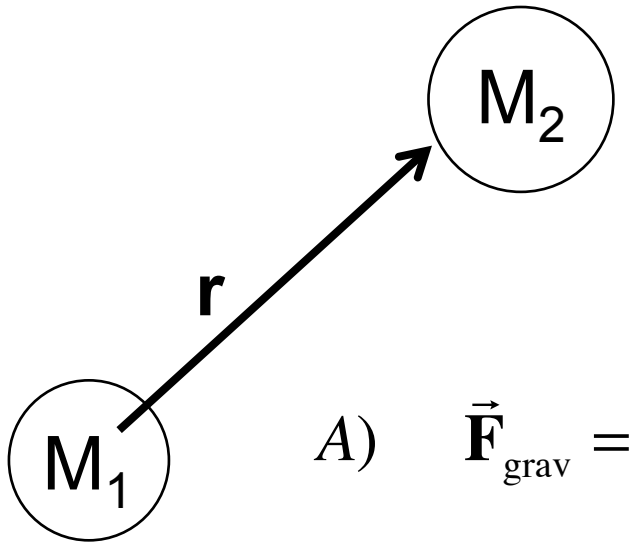
GRAVITY

GRAVITY

- Newton's law of gravity
- Gravitational force (direct integration)
- gravitational field
- Symmetry and Invariance arguments
- gravitational potential, and PE



What is the force of gravity on (pointlike) M_2 caused by (pointlike) M_1 ?

$A)\quad \vec{\mathbf{F}}_{\text{grav}} = -G\dfrac{M_1 M_2}{r^2}\hat{r}$

$B)\quad \vec{\mathbf{F}}_{\text{grav}} = +G\dfrac{M_1 M_2}{r^2}\hat{r}$

$C)\quad \vec{\mathbf{F}}_{\text{grav}} = -G\dfrac{M_1 M_2}{r^3}\vec{r}$

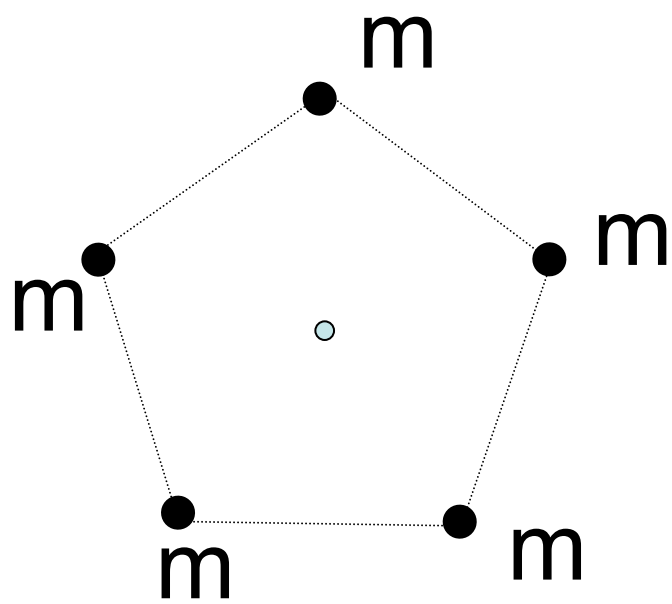
$D)\quad \vec{\mathbf{F}}_{\text{grav}} = +G\dfrac{M_1 M_2}{r^3}\vec{r}$

$E)$ None of these or MORE than one of these!!

Challenge question: Can you use your answer to predict the mathematical form of "Gauss' law for gravity"?

2.5 5 masses, m, are arranged in a regular pentagon, as shown.

What is the force of gravity on a test mass at the center?



A) Zero

B) Non-zero

C) Really need trig and a calculator to decide!

A “little man” is standing a height z above the origin. There is an infinite line of mass (uniform density) lying along the x-axis.

Which *components* of the local **g** field can little man argue against *just by symmetry alone?*

A. ONLY that there is no g_x
B. ONLY that there is no g_y
C. ONLY that there is no g_z
D. ONLY that there is neither g_x nor g_y
E. That there is neither g_x, g_y, nor g_z

<u>Challenge</u>: Which components (g_r, g_Φ, and/or g_z) of the local **g** field can the little man argue against just by symmetry if he is thinking in cylindrical coordinates – be thoughtful about what direction you have the cylindrical z-axis – is it the same as in Cartesian coordinates?

A “little man” is standing a height z above the origin. There is an infinite line of mass (uniform density) lying along the x-axis.

He has already decided $\vec{\mathbf{g}}(x,y,z) = g_z(x,y,z)\,\hat{\mathbf{k}}$

Which *dependences* of the local $g_z(x,y,z)$ field can he argue against, *just by symmetry alone!*

A. ONLY that there is no x dependence,
   i.e. only that $\vec{\mathbf{g}}(x,y,z) = g_z(y,z)\,\hat{\mathbf{k}}$
B. ONLY that there is no y dependence
C. ONLY that there is no z dependence
D. ONLY that there is neither x nor y dependence
E. That there is neither x, y, nor z dependence.

<u>Challenge</u>: Again try switching to cylindrical coordinates. He has decided that **g**= $g_r(r,\Phi,z)$ **r-hat**. Which dependences (r,Φ, and/or z) can he argue against by symmetry alone?

A “little man” is standing a height z above the origin. There is an infinite line of mass (uniform density) lying along the x-axis.

He has already decided $\vec{\mathbf{g}}(x,y,z) = g_z(x,y,z)\,\hat{\mathbf{k}}$

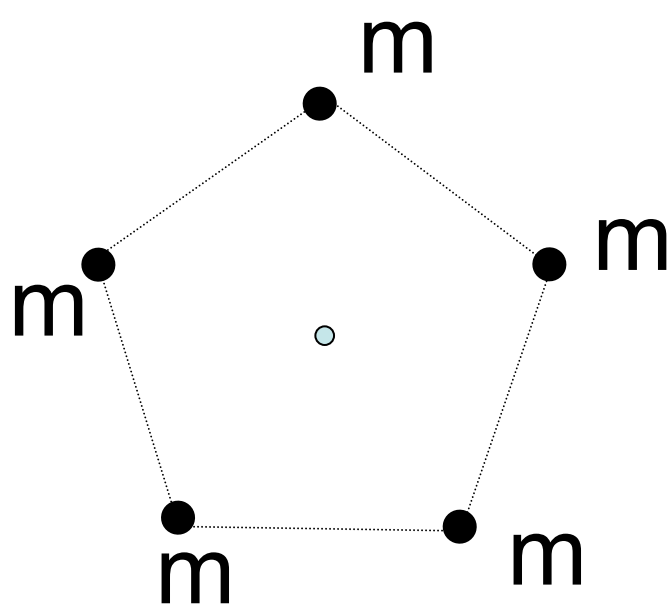
Which *dependences* of the local $g_z(x,y,z)$ field can he argue against, *just by symmetry alone!*

A. ONLY that there is no x dependence,
   i.e. only that $\vec{\mathbf{g}}(x,y,z) = g_z(y,z)\,\hat{\mathbf{k}}$
B. ONLY that there is no z dependence
C. ONLY that there is neither x nor z dependence
D. ONLY that there is neither x nor y dependence
E. That there is neither x, y, nor z dependence.

<u>Challenge</u>: Again try switching to cylindrical coordinates. He has decided that **g**= $g_r(r,\Phi,z)$ **r-hat**. Which dependences (r,Φ, and/or z) can he argue against by symmetry alone?

2.5 5 masses, m, are arranged in a regular pentagon, as shown.

What is the g field at the center?



A) Zero

B) Non-zero

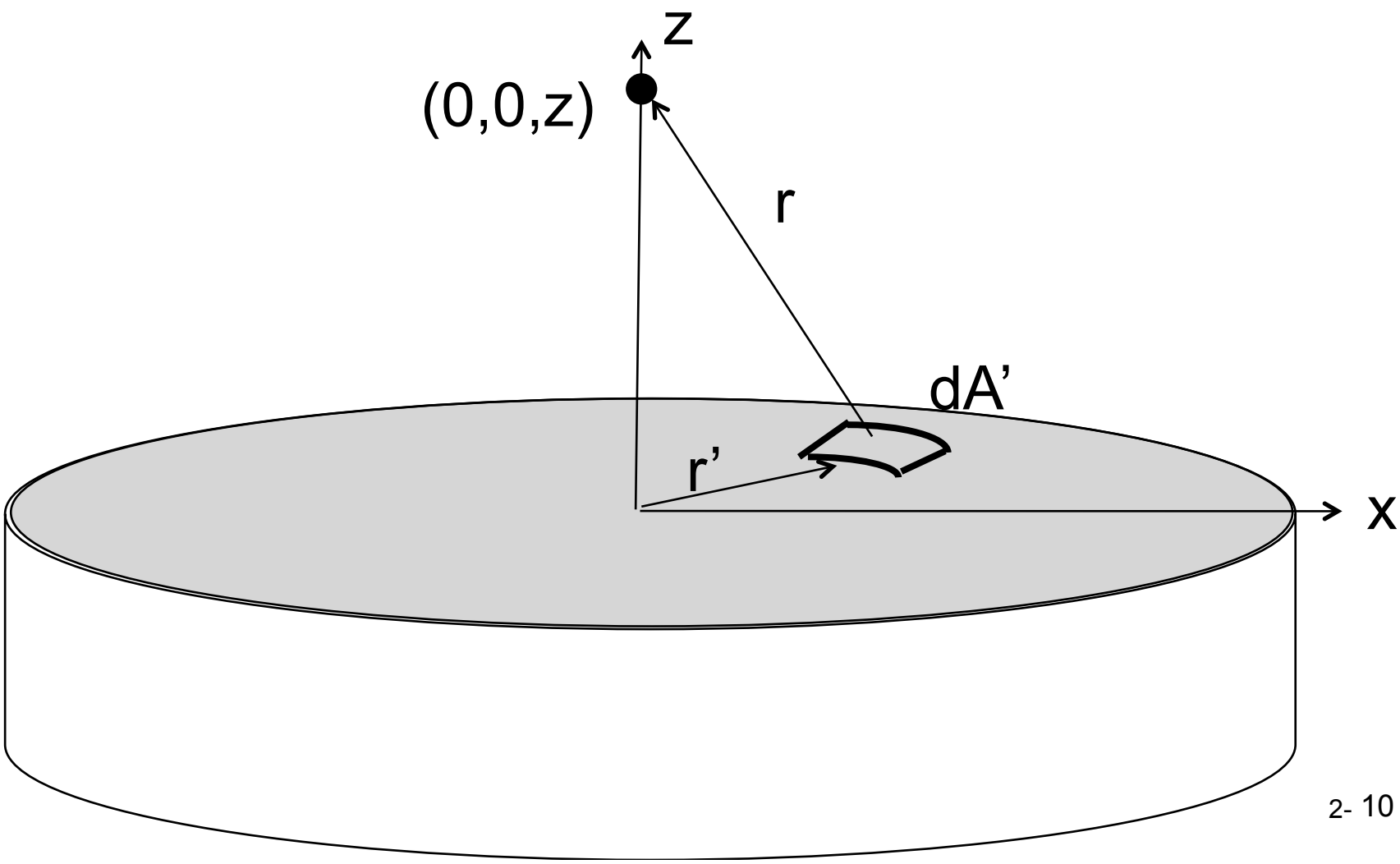
C) Really need trig and a calculator to decide

Force, potential energy, gravitational potential, gravitational acceleration

- What is the practical difference between F, U, g, and F?
  - U and F are fields while F and g are not.
  - They have different masses.
  - F, U depend on both masses, while g, and F depend on only one mass.
  - F and g are easier to calculate.
  - U and F are energies, while F and g are forces.

For **g** along the z-axis above a massive disk, we need

$$(\vec{\mathbf{g}})_z = -G\iiint \frac{\rho(r')}{r^2}(\hat{\mathbf{r}})_z\, dV' \qquad = -G\iint_{disk} \frac{1}{r^2}(\hat{\mathbf{r}})_z\, \sigma\, dA'$$



$$(\vec{\mathbf{g}})_z = -G \iint_{disk} \frac{1}{r^2} (\hat{\mathbf{r}})_z \, \sigma \, dA'$$

What is $(\hat{\mathbf{r}})_z = ?$

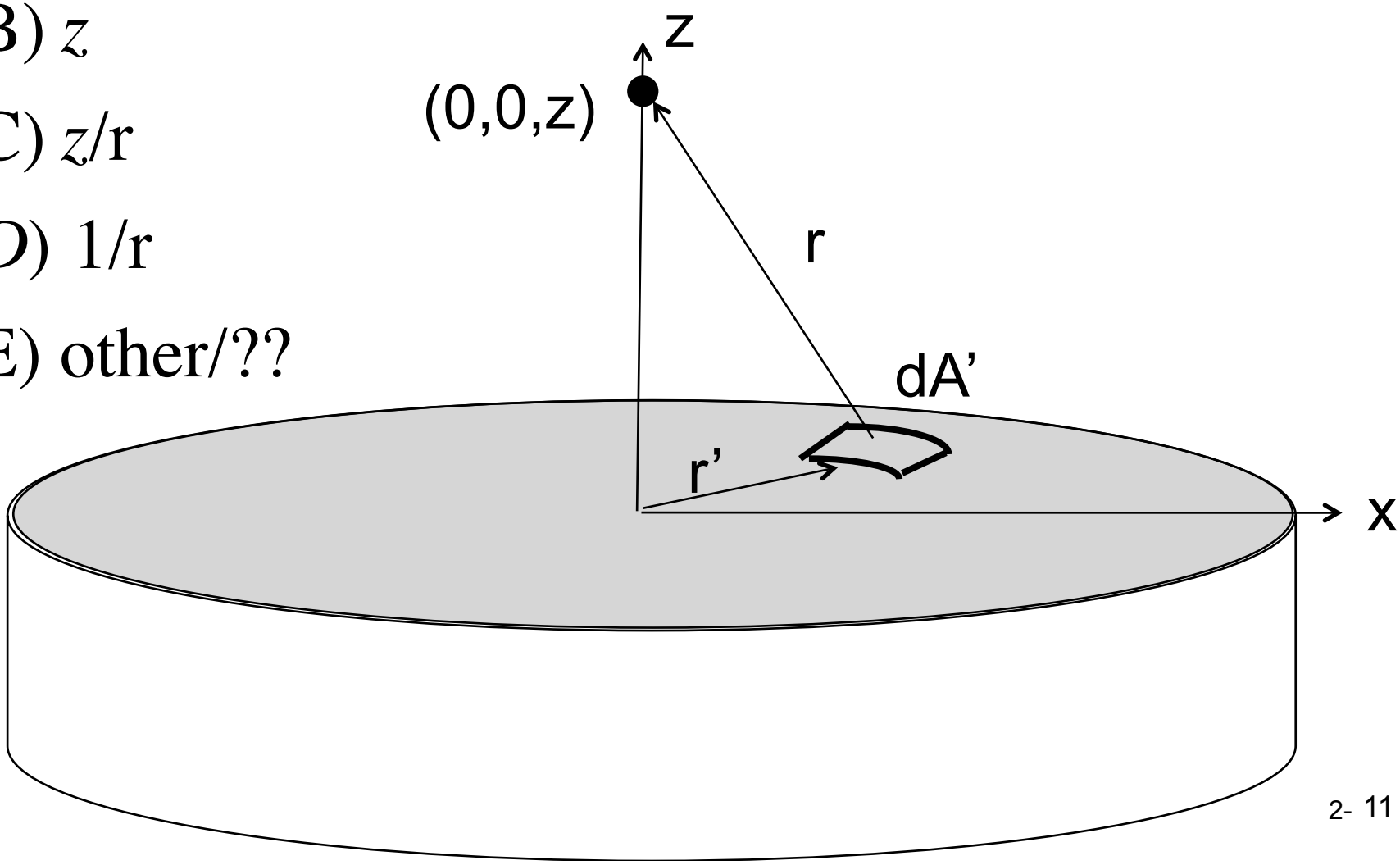
A) 1

B) z

C) z/r

D) 1/r

E) other/??



