## Problem List

## 0.1 Trig. Identity

**Given: Marino – Fall 2011**

Show that $\frac{1}{2}\sin(\frac{\pi}{2}+\alpha)+\frac{1}{2}\sin(\frac{\pi}{2}-\alpha)=\cos(\alpha)$.

## 0.2 Basic vector properties (Numeric)

**Given: Marino – Fall 2011**

(a) Given the two vectors $\vec{B} = \{2, 2, 1\}$ and $\vec{C} = \{1, 2, -6\}$, what is $\vec{B} \times \vec{C}$?

(b) Are $\vec{B}$ and $\vec{C}$ in the previous problem orthogonal to each other? How can you tell?

## 0.3 Basic vector properties (Conceptual)

**Given: Pollock – Spring 2011**

Given vectors $\vec{A}$ and $\vec{B}$. ("Given" means you know the components, or alternatively, the length and angle of the vectors)

(a) Define the dot product mathematically in two very different looking ways. (Hint: one way should involve the components, the other, the length/angles)

(b) Give a brief physical interpretation of what the dot product means or tells you (you can give a concrete example if you like)

(c) Define the vector cross product mathematically in two very different looking ways. (Hint: one way should involve the components, the other, the length/angles)

(d) Give a brief physical interpretation of what the cross product means or tells you (you can give a concrete example if you like)

*Extra credit*

Use the appropriate part of the question above (4) to figure out the angle between the diagonal of one face of a cube and the "body diagonal" of the cube. *Hint: Let your cube have side length one, with one corner at the origin. Can you write down simple expressions for the vectors that represent a face diagonal, and the body diagonal?*

## 0.4 Vector decomposition (Conceptual)

**Given: Pollock – Spring 2012**

Consider an arbitrary vector $\vec{A}$; and let the unit vector $\vec{e}$ point along some fixed coordinate direction.

Using words and figures, explain the geometrical significance of the two two terms in the following vector expansion: (You do not have to prove this relation, you just have to explain what it means!)

$$\vec{A} = \vec{e}(\vec{A} \cdot \vec{e}) + \vec{e} \times (\vec{A} \times \vec{e}) \tag{1}$$

*Extra credit*

Use your knowledge of the dot product to figure out the angle between the diagonal of one face of a cube and the "body diagonal" of the cube. *Hint: Let your cube have side length one, with one corner at the origin. Can you write down simple expressions for the vectors that represent a face diagonal, and the body diagonal?*

## 0.5 Div, grad, curl, and all that

**Given: Marino – Fall 2011**

(a) Given the function $f(x, y, z) = xy + z$, what is the gradient of $f$, $\boldsymbol{\nabla} f$? Is it a vector or scalar?

(b) Given the function $\vec{F} = x\hat{\imath} + xy\hat{\jmath} + \hat{k}$, what is the curl of $\vec{F}$, $\boldsymbol{\nabla} \times \vec{F}$? Is it a vector or scalar?

(c) Given the function $\vec{F} = x\hat{\imath} + xy\hat{\jmath} + \hat{k}$, what is the divergence of $\vec{F}$, $\boldsymbol{\nabla} \cdot \vec{F}$? Is it a vector or scalar?

## 0.6 Curl of a grad

**Given: Marino – Fall 2011**

For an arbitrary scalar function $f(x, y, z)$, evaluate the components of $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f$ in Cartesian coordinates and show that the result is 0.

## 0.7 Conservative field?

**Given: Marino – Fall 2011**

(a) Taylor Problem 4.2, part a

(b) Taylor Problem 4.2, part b

(c) From your answers to parts a and b, do you think that **F** in this problem represents a conservative force? Why or why not?

(d) Calculate the curl of F ($\nabla \times \mathbf{F}$). Is this consistent with your answer to part c? Why or why not?

(e) Using the `VectorPlot` function in Mathematica, make a plot of this force **F** over the range $x = 0 \to 1$ and $y = 0 \to 1$. (Be sure to attach your plot to this assignment.) By visually examining your graph, what can you say about the local rotation of the vector field at the point (0.6,0.6)? Is this consistent with your answer to part d? Why or why not?

## 0.8 Volume of an ice cream cone

**Given: Marino – Fall 2011**

Use an appropriate volume integral to find an expression for the volume enclosed between a sphere of radius 1 centered on the origin and a circular cone of half-angle $\beta$ with its vertex at the origin. (This intersection is sort of shaped like an ice cream cone.) Show that in the limits where $\beta = 0$ and $\beta = \pi$ that your expression gives the expected values.

## 0.9 Mass density of a star

**Given: Pollock – Spring 2011, Spring 2012**

(a) A very simplistic model of a gaseous star might give you the mass density, $\rho$ as a function of radius, $r$ as

$$\rho(r) = \begin{cases} \rho_0 e^{-r/H} & r < R \\ 0 & r \geq R \end{cases} \tag{2}$$

Just looking at the expression, try to give some simple, physical interpretation of the three parameters $\rho_0$, $H$, and $R$. What are their units? What do they mean, or tell you, physically?

(b) Make a simple sketch of the mass density as a function of radius. (Your axes, where possible, should show how $\rho_0$, $H$, and $R$ are involved)

(c) An even more simplistic model might instead give

$$\rho(r) = \begin{cases} c_0/r & r < R \\ 0 & r \geq R \end{cases} \tag{3}$$

In this case, can you compute the total mass of the star in terms of the two parameters $c_0$ and $R$? (If so, do it!) What are the units of $c_0$?

## 0.10 Integration Review

**Given: Pollock – Spring 2011**

And now, a couple of math review questions:

(a)

$$\int dx \frac{4x}{(x^2 + a^2)^{3/2}} \tag{4}$$

(where a is a known constant. Note: this is an indefinite integral. Can you do this without computer assistance?).

(b)

$$\frac{d}{dx} \int_1^x dy f(y) \tag{5}$$

(where f(y) is some given, known (well behaved) function of y)

(c)

$$\frac{d}{dx} \int_0^1 dy (y + x) \tag{6}$$

**Given: Pollock – Spring 2012**

*A couple of math/calculus review questions*

(a)

$$\int dx \frac{4x}{(x^4 + 2(bx)^2 + b^4)^{3/2}} \tag{7}$$

(where $b$ is a known constant. Note: this is an indefinite integral)

(b)

$$\frac{d}{dy} \int_c^y dx f(x) \tag{8}$$

(where $f(x)$ is some given, known (well behaved) function of $x$ and $c$ is a constant.)

(c)

$$\frac{d}{dy} \int_0^2 dx (\log x + \log y) \tag{9}$$

## 0.11 Practicing Multi-dimensional Integrals

**Given: Pollock – Spring 2011, Spring 2012**

*Practice, practice: We have noticed some difficulties in evaluating multi-dimensional integrals. Since this is one of our class learning goals, we are giving you some basic problems to practice with.*

(a) Consider a solid sphere, determined by the volume $x^2 + y^2 + z^2 \leq R^2$, with a non uniform density $\rho(x, y, z) = A|z|$. What are the units of A? Determine the total mass of the sphere in terms of $A$ and $R$. *(Hints: Use spherical coordinates. Can you write z in terms of r, $\theta$, and $\phi$? