganize to get $a = \dfrac{v}{r}\dfrac{\Delta r}{\Delta t} = \dfrac{v^2}{r}$ (the last step uses $v = \dfrac{\Delta r}{\Delta t}$) QED.

(Finally, look at the picture and convince yourself that the Δv vector, if placed where the particle is, points towards the center of the circle. This is only true, strictly, in the limit that Delta theta gets tiny, so the two dots get very very close together)