<u>CH 6: Work and Energy:</u> These are two new, closely related, concepts. Newton didn't think about these quantities, but they are very useful!

Energy is *conserved*: it changes form, but the total amount (in any isolated system) never changes. This is a central idea in all of modern physics.

Energy is a little tough to define, because of all the different forms it takes, e.g. kinetic (or "energy of motion"), chemical, nuclear, gravitational, thermal,...

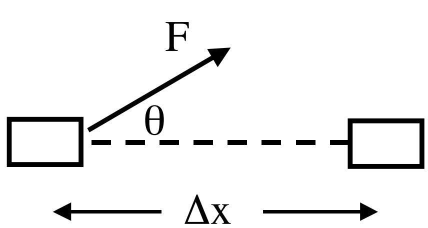
Fundamentally, *energy is the ability of a system to do work*. Whenever work is done, energy is *transformed* from one form to another, but you can never create or destroy energy, only change its form. This is of central importance in understanding a variety of phenomena (including the increasingly relevant issue of our finite supplies of energy reserves in the form of fossil fuels)

We must begin with the concept of *work*. Think of what it feels like to do work - plow a field, let a motor speed a car up, climb the stairs. There's always a force involved, and it does something, moving an object.

Consistent with this, we make a formal **<u>Definition of work</u>**

The work, W, done by a constant force **F** *moving an object through a displacement* $\Delta\vec{\mathbf{x}}$ *is defined to be* $W = F_x \Delta x$

Graphically:



$$W = F\cos\theta \ \Delta x$$

Work is a *scalar*, a number, not a vector. (It can be positive or negative.)

Units of work: [1 N*m] = [1 kg m^2/s^2] = [1 Joule]= [1 J]

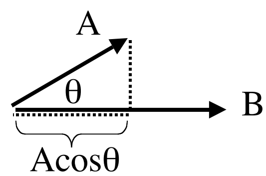
From the picture (or formula), only the component of force in the direction of motion is doing work. (Only the "projection of force along the motion" does work.)

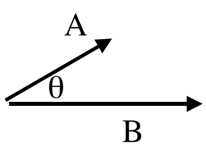
In 2- or 3-D, we write the equation as $W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}}$.

That notation is called the "dot product" or "scalar product" (because the *answer* is a scalar). It's defined by

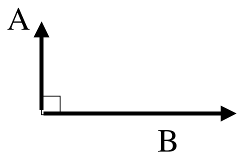
$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \equiv |\vec{\mathbf{A}}||\vec{\mathbf{B}}| \cos\theta = AB\cos\theta,$$

where θ is the angle between **A** and **B**.

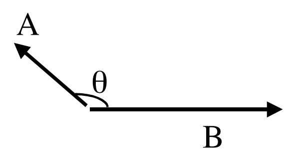




$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} > 0$



$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = 0$



$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} < 0$

In this way, W>0 if force and displacement point in the same directions,

W=0 if force is perpendicular to displacement,

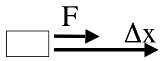
W<0 if force opposes displacement. (e.g. a frictional force.)

Think about unit vectors, convince yourself that $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1$, but $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$, and so

$$\begin{aligned} \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} &\equiv (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}}) \cdot (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}}) \\ &= (A_x B_x \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + A_y B_y \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} + A_x B_y \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + B_x A_y \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}) \\ &= A_x B_x + A_y B_y \end{aligned}$$

(This last result is a *very* handy alternative formula to use for dot products if you know the components.)

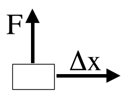
Example: An object moves right, and you push on it to the right. .



$W = F\ |\Delta x|(\cos 0) = +F\ |\Delta x|$

The work you do is + (you are adding energy, speeding it up)

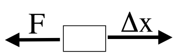
Example: An object moves right, and you lift upwards on it.



$W = F\ |\Delta x|(\cos 90) = 0$

The work you do is zero.