$$(\vec{\mathbf{g}})_z = -G\sigma \iint_{disc} \sigma \frac{z}{r^3}\, dA'$$

What is $\frac{1}{r^3} = ?$

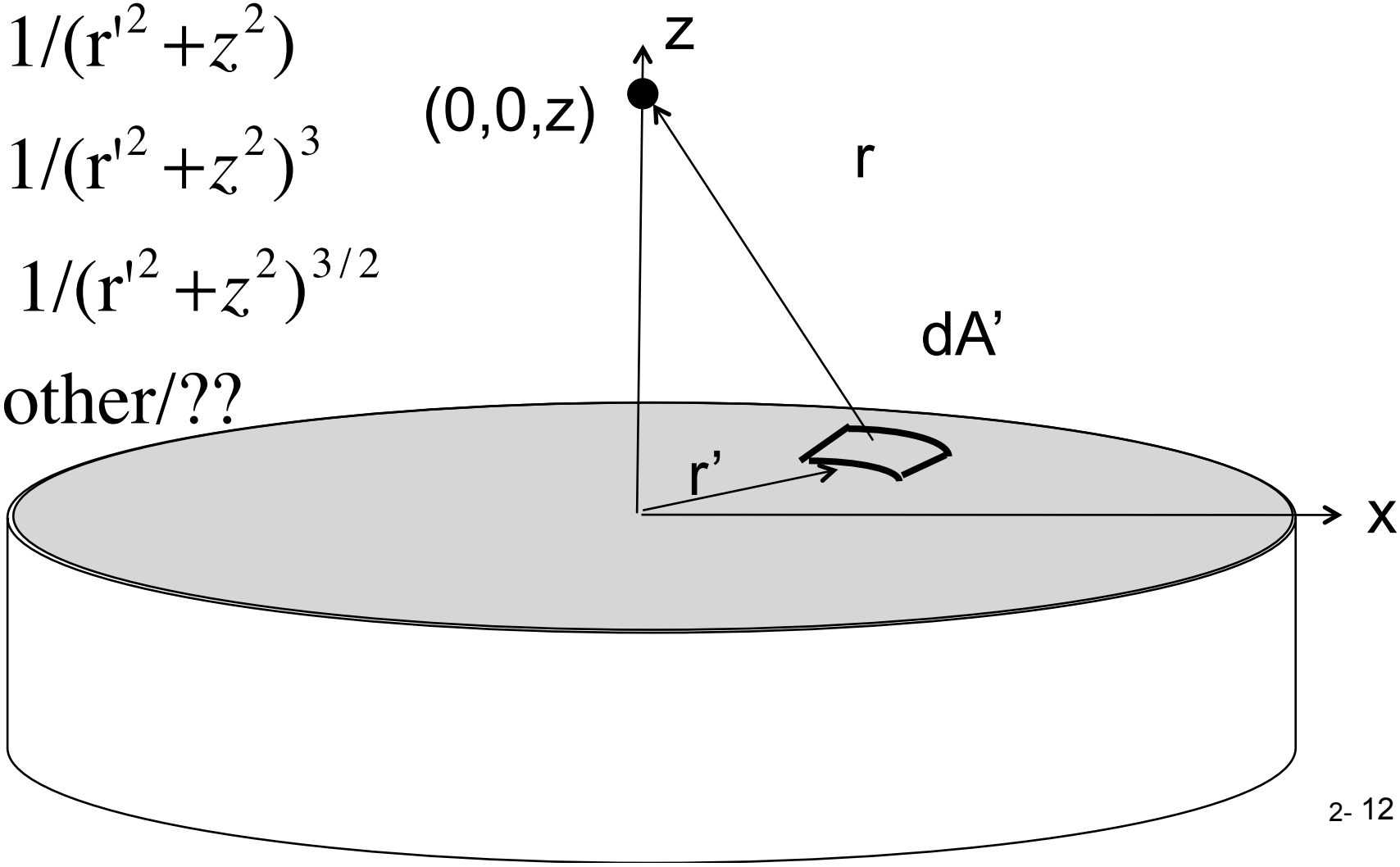
A) $1/(r'^3 + z^3)$

B) $1/(r'^2 + z^2)$

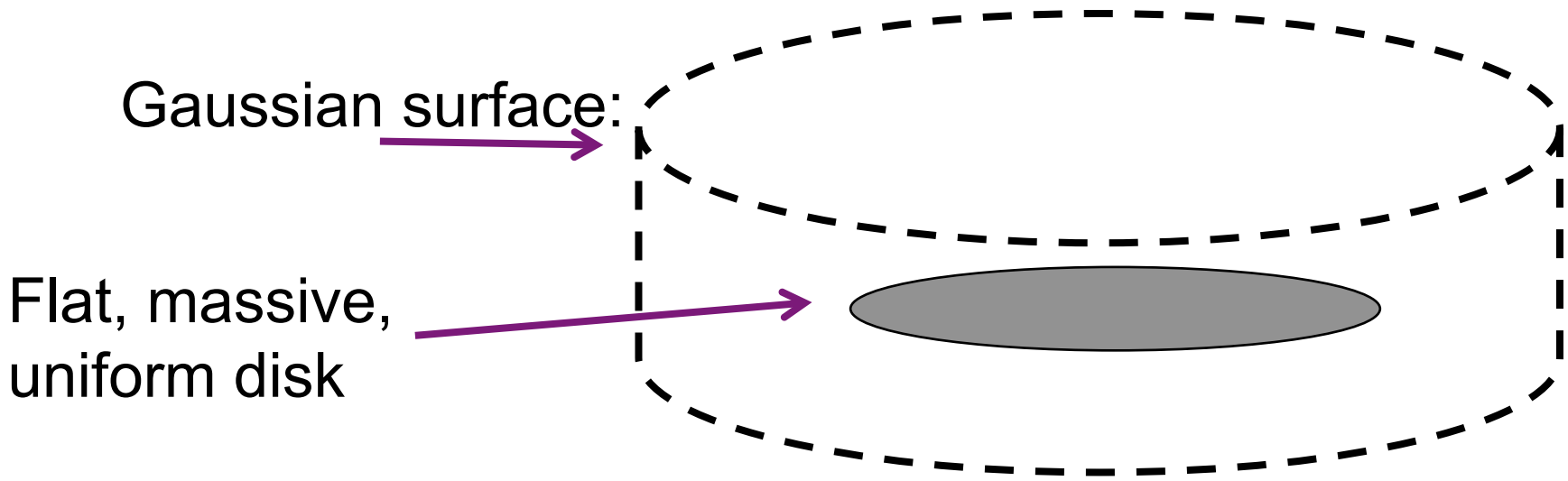
C) $1/(r'^2 + z^2)^3$

D) $1/(r'^2 + z^2)^{3/2}$

E) other/??



Last class we started calculating the g field above the center of a uniform disk.
Could we have used Gauss' law with the Gaussian surface depicted below?



A) Yes, and it would have made the problem easier!!!
B) Yes, but it's not easier.
C) No, Gauss' law applies, but it would not have been *useful* to compute "g"
D) No, Gauss' law would not even apply in this case

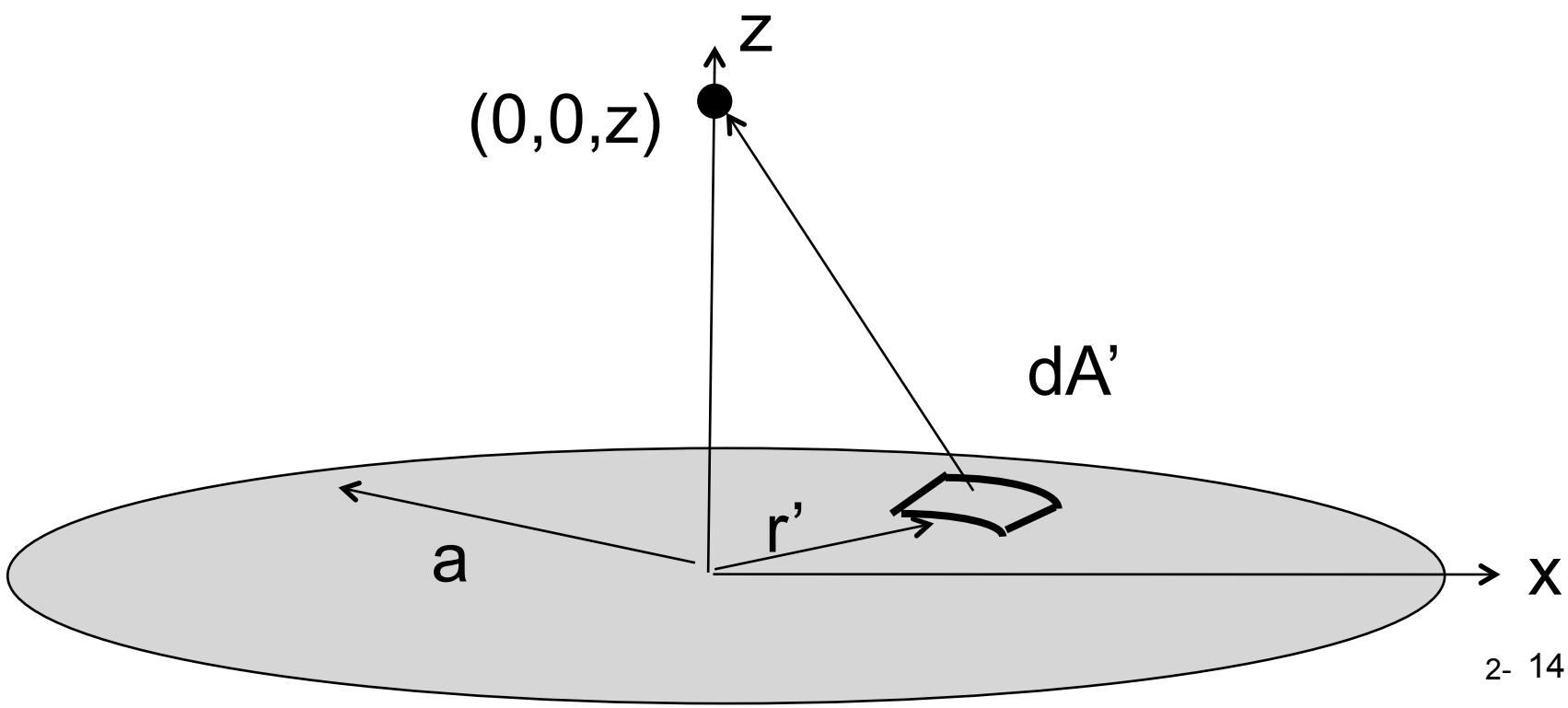
Last class we started working out

$$(\vec{\mathbf{g}})_z = -G\sigma \iint_{disc} \frac{z}{(r'^2 + z^2)^{3/2}}\, dA'$$

What is dA' here?

A) $\pi r'^2$ B) $dr'\, d\phi'$

C) $r'\, dr'\, d\phi'$ D) $a\, dr'\, d\phi'$ E) Something else!



Above the center of a massive disk (radius a), $g_z(0,0,z)$ is

$$(\vec{\mathbf{g}})_z = +2\pi G\sigma\, z\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{z}\right] \quad (\text{if } z > 0)$$

If z>>a, let's Taylor expand. What should we do first?

Above the disk, $g_z(0,0,z)$=

$$(\vec{\mathbf{g}})_z = +2\pi G\sigma\, z\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{z}\right] \quad (\text{if } z > 0)$$

If z>>a, let's Taylor expand. What should we do first?

A) Find d/dz of this whole expression, and evaluate it at z=0

B) Rewrite $\frac{1}{\sqrt{a^2+z^2}} = \frac{1}{a\sqrt{1+(z/a)^2}}$ and then use the "binomial" expansion $(1+\varepsilon)^n \approx (1+n\varepsilon+\ldots)$

C) Rewrite $\frac{1}{\sqrt{a^2+z^2}} = \frac{1}{z\sqrt{1+(a/z)^2}}$ and then use the "binomial" expansion.

D) Just expand $(a^2+z^2)^{-1/2} \approx a^2 - (1/2)z^2 + \ldots$

The binomial expansion is:

$$(1+\varepsilon)^n \approx 1 + n\varepsilon + \ldots$$

On Tuesday, a clicker question suggested we might try :

$$(a^2 + z^2)^{-1/2} \overset{?}{\approx} a^2 - (1/2)z^2 + \ldots$$

When z is small. We didn't talk about it. So, what do you think? Is that…

A) It's fine, it's correct to all orders, it's the binomial expansion!
B) Correct, but only to leading order, it will fall apart in the next term
C) Utterly false, even to leading order.

Given that $\vec{\mathbf{F}}_{\text{grav, point masses}} = -G\dfrac{M_1 M_2}{r^2}\hat{r}$

What should we write for P.E.: $\mathrm{U(r)}_{\text{for point M2 near point M1}}$

A) $\mathrm{U(r)} = -G\dfrac{M_1 M_2}{r}$

B) $\mathrm{U(r)} = +G\dfrac{M_1 M_2}{r}$

C) $\mathrm{U(r)} = -G\dfrac{M_1 M_2}{r^3}$

D) $\mathrm{U(r)} = +G\dfrac{M_1 M_2}{r^3}$

E) Something totally different,

this is 3-D spherical coordinates!!

g tutorial

A) Done with side 1 of part 1

B) Done with side 2 of part 1

C) Done with side 1 of part 2

D) Done with side 2 of part 2

E) Done with the question below too!

To think about if you're done:
For the "massive disk" problem we just worked out, set up the integral to find $\Phi(0,0,z)$

- Do you expect $g = -d\Phi(0,0,z)/dz$ here?

