

Systems of particles, and momentum: So far, we've made a lot of use of Newton's II law, $\mathbf{F}_{\text{net}} = m \mathbf{a}$. This holds for pointlike particles - clearly something a little *idealized*. But the law is very general and very powerful - it holds even for more complicated (and thus realistic) objects. That's the topic of this chapter. The way to figure out what to do when you have more realistic (complex shaped) objects is to think of any real object as built up out of a collection of tiny (pointlike!) particles. (Think of atoms, if you like).

Claim: Any object, viewed as a collection of small (pointlike) objects, has a special point called the **center of mass** ("CM"). For many purposes, the object behaves AS THOUGH it was itself one pointlike object, located at the center of mass! So, even though e.g. gravity acts on ALL the points in the body, the body as a whole acts in many ways the exact SAME as though all its mass was concentrated at the CM. (E.g, when you toss a wrench, although its motion may look weird, the CM of the wrench follows a simple, parabolic trajectory, *exactly* the same as if all the mass of the wrench had been concentrated at that one point)

The CM is not always physically the "center" of the body (so, it's not the best name in the world). But it's fairly intuitive, a kind of "balance point". For a nicely symmetric object, the CM will be in the center. For a funny shape, it will be located more where the mass is concentrated. The formula

to find the CM is given by $\mathbf{r}(\text{CM}) = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}$.

To get the x coordinate of the cm, just take the "x-component" of that formula,

e.g.. $x(\text{cm}) = \frac{\sum_i m_i x_i}{\sum_i m_i}$. (Ditto for y, or z)

Let's stare at the equation(s) a bit, because they are kind of nasty looking at first glance.

I'm SUMMING over "i", which labels all the little points that make up the body.

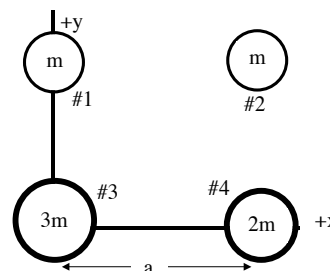
(Think of the complicated body as being built out of $i=1$ to N little points. It's like we numbered the parts, part #1, part #2, part #3, ... etc.!)

The denominator of that formula (sum of m_i) is just the SUM of all the masses of the little points, which would just be M , the total mass of the big object.

The numerator is a "weighted sum". That is, it looks KIND of like an average of the x -positions of all the little pieces that built up the object (like, the "average position"!) except that if some pieces are more massive, they count more.

If the body is a solid, i.e. a continuous collection of "points", then the sum becomes an *integral*. (We'll come back to this idea soon, and develop it more fully later.)

Example: Consider a complicated object built out of 4 smaller masses, (2 of them "m", one "2m", and one "3m") configured in a square of side "a" as shown.



Note: this might be a solid object, where the masses are all hooked together by rigid (superlight!) rods which I haven't shown. Or, they might be atoms connected by electromagnetic forces. Or, they might not be connected at all - you can consider the CM of ANY collection of objects, even if they don't happen to form a rigid solid!

Just think for a second - the CM is NOT going to be in the "geometric center", because the bottom is heavier, and the bottom left is heaviest of all! Let's work it out:

$$\begin{aligned}
 x(\text{cm}) &= (m_1 \cdot x_1 + m_2 \cdot x_2 + m_3 \cdot x_3 + m_4 \cdot x_4) / (m_1 + m_2 + m_3 + m_4) \\
 &= (m \cdot 0 + m \cdot a + 3m \cdot 0 + 2m \cdot a) / (m + m + 3m + 2m) \\
 &= 3 m a / (7 m) = 0.43 * a. \quad (\text{A little LESS than halfway across!})
 \end{aligned}$$

(Please look CAREFULLY at line two. Each mass multiplies ITS x -position! But ALL the masses contributed to the denominator, the total mass)

$$\begin{aligned} \text{Similarly, } y(\text{cm}) &= (m_1*y_1 + m_2*y_2 + m_3*y_3 + m_4*y_4) / (m_1+m_2+m_3+m_4) \\ &= (m*a + m*a + 3m*0 + 2m*0) / (7 m) = 2 ma / 7 m = .29 a. \end{aligned}$$

The CM is located a little left of center (because the left side is a bit heavier), and a LOT below the center in the y direction (because the bottom is a lot heavier!) Makes sense.

What we're claiming is the CM of a group of particles moves along LIKE a point particle would. Even if the masses are not connected! Think of a firework - traveling up in a lovely parabolic arc, like a point mass. Then BOOM it explodes... fragments go everywhere... yet, I claim that if you could track the CM of all the fragments, they will CONTINUE TO FOLLOW THE SAME, ORIGINAL parabolic arc!! This is very cool, and very powerful. It lets us understand very complex systems by using all the (relatively simple) ideas we've covered this semester.

Here's the bottom line, then. EVEN if the object is funny shaped, with mass distributed all around, it is still true that $\mathbf{F}(\text{net}) = M \mathbf{a}$.