Example: An object moves right, and you push leftwards on it . (E.g. slow it down)

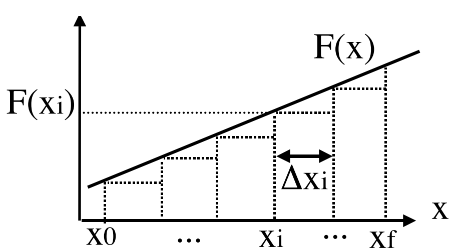


$W = F|\Delta x|(\cos 180) = -F|\Delta x|$

The work you do is negative (you are removing energy)

Variable Forces: What if **F** is *not* constant? We can't use $W = \vec{\mathbf{F}}\cdot\Delta\vec{\mathbf{r}}$ because we don't know what **F** to plug in. The idea then is to break the motion up into tiny chunks. For each small piece of the motion, the force will be approximately constant, and we can figure out the small amount of work involved with our old definition. (Then we *add* up all these small works to get the total.)

Consider 1-D motion, where a tiny displacement means $\Delta W = F_x \Delta x$.



For each little chunk of motion,

F is nearly constant, $\approx F(x_i)$

so $\Delta W_i = F(x_i)\Delta x$.

The total work done by F is then

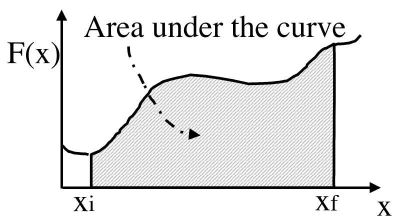
$$W_{\text{by }F} = \lim_{\Delta x\to 0}\sum_i \Delta W_i = \lim_{\Delta x\to 0}\sum_i F(x_i)\Delta x \equiv \int_{x0}^{xf} F(x)dx$$

Math digression: (Integrals)

“Integral” really means “sum”. (The symbol is even a curvy “s” shape).

The integral of F(x) dx means: sum up F(x)*Delta x for each little Delta x chunk (starting at x0 , going to xf.)

Any reasonable function can be integrated graphically. Stare at the sketch on the previous page, convince yourself that our definition of the integral just gives the area under the curve:



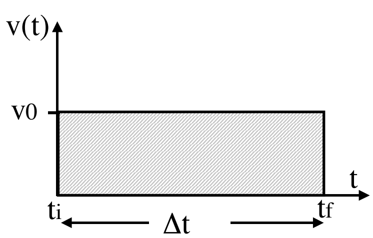
$$\text{Area} = \int_{x1}^{x2} F(x)dx$$

If F represents "force", this is the work done by F.

Here’s another example of integrals, that doesn't involve forces.

Consider an old familiar result: $\Delta x = v\,\Delta t$.

(Remember, this eq’n is only true if v is constant.)

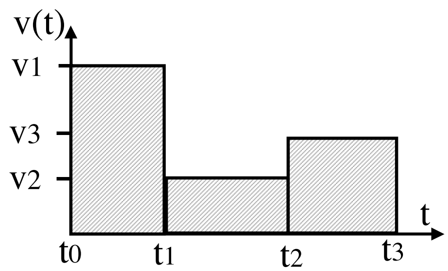


Now think of the equation graphically:

By inspection, the area under this graph of v(t) vs t (when v(t) is constant) is simply $v_0 \Delta t$, which is Δx.

In other words, in this simple case of constant v, the total distance traveled is the area under the v(t) vs t curve.

Next suppose v(t) is *not* constant, but changes as shown below.



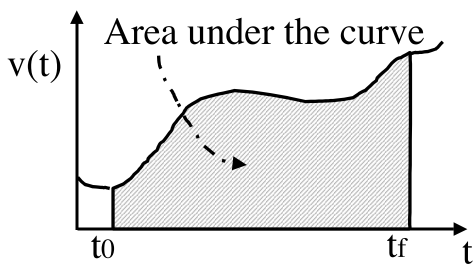
The total distance traveled is simply:

$$\Delta x = v_1(t_1 - t_0) + v_2(t_2 - t_1) + v_3(t_3 - t_2)$$

$$= \sum_{i=1}^{3} v_i \, \Delta t_i$$

(Again, it's the area under the v(t) vs t curve)

Finally, suppose v is varying the whole time.