Gauss' law:

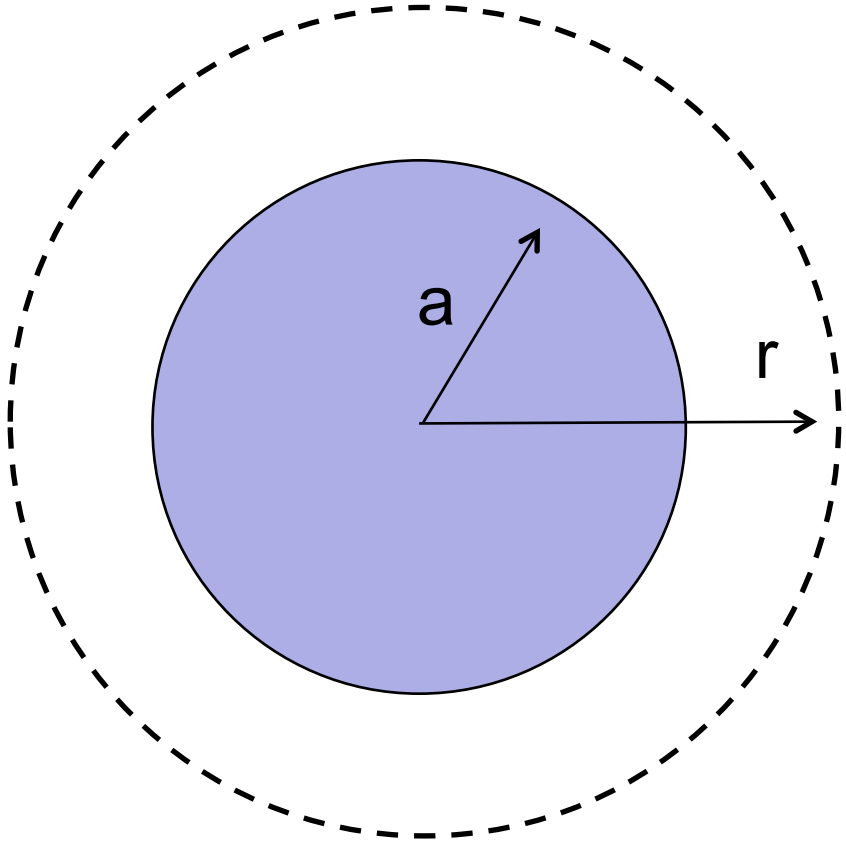
For electricity : $\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q_{enclosed} / \varepsilon_0$

For gravity : $\oiint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}} = -4\pi G M_{enclosed}$

The uniform solid sphere has mass density ρ, and radius a.

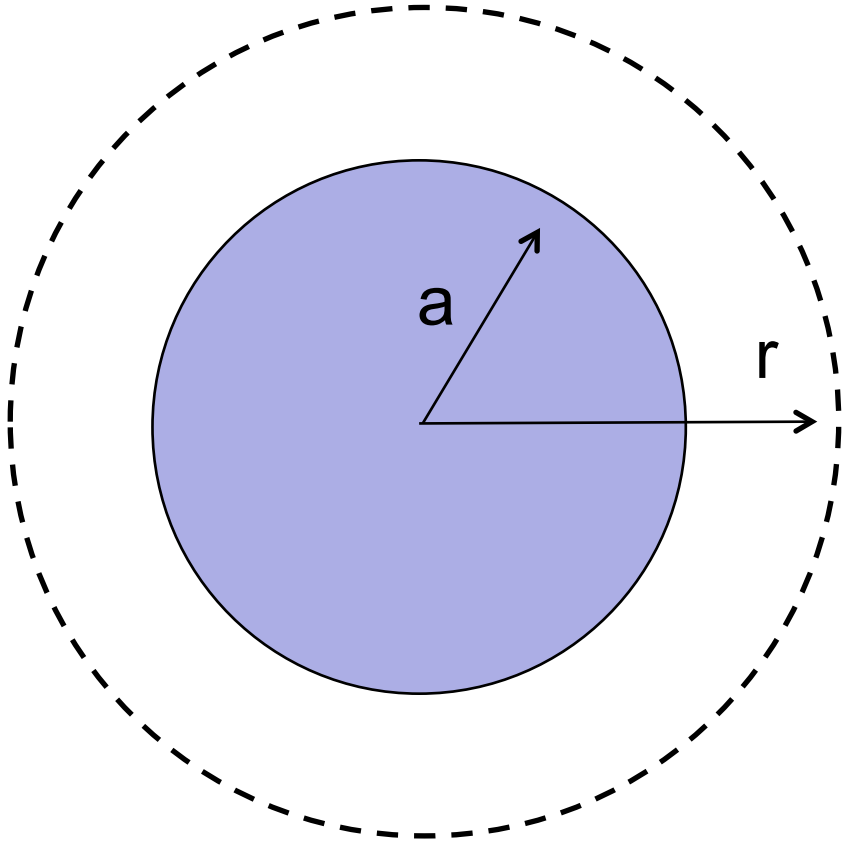
What is Menclosed, for r>a?

A) $4\pi r^2\rho$
B) $4\pi a^2\rho$
C) $(4/3)\,\pi r^3\rho$
D) $(4/3)\,\pi a^3\rho$
E) Something else entirely!



The uniform solid sphere has mass density ρ, and radius a.

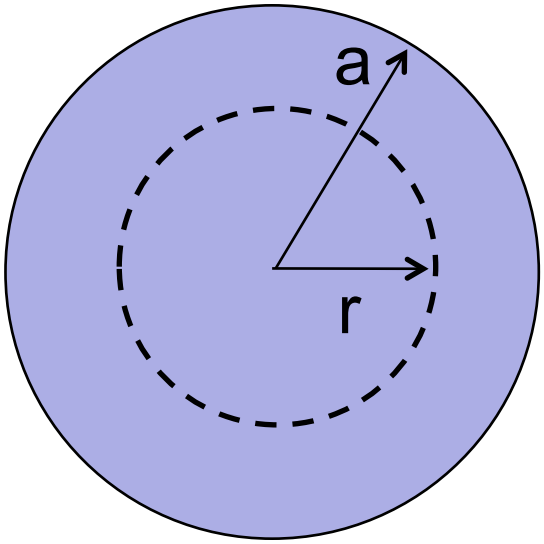
What is $\oiint_{\substack{\text{Gaussian}\\ \text{surface}}} \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}}$?



A) $-g\ 4\pi r^2$
B) $-g\ 4\pi a^2$
C) $-g\ (4/3)\ \pi r^3$
D) $-g\ (4/3)\ \pi a^3$
E) Something else entirely! (e.g, signs!)

For gravity : $\oiint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}} = -4\pi G M_{enclosed}$

The uniform solid sphere has mass density ρ, and radius a.



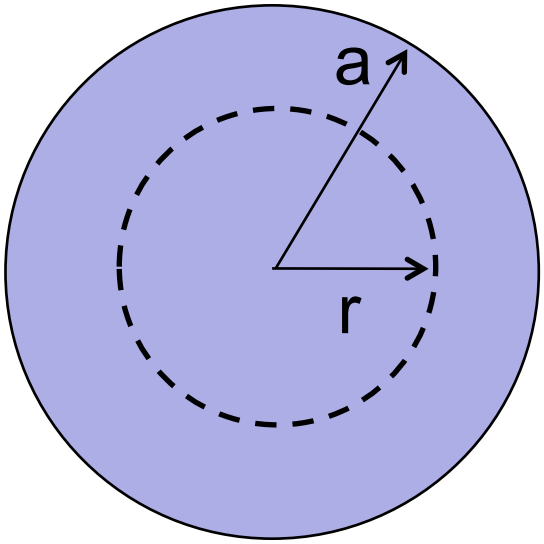
What is Menclosed, for r<a?

A) $4\pi r^2\rho$
B) $4\pi a^2\rho$
C) $(4/3)\ \pi r^3\rho$
D) $(4/3)\ \pi a^3\rho$
E) Something else entirely!

The uniform solid sphere has mass density ρ, and radius a.

What is $\oiint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}}$?

Gaussian surface



A) $-g\ 4\pi r^2$
B) $-g\ 4\pi a^2$
C) $-g\ (4/3)\ \pi r^3$
D) $-g\ (4/3)\ \pi a^3$
E) Something else entirely! (e.g, signs!)

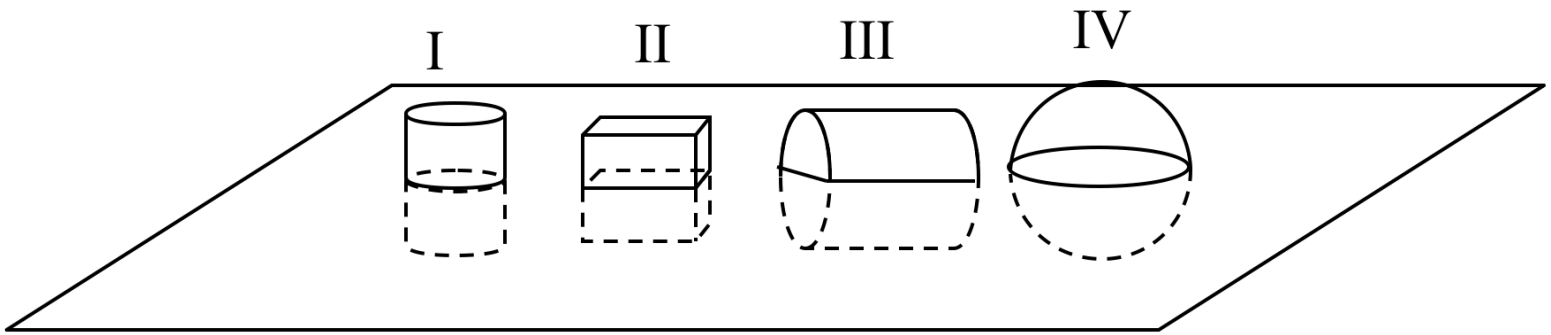
Consider these four closed gaussian surfaces, each of which straddles an infinite sheet of constant areal mass density.

The four shapes are

I: cylinder II: cube III: cylinder IV: sphere

For which of these surfaces does gauss's law,

$$\oiint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}} = -4\pi G M_{\text{enclosed}}$$ hold ?



A) All B) I and II only C) I and IV only D) I, II and IV only

E) Some other combo

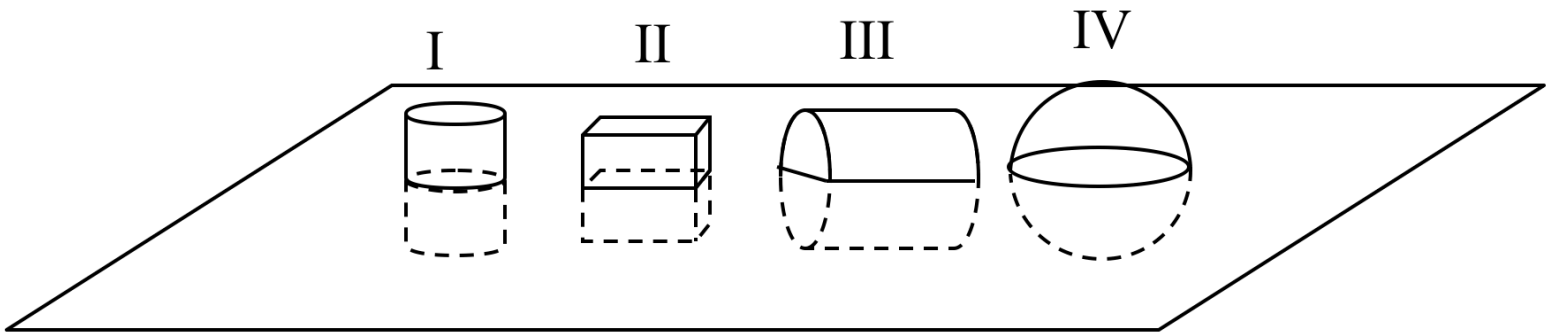
Consider these four closed gaussian surfaces, each of which straddles an infinite sheet of constant areal mass density.

The four shapes are

I: cylinder II: cube III: cylinder IV: sphere

For which of these surfaces does gauss's law,

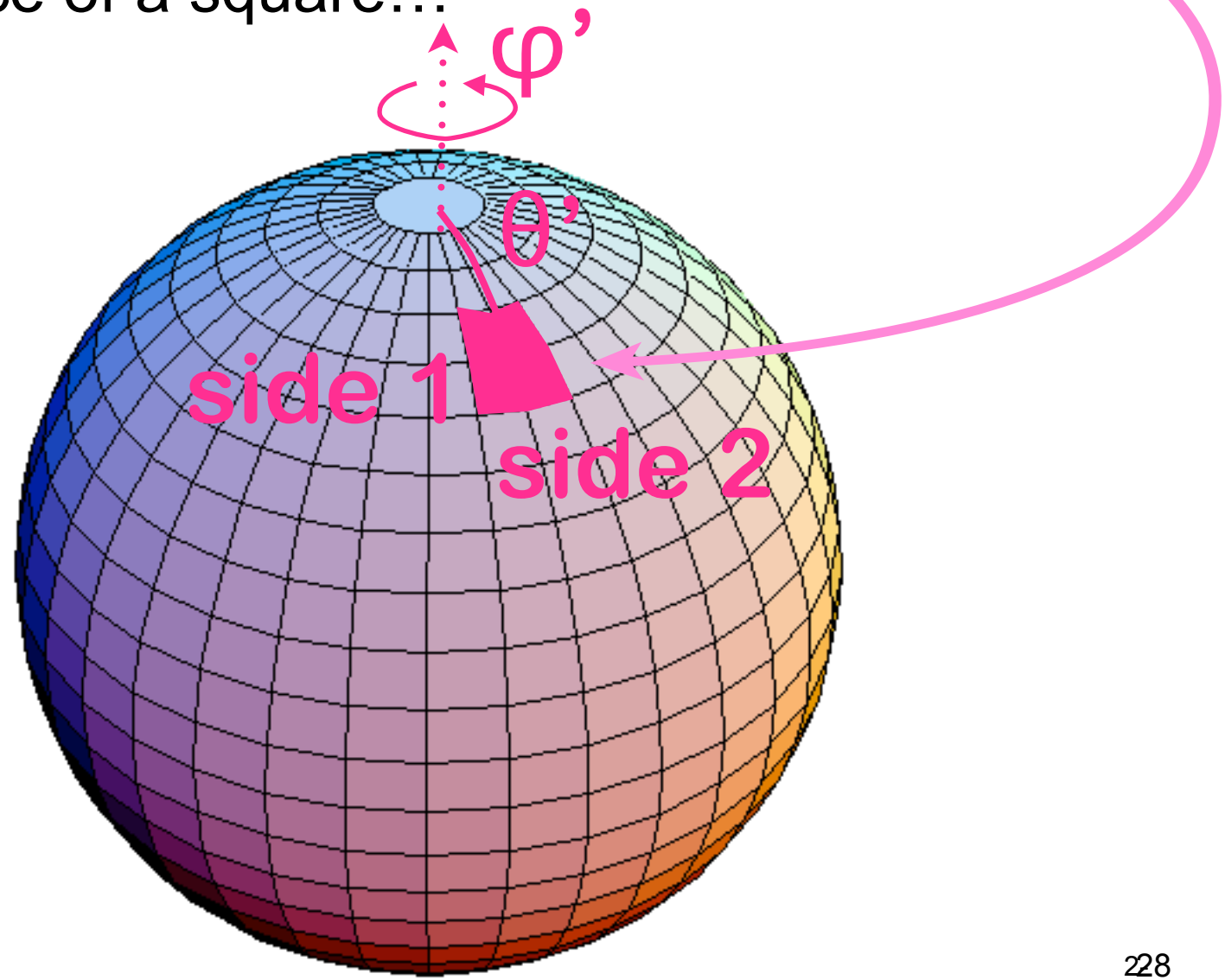
$\oiint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}} = -4\pi G M_{\text{enclosed}}$ help us find g near the surface??