Here, $\mathbf{F}(\text{net})$ would be the TOTAL external force on the object, which can be found by summing the external forces on all the pieces (Sum of \mathbf{F}_i). You don't need to worry about any INTERNAL forces between the pieces of your object, because those ALWAYS come in N-III pairs, and when you add those all up, they will cancel out. So $\mathbf{F}(\text{net})$ here really just means OUTSIDE forces acting on your system or object.

The right side has M, the TOTAL mass of the whole object, and \mathbf{a} , which here would have to represent $\mathbf{a}(\text{CM})$, the acceleration of the center of mass. So you have to watch where the CM goes, and $\mathbf{a}(\text{CM}) = d\mathbf{v}(\text{CM})/dt = d^2 \mathbf{r}(\text{CM}) / dt^2$. (Just like for a point, it's the rate of change of the velocity of the center of mass, or the second derivative of the CM position vector).

We finish this chapter with one new defined term. It's called **MOMENTUM**.

The momentum of a pointlike particle of mass m is given the symbol " p " (I don't know why!)

and it's a vector: **Momentum $\mathbf{p} = m\mathbf{v}$.** (Mass times velocity).

It's NOT the same as energy (which was $1/2 m v^2$, and was just a number).

Momentum is a measure of "oomph". More massive objects have more "oomph" (like bowling balls), and fast moving objects do, too (a light bullet has a lot of "oomph"!)

If you have a complex object, its momentum is simply the sum of the momenta of all the pieces, so we can talk about **$\mathbf{p}(\text{CM}) = M \mathbf{v}(\text{CM})$** . (This is the momentum of the center of mass, also referred to simply as "the momentum of the system")

Now look back at our favorite law of nature, N-II, **$\mathbf{F}(\text{net}) = m\mathbf{a} = m d\mathbf{v}/dt$** .

Notice that, if I slide the mass into the derivative, I get **$\mathbf{F}(\text{net}) = d(m\mathbf{v})/dt = d\mathbf{p}/dt$** .

That's Newton's law, as actually written by Newton! Force is the rate of change of momentum

It's the same formula we've been using, but actually a little more general and powerful still.

For example, if you have a complex object, but the net **EXTERNAL** force is zero (that doesn't mean the **INTERNAL** forces have to vanish!), then we get (Remember, I'm now assuming **$\mathbf{F}(\text{net})=0$**):

$d\mathbf{p}/dt = 0$, which is another way of saying " **\mathbf{p}** " doesn't change, p is conserved, it's a constant.

If there is no net EXTERNAL force on a system, the momentum of the system never changes.

Think of a firework exploding in space. If it begins at rest, (so, no momentum to start with, 'cause $v=0$) and blows up, there are LOTS of internal forces - but no **OUTSIDE** force. So, all the bits fly apart, in a complex pattern... but if you find the motion of the **CM**, you'll discover it's still at rest!

The momentum of the **SYSTEM** didn't change! (Each **PIECE** has changed momentum, but they're vectors, and they all cancel out in the sum. The **TOTAL** momentum is conserved, it stays the same)

Example: Consider two cars on an air track. There is no external friction. The cars are connected by a spring, and at $t=0$, the spring is allowed to expand, and the two cars fly apart. Suppose car one has mass "m", and car two has mass "4 m". How do the speeds of the two cars compare afterwards?

This may seem rather complicated (and, I haven't told you anything about the spring!) but we can use this new idea of CONSERVATION OF MOMENTUM to help us out.

Initially, we have a total momentum of $m*0 + 4m * 0 = 0$. The system began at rest, $\mathbf{p}=0$.

Then there's this "spring expanding event". I don't want to think about it, it's too complicated! It might be a real-life spring with a complicated force law. But in the end, as the two cars fly apart, no longer touching each other (or the spring), they **STILL MUST HAVE THE SAME** total momentum. Why? Because, consider the system (cars+spring). There are plenty of internal forces, but **NO NET EXTERNAL FORCE**. Friction is zero. There *is* gravity, and normal forces (up), but those cancel each other (there's no motion up or down, here). So, $\mathbf{F}(\text{net}) = 0$, which means $\mathbf{p}(\text{system}) = \text{constant}$. Since it started at zero, it remains at zero! (Forever, until external forces act!)

In the end, suppose car one moves left with speed v_1 , and car two moves right with speed v_2 .

The formula for total momentum would then be

$$p(\text{in the x direction}) = - m_1*v_1 + m_2*v_2$$

$$= -m v_1 + 4m v_2. \text{ (Substituting in the given masses)}$$

Notice the minus sign on the first term. Car one is going **LEFT**, it has a **NEGATIVE** momentum!

By conservation of \mathbf{p} , this must still be zero. So $m v_1 = 4m v_2$, or $v_2 = (1/4) v_1$.

Makes sense, the more massive one goes more slowly, the light one recoils a lot.

We'll start applying this idea in the next chapter. It's one of the more useful ideas in physics!