The total distance traveled is still the area under the v(t) vs t curve:

$\Delta x =$ Area under curve

$$= \sum_{i=1}^{N\to\infty} v_i \, \Delta t_i = \int_{t0}^{tf} v(t)\,dt$$

Notice that $v(t) = dx/dt$, while $\Delta x = \int_{t0}^{tf} v(t)\,dt$.

I.e. v is the derivative of x, and x is the integral of v.

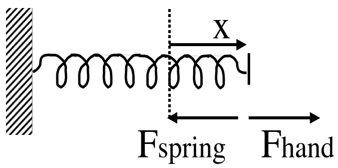
This is why we say the integral is the "anti-derivative".

Finally: here's an essential result from Calc I to use:

$$\int_{xi}^{xf} c x^n \, dx = \left. \frac{c x^{n+1}}{n+1} \right|_{xi}^{xf} = \frac{c}{n+1}\left(x_f^{\,n+1} - x_i^{\,n+1}\right)$$

Example: You slowly, steadily stretch a spring, from unstretched (xi=0) out to xf.

How much work do you do in the process?



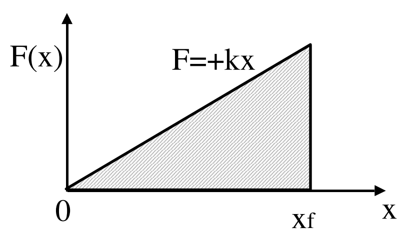
("Steady" means Fnet=0, because the end of the spring moves with zero acceleration.)

Hooke's law: **F**_spring = (-k**x**)

N-II: **F**_(by hand) + **F**_spring = 0, so

F_(by hand) = +k**x**

Work done by YOUR HAND= area under F_(by hand) curve = $\int_0^{xf} F_{hand,x}(x)\,dx$.



Graphical solution:

Work = Area of triangle = 1/2 base*height

= 1/2 (xf)*(+k xf)

Which means work (by you) = $\frac{1}{2} k x_f^2$.

or, use the Calc 1 solution

$$W=\int_0^{xf} F_{hand,x}(x)\,dx = \int_0^{xf} (+kx)\,dx = +k \left.\frac{x^2}{2}\right|_0^{xf} = \frac{1}{2} k x_f^2$$

(same answer, of course)

Note: Work done by the *spring* is different than the work done by *you*.

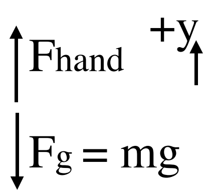
The spring's force has the exact opposite sign, so the work will, too:

W_by spring = $-\frac{1}{2} k x_f^2$

For future reference, notice that in this problem, W(net) = W(spring)+W(you) = 0

(Note: work is a scalar, not a vector. Those are numbers we're adding, not vectors)

Example: You lift (or lower) an object slowly and steadily. How much work do you do? (Steady means Fnet=0, because the object moves with zero acceleration.)



N-II says **F**_(by you) + **F**g=0, i.e. F_(by you) = +mg **j**

This is a constant force, so work done by you is

$W = F\cos\theta\ \Delta y$,

where θ is the angle between the force and the vertical displacement (0 degrees)

In this case, if you lift it up by height h, W(by you) = +mgh.

(You could *integrate* to get it, $W(\text{you}) = \int_{y0}^{yf} +mg\,dy$. This gives the same result.)

If you lower it by h, the answer will be *negative* mgh.

Indeed, the integral gives you the proper answer, *with signs*, for the general case:

W(by you) = +m g Δy. If you lift it UP, you do POSITIVE work. (Makes sense.)

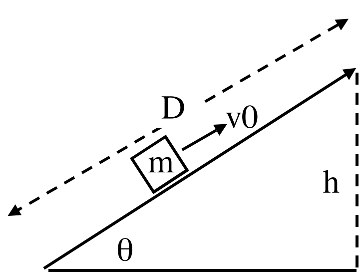
Notice that W(by gravity) = - m g Δy. (Because F_g is opposite in direction.)

Once again, for future reference, W(net) = W(by you) + W(by grav) = 0.

What if this problem is repeated, but the motion is a little more complicated?