A) All B) I and II only C) I and IV only D) I, II and IV only

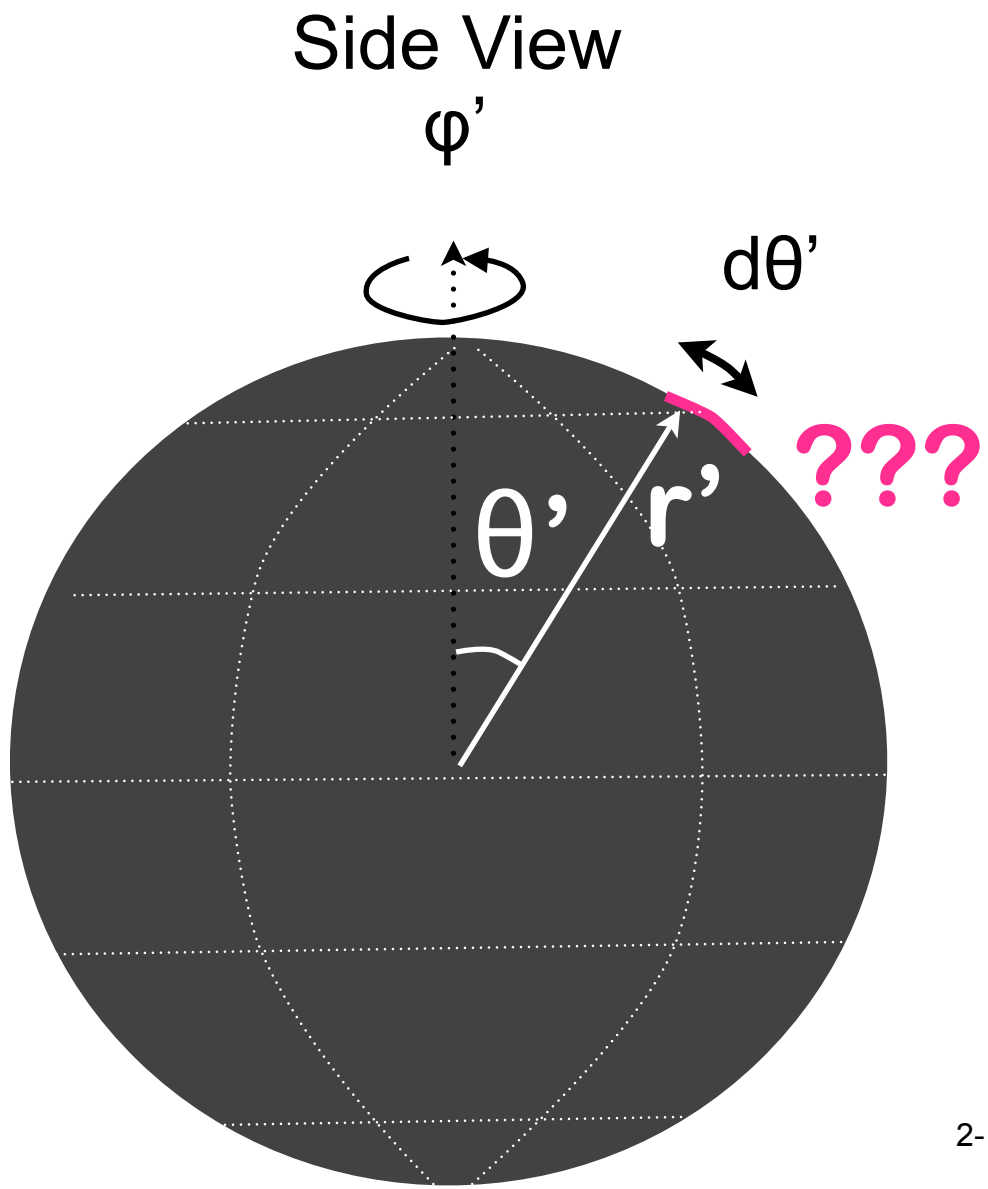
E) Some other combo

The next 2 clicker questions involve finding the area of a little square of area on the surface of a square…



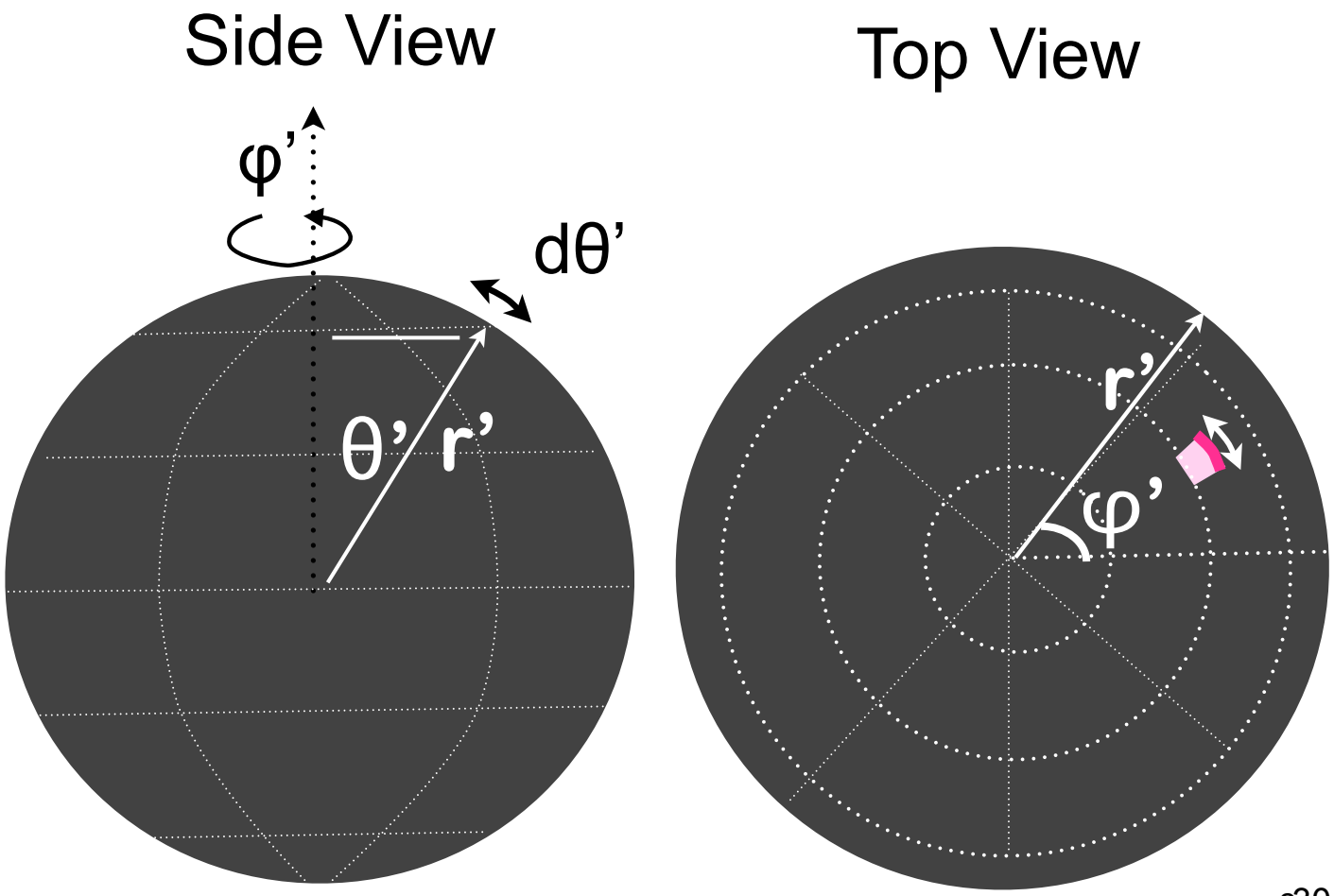
What is the length of the highlighted edge (side 1) of this area on the surface of a sphere of radius r'?

A) dθ'
B) dφ'
C) r'dθ'
D) r'dφ'
E) r' sinθ' dθ'



What is the length of the highlighted edge (side 2) of this area on the surface of a sphere of radius r'?

A) r' sinθ' dθ'
B) r' sinθ' dφ'
C) r'dθ'
D) r'dφ'
E) ???



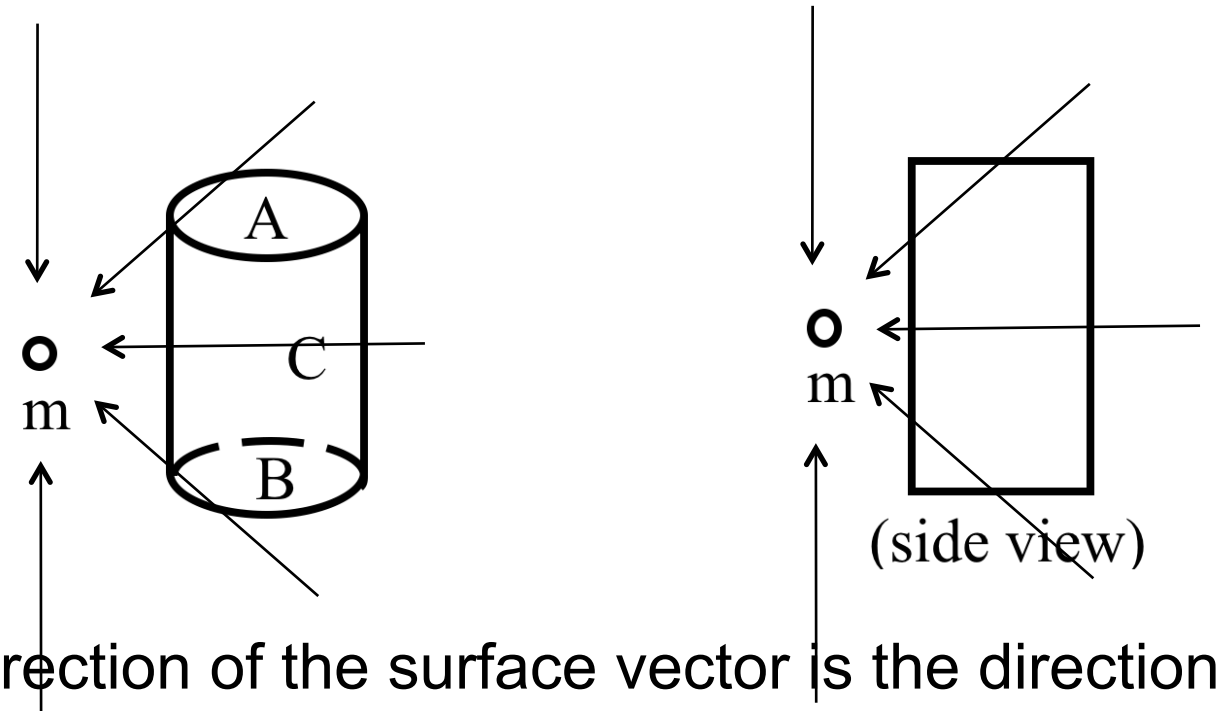
A point mass m is near a closed cylindrical gaussian surface. The closed surface consists of the flat end caps (labeled A and B) and the curved barrel surface (C). What is the sign of $\iint_C \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}}$ through surface C?

A) + B) - C) zero D) ????

(the direction of the surface vector is the direction of the *outward* normal.)

A point mass m is near a closed cylindrical gaussian surface. The closed surface consists of the flat end caps (labeled A and B) and the curved barrel surface (C). What is the sign of $\iint_C \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}}$ through surface C?

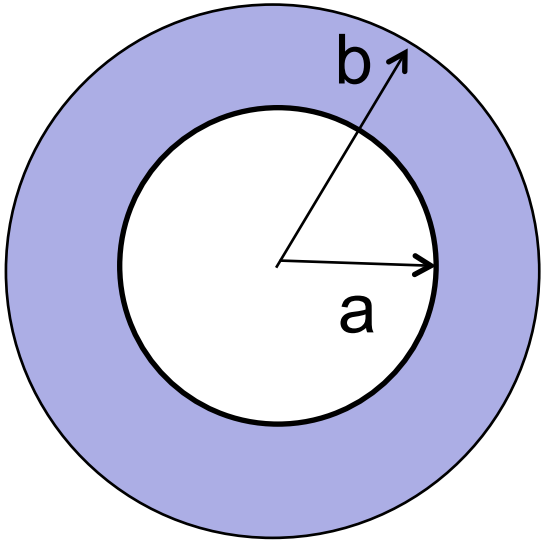
A) + B) - C) zero D) ????



(the direction of the surface vector is the direction of the *outward* normal.)

The spherical *shell* has mass density ρ, inner radius a, outer radius b.

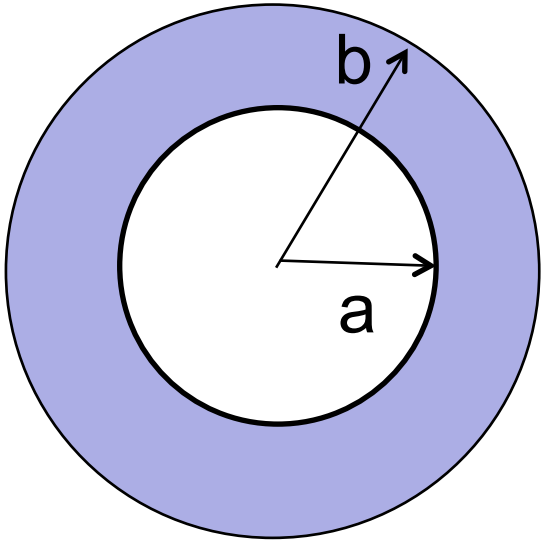
How does the gravitational potential ϕ depend on r, for r>b?



A) $\sim r$
B) $\sim r^2$
C) $\sim r^{-1}$
D) $\sim r^{-2}$
E) Something else entirely!

The spherical *shell* has mass density ρ, inner radius a, outer radius b.