The answer is kind of remarkable: *the path doesn't matter*, if there's no friction.

The answer is what we just got, W(by you) = mg Δy.



Let's check a special case:

E.g. You slide an object steadily up a frictionless ramp, as shown. How much work do you do?

Remember, in 2-D, $W = \int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

One way to solve for W is to draw a force diagram and deduce what your force is, then do the required integral. This force diagram was effectively done back in Ch. 6, (see the section on static friction) where we found that you must apply a force $F = mg\sin\theta$, in order that F_net=0. (Once again, *steady* linear motion means **a**=0) This F is constant, so the "integral" is easy, we just have

$$W = \vec{\mathbf{F}}\cdot\Delta\vec{\mathbf{r}} = |\vec{\mathbf{F}}||\Delta\vec{\mathbf{r}}|\cos\theta_{\text{between F and }\Delta\vec{\mathbf{r}}}$$

Here, $|F| = mg\sin\theta$ (see Ch. 6 note) and $|\Delta\vec{\mathbf{r}}| = D$ (see the picture), and the force and displacement are in the same direction, so the angle between F and $\Delta\vec{\mathbf{r}}$ is zero (*not* θ !)

So W(by you) = mg $\sin\theta$ * D. Look at the sketch, and notice $D*\sin\theta = h = \Delta y$, so W(by you) = $mg\Delta y$, exactly as we had before for direct vertical motion.

The work done by *gravity* in this example is

$$W_{\text{by gravity}} = \int \vec{\mathbf{F}}_{\text{grav}}\cdot d\vec{\mathbf{r}} = \int(-mg\hat{\mathbf{j}})\cdot(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}) = \int_{y0}^{yf} -mg\,dy = -mg\Delta y$$

Also as before. (Lifting things up, you do + work, but gravity does - work.)

<u>Work and Energy</u>:

Any moving object (mass m, velocity v) can do work. Let it run into something else - it'll apply a force, over a distance, until it has stopped (or bounced). So *how much* work can it do in this process? More important - what's the MOST work it can do? The answer to this question is $\frac{1}{2}mv^2$. (Proof is later, in the appendix)

We call this result the *energy of motion* or *kinetic energy* or *K* of the particle.

$K = \frac{1}{2} m v^2$ tells how much work an object can do, just by virtue of its having motion. It's a scalar, a number. It's always positive (unlike work, which can be -) It has the same units as work, Joules. (Check yourself.)

Here's an important theorem which will show how useful this quantity K is:

Work Energy Theorem: For any object feeling any kinds of forces, $\boxed{W_{net} = \Delta K}$

This theorem is proven in your textbook. It follows from Newton's laws and the definition of work, so it's a deep and fundamental law of physics - no approximation (beyond classical physics.) It's true if the forces are friction, springs, gravity, anything. It doesn't say YOUR work is ΔK, it says the NET work is ΔK. Since the right hand side is simple kinematics (you just need to know the final and initial speed), it can be very useful to learn quickly, and easily, about net work done without having to compute any nasty integrals.

Remember: *In general, ENERGY is the capacity to do work.* Kinetic energy *is the capacity to do work merely by virtue of the fact that the object is moving.*

Examples: In the last several worked examples (stretching a spring, moving an object around) the motion was "steady", and in each case we explicitly discovered :

W(net) = W(by all forces) = W(by you) + W(by the other force in the problem) = 0.

And of course, steady motion meant v was constant, so K was constant: $\Delta K = 0$.

So the W.E.Theorem clearly worked in all those cases.

Another example: Drop a book from a starting height h. What will its speed be, when it hits the ground?

Old way to solve this problem: It's a free fall problem, call up "+", so a_y = -g is constant. Use the equation of constant acceleration, $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$.

Since it starts from rest, v_0=0, giving $v_y^2 = 2(-g)(0 - h) = 2gh$.

(Check those signs yourself! Of course, take the square root to get the speed)

New way: There is only one force acting - gravity - so W(net) = W(gravity)

= |F_g|*|displacement|*cos(angle between force and displacement)

= (mg)*(h)*(+1) (both the force, and the displacement, are down in this problem)

but $W_{net} = \Delta K$, so plugging in: $mgh = \frac{1}{2} m v_f^2 - 0$.