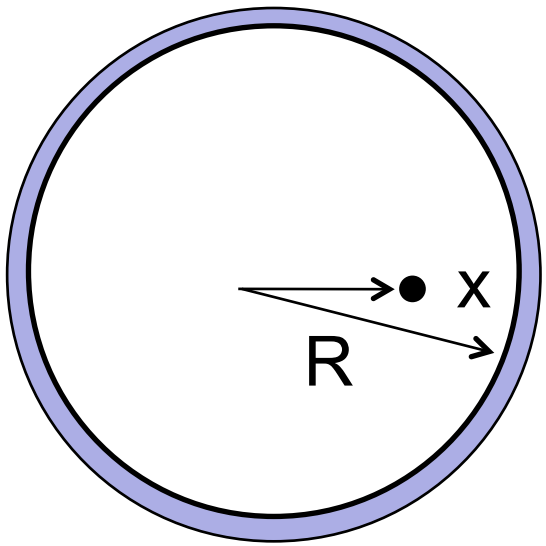
How does the gravitational potential ϕ depend on r, for r>a?



A) $\sim r$
B) $\sim r^2$
C) $\sim r^{-1}$
D) $\sim r^{-2}$
E) Something else entirely!

You have a THIN spherical uniform mass shell of radius R, centered on the origin.
What is the g-field at a point “x” near an edge, as shown?

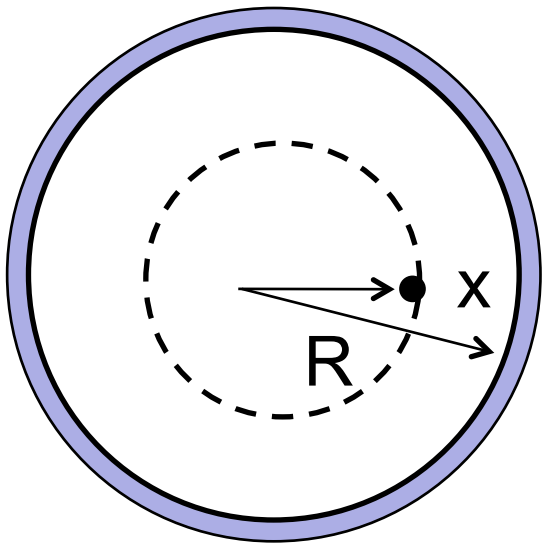
A) zero
B) nonzero, to the right
C) nonzero, to the left
D) Other/it depends!



Challenge question: Can you prove/derive your answer by thinking about Newton’s law of gravity directly?

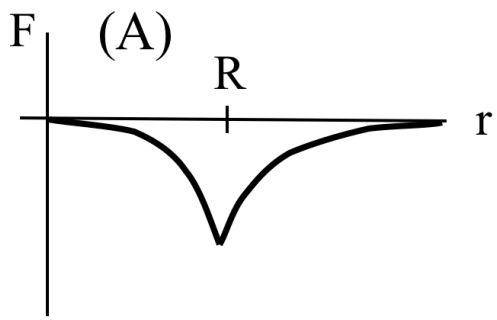
You have a THIN spherical uniform mass shell of radius R, centered on the origin.
What is the g-field at a point “x” near an edge, as shown?

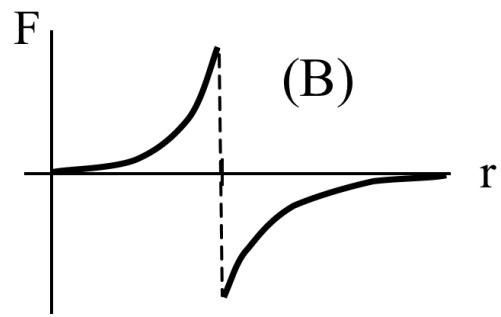
A) zero
B) nonzero, to the right
C) nonzero, to the left
D) Other/it depends!

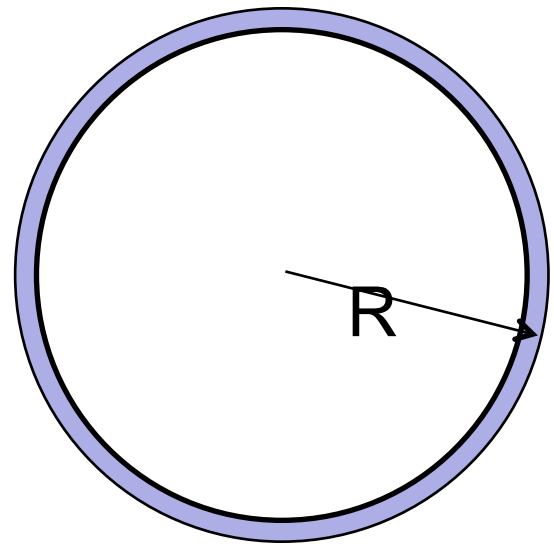


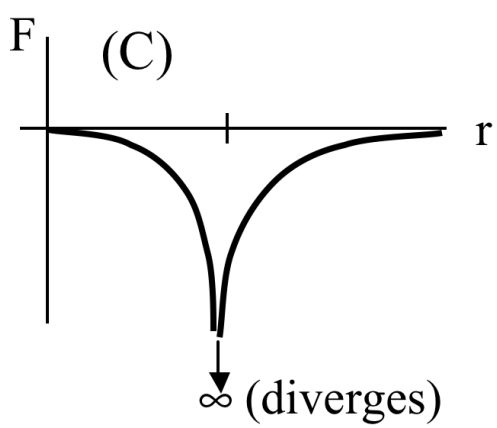
Challenge question: Can you prove/derive your answer by thinking about Newton’s law of gravity directly?

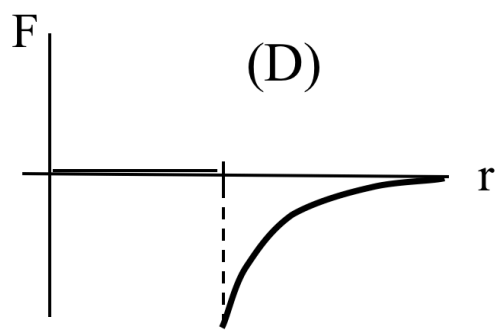
A test mass m moves along a straight line toward the origin, passing through a THIN spherical mass shell of radius R, centered on the origin.
Sketch the force F on the test mass vs. position r.







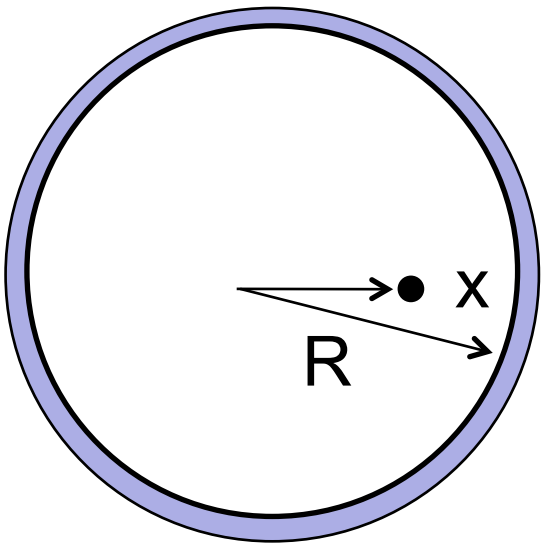




E) Other!

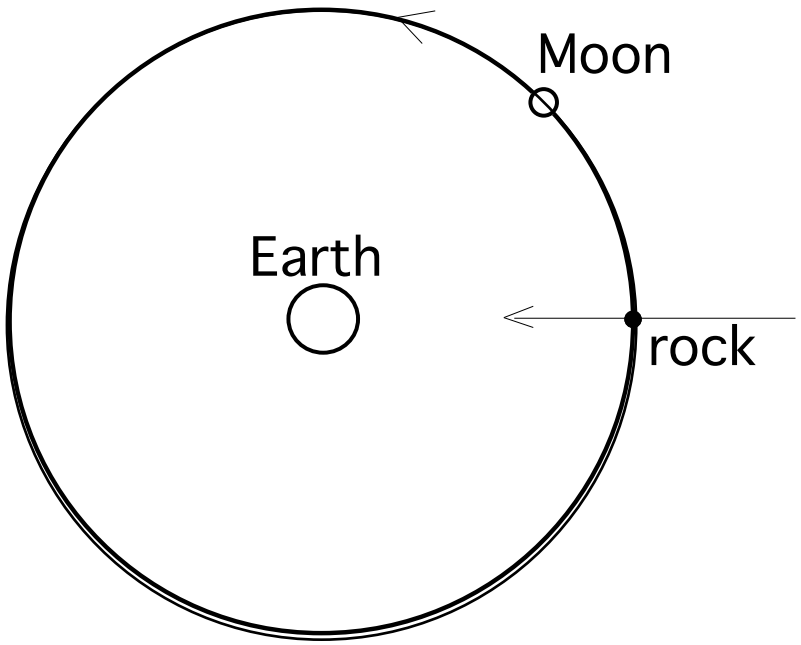
Assuming that $U(\infty)=0$, what can you say about the potential at point x?

A) It is zero (everywhere inside the sphere)
B) It is positive and constant everywhere inside the sphere
C) It is negative, and constant everywhere inside the sphere
D) It varies within the sphere
E) Other/not determined



Challenge question: What is the formula for $U(0)-U(\infty)$?

A rock is released from rest at a point in space far beyond the orbit of the Moon. It falls toward the Earth and crosses the orbit of the Moon.
When the rock is the same distance from the Earth as the Moon, the acceleration of the rock is:
(Ignore the gravitational force between the rock and the Moon.)



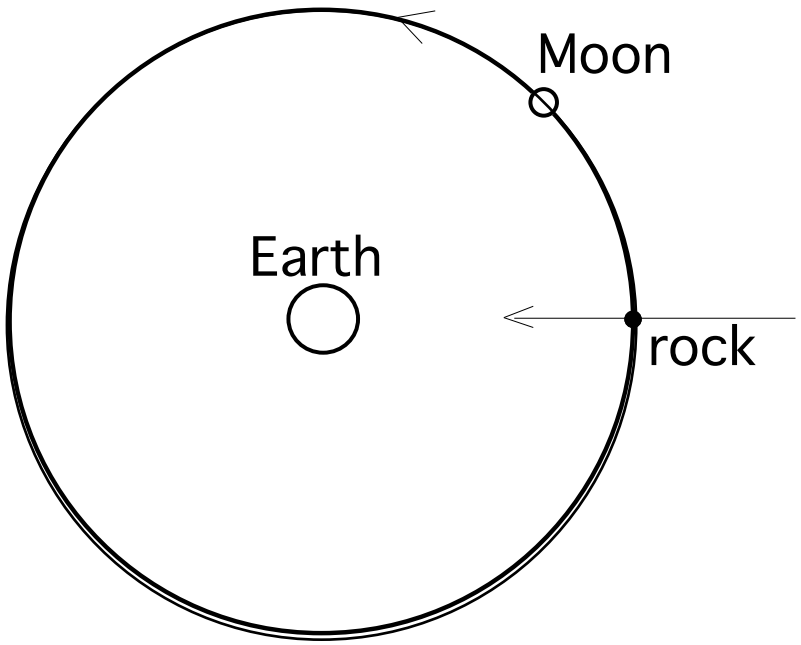
A) greater than
B) smaller than
C) the same as

the acceleration of the Moon.

As the rock falls toward the Earth, its acceleration is:

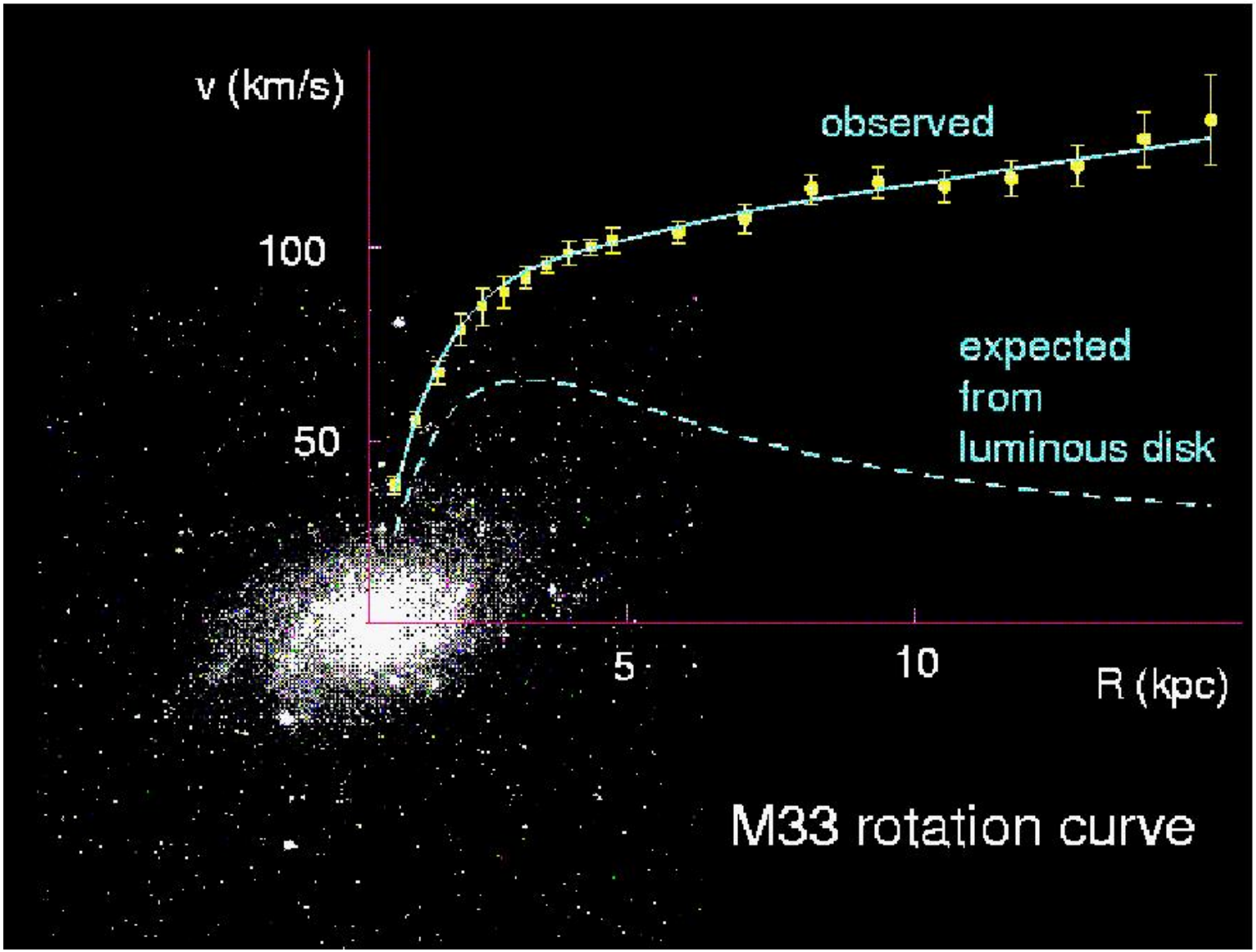
A) constant.

B) not constant.



Summary: Gauss' law

For gravity : $\oiint \vec{\mathbf{g}} \cdot d\vec{\mathbf{A}} = -4\pi G M_{enclosed}$



Gravity Anomaly Map from GRACE satellite

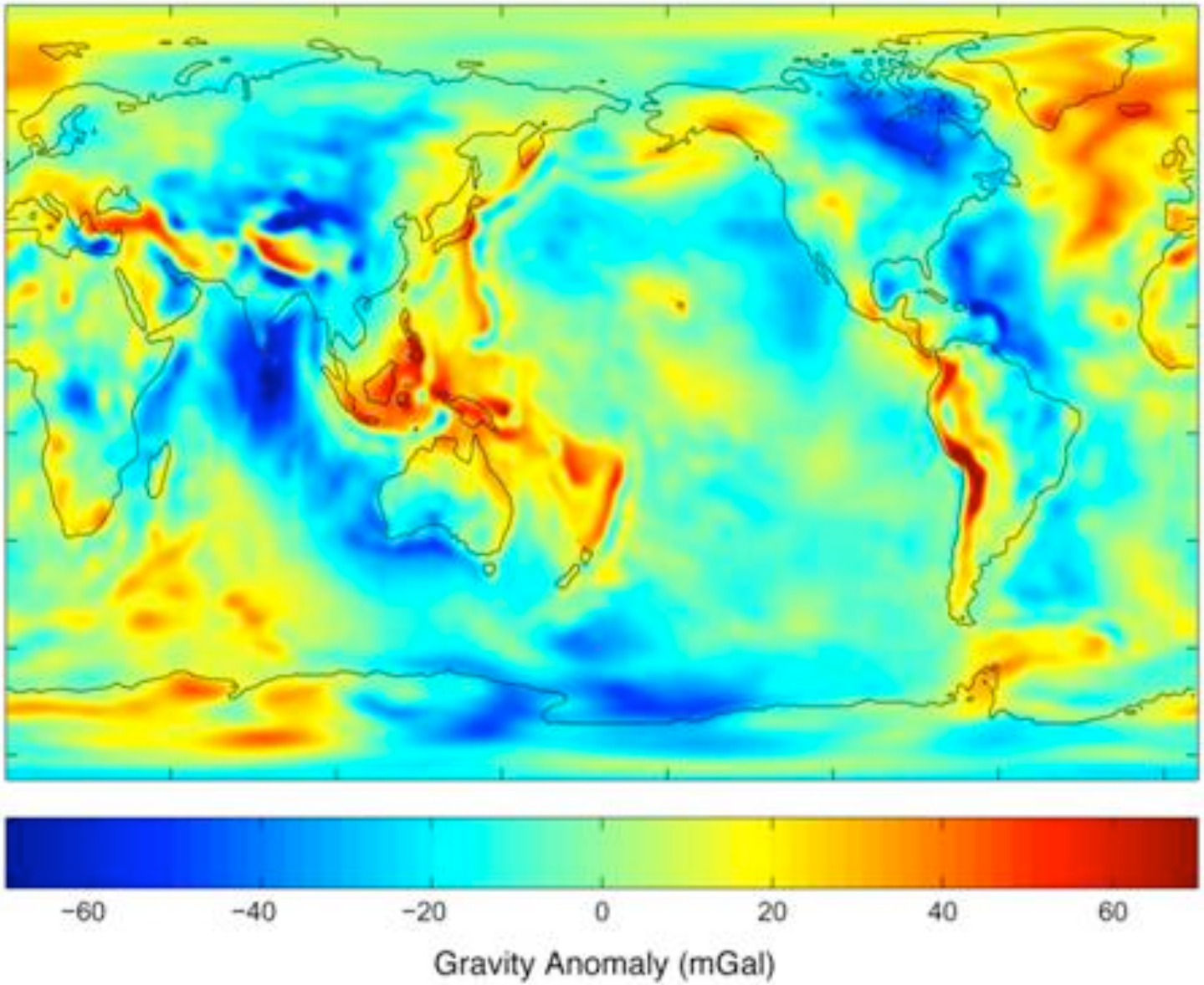


Image from U Texas Center for Space Research and NASA

Gravity Anomaly Map from GRACE satellite

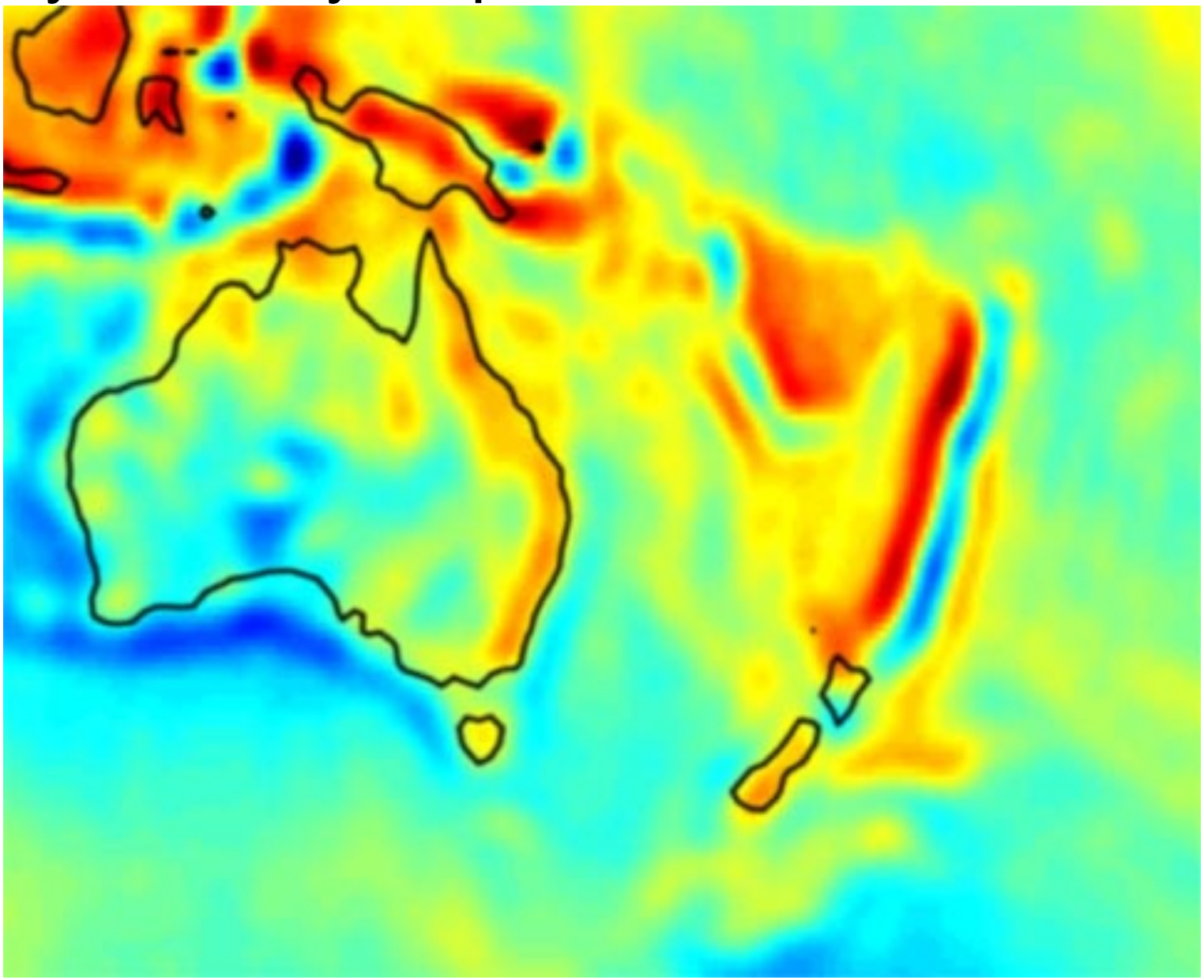
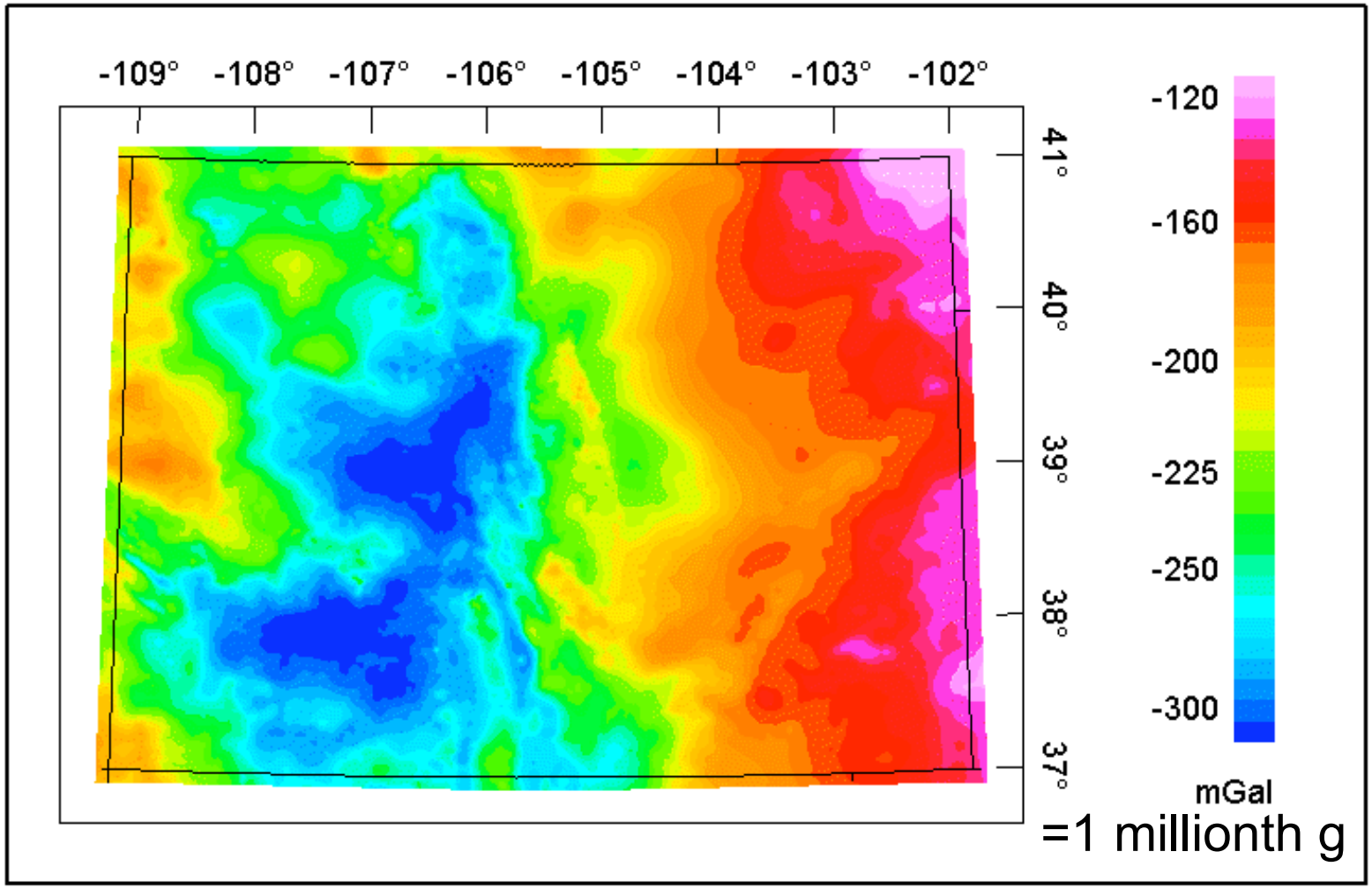


Image from U Texas Center for Space Research and NASA

Gravity Anomaly Map for Colorado Bouguer model

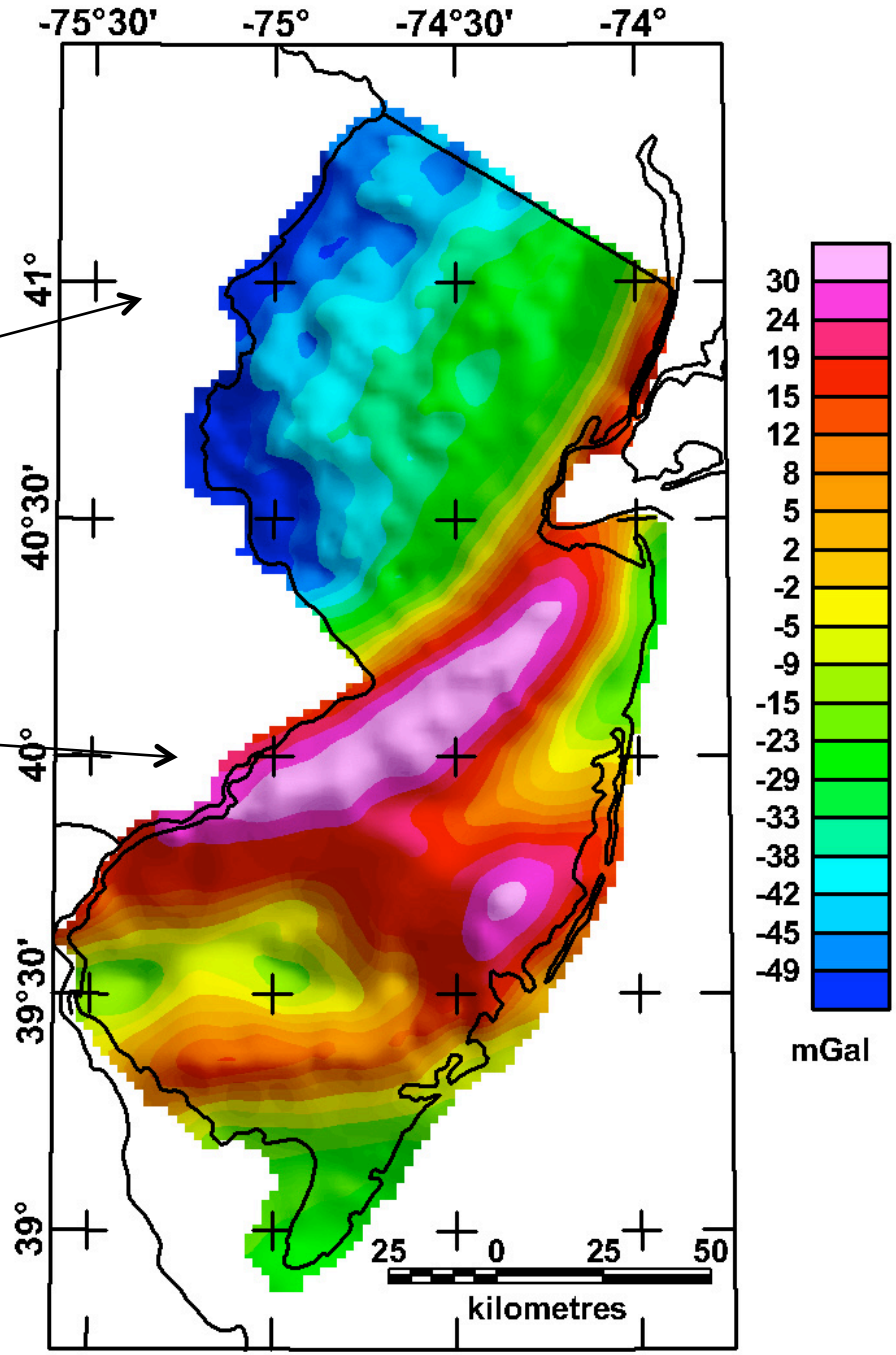


USGS

Gravity Anomaly Map for New Jersey

Appalachians

Dense Basalt Rock



USGS

Summary

$$\vec{\mathbf{F}}_{\text{grav, points}} = -G\frac{M_1 M_2}{r^2}\hat{r}$$

$$\vec{\mathbf{F}}_{\text{grav, M2 a point}} = -G\iiint_{V'}\frac{\rho(r')}{r^2}dV'\,\hat{r}$$

$$\vec{\mathbf{g}} = \vec{\mathbf{F}}_{\text{grav, on point m}}/m$$

$$\text{PE}: \quad \text{U(r)}_{\text{for point M2 near point M1}} = -G\frac{M_1 M_2}{r}$$

$$\text{Grav. potential}: \quad \Phi\text{(r)}_{\text{near point M1}} = -G\frac{M_1}{r}$$

$$\text{Gauss'law}: \quad \oiint \vec{\mathbf{g}}\cdot d\vec{\mathbf{A}} = -4\pi G M_{enclosed}$$