Solve for vf, it's the same answer as we got above. (The m's cancel out)

So far it doesn't look like much easier, but you'll soon discover many problems where the "work" way of solving problems is MUCH easier than the "force way".

Comments: If you do net + work, W.E.Theorem says the object speeds up. (ΔK>0)

If you do net - work, W.E.Theorem says the object slows down (ΔK>0)

If the net force is zero, or is perpendicular to the motion, W.E.Theorem says it will keep a steady speed.

Kinetic friction always opposes the motion => it always does negative work, which means (W.E.Theorem says) kinetic friction tries to slow things down. Makes sense!

Example: A rubber band launches an 0.1 kg cart up to some (given) v_f.

How much work did the rubber band do?

Method #1: W(by band) = $\int_{xi=??}^{xf=??} F_{band,x}(x)\, dx = \int_{xi=??}^{xf=??} -k\, x\, dx = ??$

I'm stuck. I don't know how far the rubber band was stretched (though that should be pretty easy to measure), and I don't know what "k" for the rubber band is...

Let's use the W.E.Theorem -

Method #2: W(by band) = $\Delta K = \frac{1}{2} m v_f^2 - 0$. Done! The answer is +, which makes sense, the rubber band did + work on the cart. (You could now USE this to deduce what k was, if you measured xi and xf and applied Method #1...)

Suppose I slide an object at constant speed along a table, pushing with a constant (known) force. Suppose there *is* friction. I am doing positive work (W= F*distance), and friction is doing negative work (because the angle between friction F and displacement is 180 degrees) The net work is zero.

You can see this from N-II (constant speed => F_net = 0, so my force and the friction force are equal and opposite, therefore the two works are equal and opposite in sign)

or from the W.E. Theorem, since ΔK=0 (if speed is constant.)

Where did my work go, by the way? It did NOT go into speeding up the object. It went into heating up the table. More on this idea soon...

Power: Sometimes you don't care *just* how much work you did, but also how long you did it in. E.g, climbing up Gamow tower takes W=+mgh, no matter whether you walk up slowly or run up. But it sure *feels* different if you do it in under 60 seconds! In this case, the work done (or energy expended) is the same no matter what the time taken, but the *power* is different.

Definition: Average power = $\overline{P} = \frac{\text{work}}{\text{time}} = \frac{\Delta W}{\Delta t}$ (Instantaneous power P = dW/dt.)

Units: [P] = [Energy/time] = [J/s]

1 Watt = 1 W = 1 J/s = 1 N m/s = 1 kg m^2/s^3 (ack!)

<u>Example</u>: I run up Gamow tower in 60 sec flat. How much work did I do?

How much average power did I expend?

The tower is 10 floors, about 3.4 m/floor means H=34 m. That means

W = (my mass)*g*H = 60 kg * 9.8 m/s^2 * 34 m = 20,000 J.

Power = work/time = 20,000 J / 60 sec = 333 W (like a VERY bright light bulb!)

Non-metric *power* units: 1 horsepower = 1 hp = 746 W.

Non-metric *energy* units: 1 British thermal unit = 1 Btu = 1055 J.

1 cal = 4.184 J (used by chemists) 1 Cal = 1000 cal = 4,184 J (used for food energy)

1 electron Volt = 1 eV = 1.6E-19 J (used in nuclear and atomic physics)

1 kiloWatt hour = 1 kWh = 3.6E6 J (Power*time=energy)

1 foot pound = 1 ft lb = 1.36 J

Example: I like to eat "Power Bar"'s after lecture. They are poorly named - they contain (and give you) energy, not power per se. Someone must have told them this, because they now say "Energy bars" in fine print on the labels. I have one here that says "250 Cal", less than a typical candy bar. Using our unit conversion above, that's 250 Cal * (4,194 J/Cal) = 1 MJ, a million Joules! Wow.