Summary

$$\vec{\mathbf{F}}_{\text{grav, 2 point masses}} = -G\frac{M_1 M_2}{r^2}\hat{r}$$

$$\vec{\mathbf{F}}_{\text{grav, M2 a point}} = -GM_2\iiint_{V'}\frac{\rho(r')}{r^2}dV'\,\hat{r}$$

$$\vec{\mathbf{g}} = \vec{\mathbf{F}}_{\text{grav, on point m}}/m$$

$$\text{PE}: \quad \text{U(r)}_{\text{for point M2 near point M1}} = -G\frac{M_1 M_2}{r}$$

$$\text{Grav. potential} = \text{PE/m} = \Phi(\text{r})_{\text{near point M1}} = -G\frac{M_1}{r}$$

$$\Phi = -G\iiint_{V'}\frac{\rho(r')}{r}dV'$$

Oscillations

How many initial conditions are required
to fully determine the general solution
to a 2nd order linear differential equation?

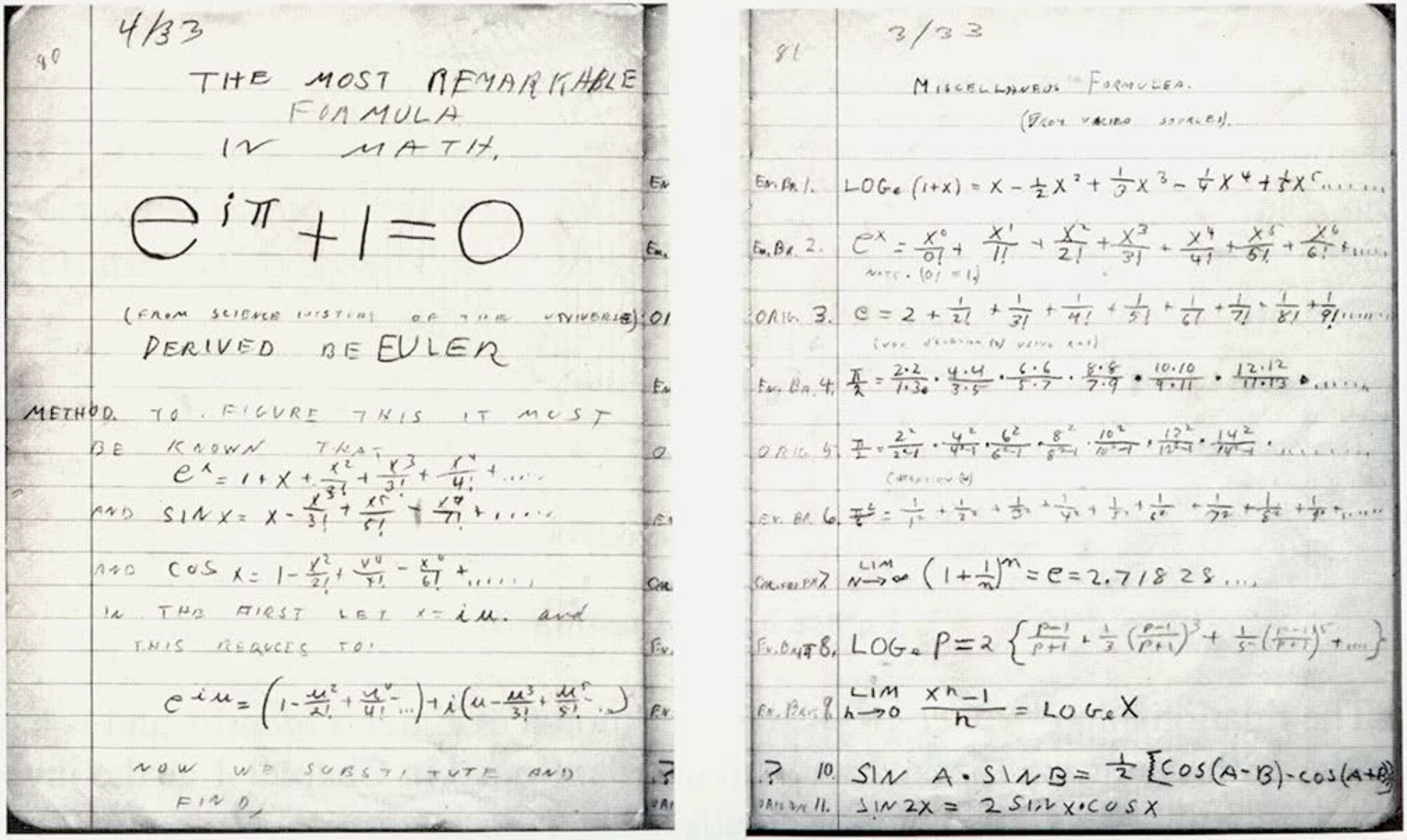
Since cos(ωt) and cos(-ωt) are both solutions of

$$\ddot{x}(t) = -\omega^2 x(t)$$

can we express the general solution as
$x(t)=C_1\cos(\omega t) + C_2\cos(-\omega t)$?

A) yes
B) no
C) ???/depends

Richard Feynman's notebook (Age 14)



4/33

THE MOST REMARKABLE FORMULA IN MATH.

$$e^{i\pi}+1=0$$

(FROM SCIENCE LISTING OF THE UNIVERSE)

DERIVED BE EULER

METHOD. TO FIGURE THIS IT MUST BE KNOWN THAT

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

AND $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

AND $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

IN THE FIRST LET $x = iu$. and THIS REDUCES TO:

$$e^{iu} = \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots\right) + i\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} \dots\right)$$

NOW WE SUBSTITUTE AND FIND,

3/33

MISCELLANEOUS FORMULEA.

(FROM VARIOUS SOURCES).

EX. PR. 1. $\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \dots$

EX. PR. 2. $e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$

NOTE. $(0! = 1)$

ORIG. 3. $e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \dots$

(VER. DERIVED BY VERY EAS.)

EX. PR. 4. $\frac{\pi}{2} = \frac{2\cdot2}{1\cdot3}\cdot\frac{4\cdot4}{3\cdot5}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{8\cdot8}{7\cdot9}\cdot\frac{10\cdot10}{9\cdot11}\cdot\frac{12\cdot12}{11\cdot13}\cdot\dots$

ORIG. 5. $\frac{\pi}{2} = \frac{2^2}{2^2-1}\cdot\frac{4^2}{4^2-1}\cdot\frac{6^2}{6^2-1}\cdot\frac{8^2}{8^2-1}\cdot\frac{10^2}{10^2-1}\cdot\frac{12^2}{12^2-1}\cdot\frac{14^2}{14^2-1}\cdot\dots$

(DERIVED BY)

EX. PR. 6. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \dots$

CONVERGENT 7. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\dots$

EX. PR. 8. $\log_e p = 2\left\{\frac{p-1}{p+1} + \frac{1}{3}\left(\frac{p-1}{p+1}\right)^3 + \frac{1}{5}\left(\frac{p-1}{p+1}\right)^5 + \dots\right\}$

EX. PR. 9. $\lim_{h\to 0}\frac{x^h-1}{h} = \log_e x$

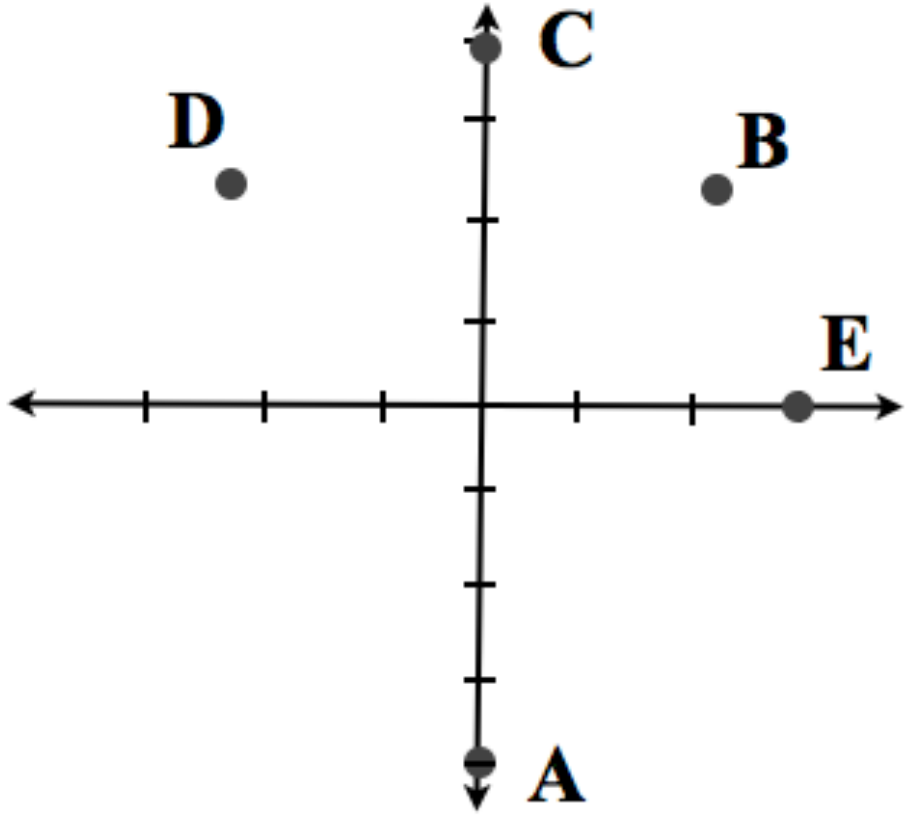
? 10. $\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

IDENTITY 11. $\sin 2x = 2\sin x \cdot \cos x$

Courtesy of the Archives, California, Institute of Technology.

Figure 3.1–3 Pages From Feynmann's Notebooks.

Which point below best represents $4e^{i3\pi/4}$ on the complex plane?



Challenge question: Keeping the general form $Ae^{i\theta}$, do any OTHER values of θ represent the SAME complex number as this? (If so, how many?)

On a white board, draw points showing

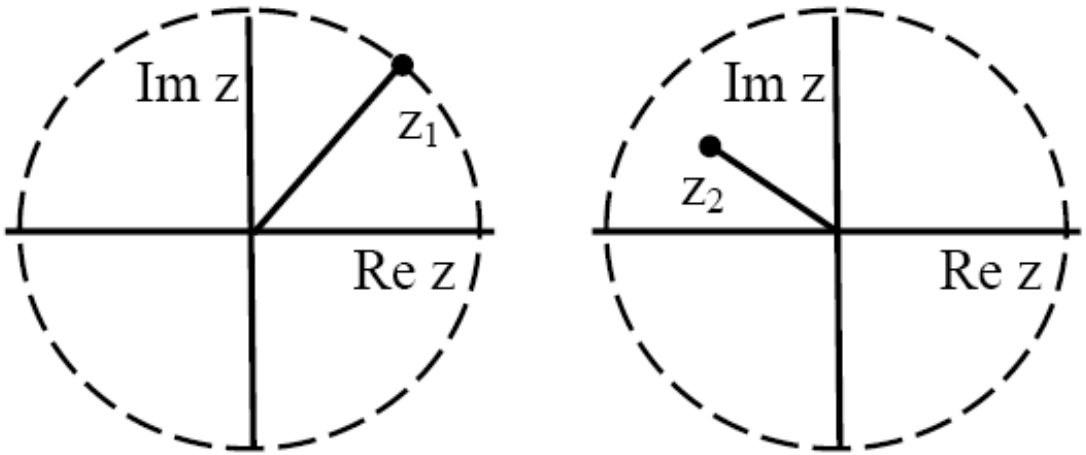
i) $e^{i\pi/6}$

ii) $e^{i\pi/3}$

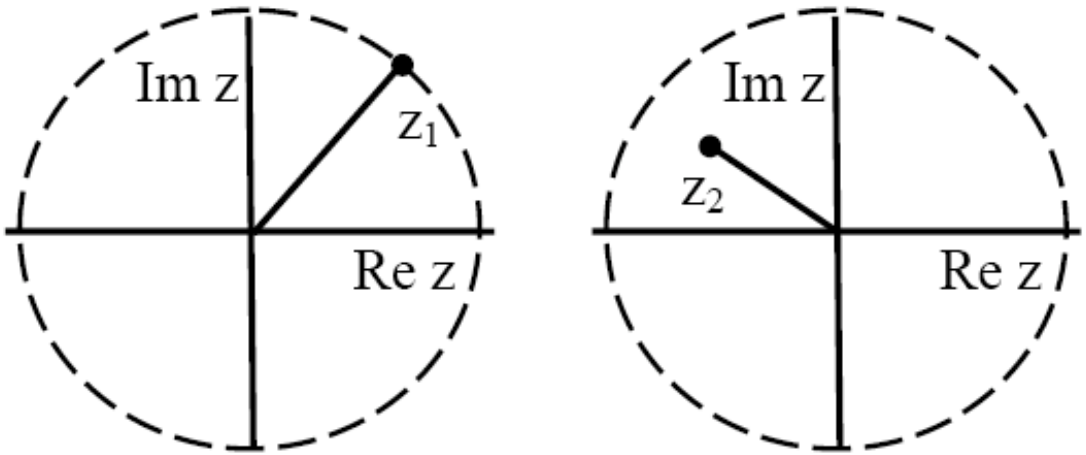
iii) $e^{i\pi/3}\ e^{i\pi/6}$

iv) $e^{i\pi/3}\ /\ e^{i\pi/6}$

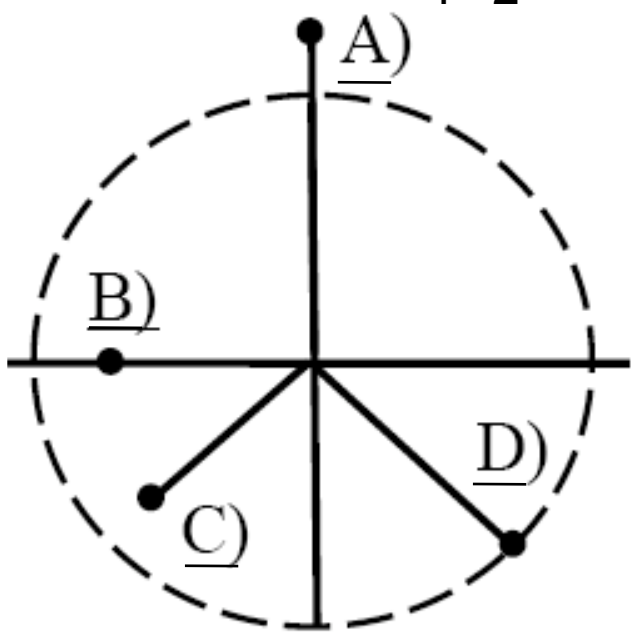
Consider two complex numbers, z_1 and z_2.
The dotted circle shows the unit circle, where $|z|=1$.



Consider two complex numbers, z_1 and z_2.
The dotted circle shows the unit circle, where $|z|=1$.

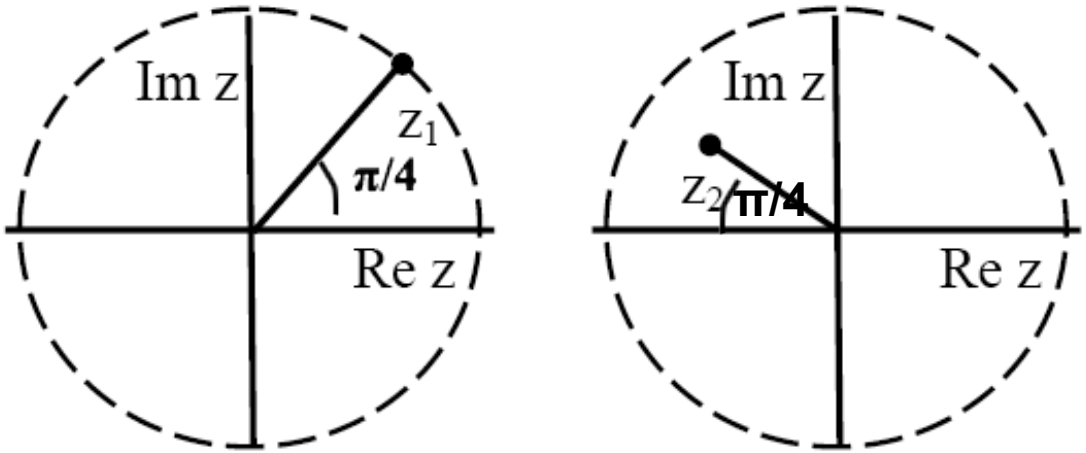


Which shows the product $z_1 z_2$?

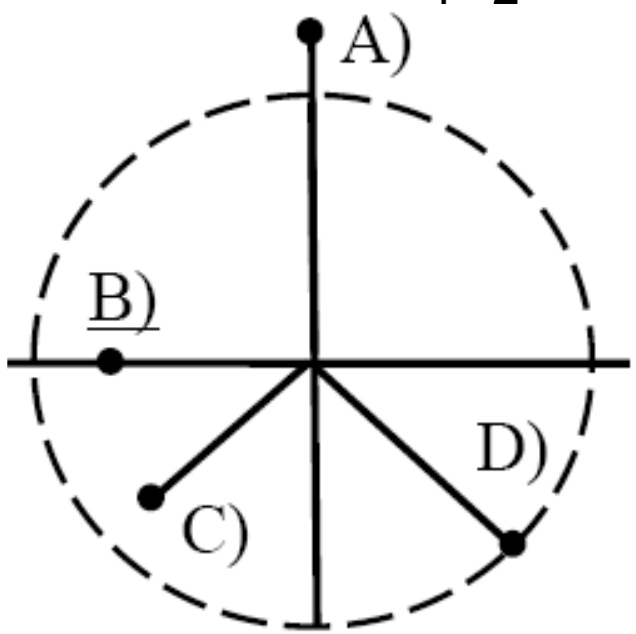


E) I have no idea

Consider two complex numbers, z_1 and z_2.
The dotted circle shows the unit circle, where $|z|=1$.



Which shows the product $z_1 z_2$?



E) I have no idea

What is 1/(1+i) in “polar” form?

A) $e^{i\,\pi/4}$

B) $e^{-i\,\pi/4}$

C) $0.5\,e^{i\,\pi/4}$

D) $0.5\,e^{-i\pi/4}$

E) Something else!

What is $(1+i)^2/(1-i)$

A) $e^{i\,\pi/4}$

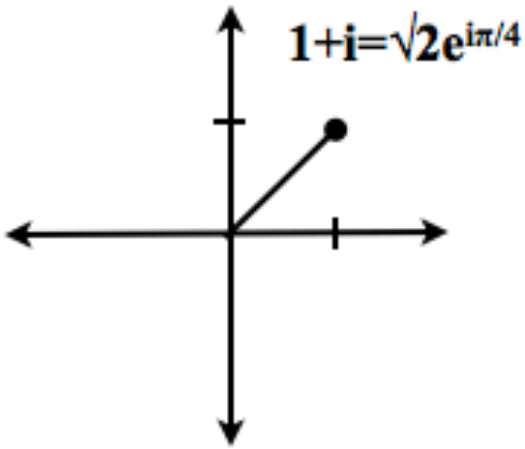
B) Sqrt[2] $e^{i\,\pi/4}$

C) $e^{i\,3\pi/4}$

D) Sqrt[2]$e^{i\,3\pi/4}$

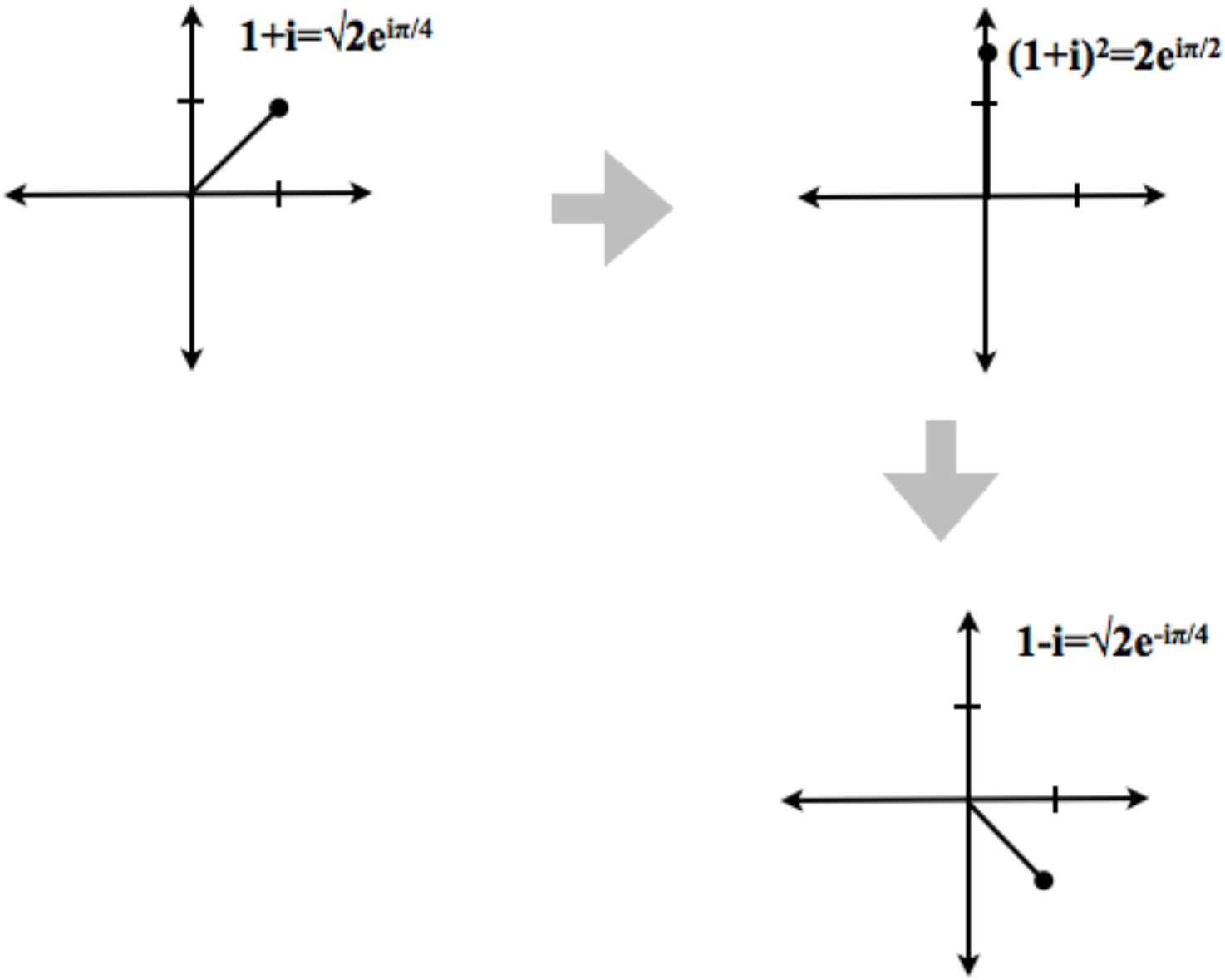
E) Something else!

What is $(1+i)^2/(1-i)$

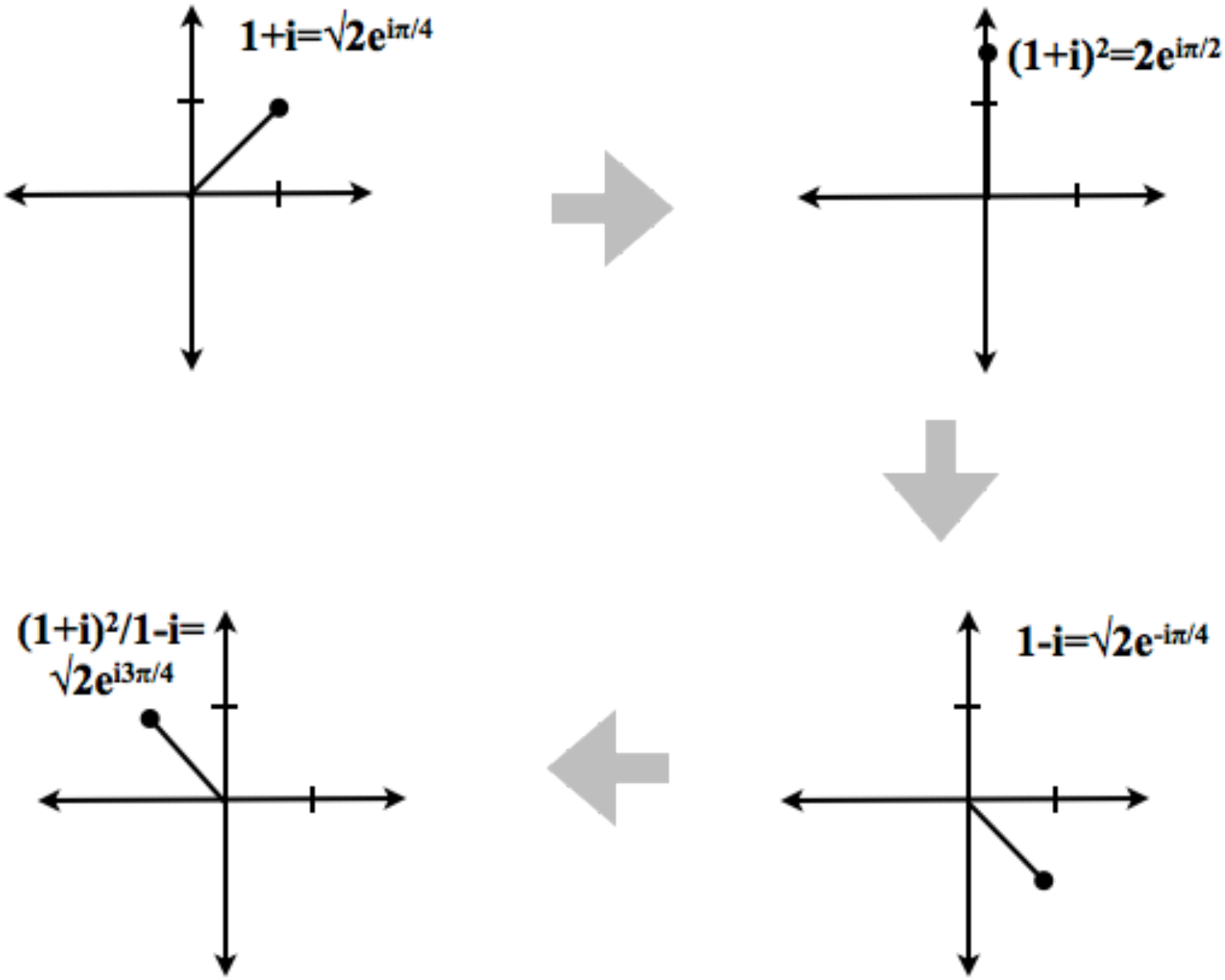


What is $(1+i)^2/(1-i)$

What is $(1+i)^2/(1-i)$



What is $(1+i)^2/(1-i)$



What is the general solution to the ODE $\ddot{x}(t) = -\omega^2 x(t)$ where ω is some (known) constant?

A) $x(t) = A\cos\omega t + B\sin\omega t$

B) $x(t) = Ce^{i\omega t} + De^{-i\omega t}$

C) $x(t) = A\cos\omega t + B\sin\omega t + Ce^{i\omega t} + De^{-i\omega t}$

D) None of these is fully general!

E) More than one of these is fine

Challenge question: Is there any OTHER general form for this solution?

Based on the pictures what is the period of motion of the block?

A) .2 s
B) .4 s
C) .6 s
D) .8 s
E) None of these/ not enough info!