The human body is, at best, about 20% efficient in converting food energy into useful work. (The rest goes into wasted heat) So, after eating this bar, I could do about 200,000 J of useful work. From the previous example, it means that running up to the top of the Physics tower (20,000 J) burns up about 1/10 of one bar's calories, taking human efficiency into account. A little depressing if you're trying to lose weight... By the way, that last problem showed I consumed 333 W if I ran up in a minute, which is about .45 hp. That's pretty good for a person! A horse, of course, should be able to produce 1 hp for a reasonably long period of time :-)

Example: Public Service (the power company) bills me for energy, *not* power!

I have "windpower", which is more expensive, my bill says I pay $0.10/kW hr.

Remember, Energy = P*t (or $\int P(t)\, dt$, if power is not constant) so

1 kW hr = (1000 J/s) * (3600 sec) = 3.6E6 J. I pay 10 cents for that much energy.

A 100 W bulb consumes 100W = 100 J/s, so after 10 hours, a 100 W bulb has consumed 100W * 10 hr = 1000 W hr = 1 kW hr, which costs me 10 cents.

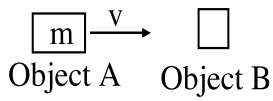
The longer you leave it on, the more you pay. The POWER is constant, but the energy consumed is steadily increasing with time. (Energy = power*time)

In the US, we collectively consume about 1E20 J/year of fossil fuel energy. About 1/3 of that in the form of oil, the rest from coal and natural gas, mostly. This is about 1/4 of the planet's energy use, from a few percent of the global population! We are incredibly wasteful in this respect - western European countries typically use about half as much power per person, and yet several of them have a *higher* GDP per person (gross domestic product, a measure of wealth, or standard of living) than we do. (e.g. Switzerland, Sweden, Norway, Germany...)

The current best estimate for the amount of energy stored underground, *in the form of oil*, is around 7E20 J in the US (including Alaska). This includes "undiscovered reserves" that people are hoping we'll find someday... Compare that with the 1E20 J/year we consume, and it doesn't look very big, does it? Also, compare that with an estimate that if we completely mine the Arctic National Wildlife Refuge in Alaska, it'll provide about 10 Billion barrels of oil, about 0.6E20 J of energy, about 7 months worth of present (*total*) US energy demand.

The entire planet is estimated to have about 100E20 J of energy stored in the form of oil, again including "undiscovered but hoped for" supplies. That's about 25 years of total present global energy demand. Of course, there are other sources of energy besides oil (coal reserves in the US are pretty big in comparison), but coal has lots of serious pollution issues that make it less desirable. When energy is not stored in a useful form (like oil), it's harder to make use of it. I predict that finite (fossil fuel) energy supply issues are going to be a *very* central issue in politics, science, engineering, and your own lives, for as long as you live, starting very soon!

Appendix: How much work can a moving object do?



Consider object A with K = (1/2) m v^2, crashing into object B. In the process of stopping, it's pushing on B, and thus doing work on B.

What is the MAXIMUM amount of work that object A can do on B?

The most it can do is when A loses ALL its kinetic energy, which means it comes to a complete halt. We said in the notes, the answer to this question is Ki. Why?

The W.E. Theorem says Wnet (on A) = ΔK (of A) = 0 - (1/2) m v^2.

Who did this work on A? It was object B, pushing backwards on A.

How would we calculate this work? It's force*distance*cos(theta), or here

W (on A) = $|F_{\text{by B on A}}|\,\Delta x\,\cos(180)$. That's it, because there are no other forces.

The angle is 180, because F(by B on A) is to the LEFT, but Delta x is to the right during the crash.

I asked how much can A do on B? Well, that would be

W (on B) = $|F_{\text{by A on B}}|\,\Delta x\,\cos(0)$.

(The angle is 0, because F(by A on B) is to the right, and so is Δ x during the crash)

By N-III, $\vec{F}_{\text{by B on A}} = -\vec{F}_{\text{by A on B}}$, the magnitudes are equal, and so we've found

W (by A, on B) = - W(by B, on A) = - - (1/2) mv^2

This is what we wanted to show. The - signs cancel, the MAX work A can do on B, which happens when A comes to a complete halt, is 1/2 mv^2. The Kinetic energy of an object tells you the most work an object can do on another..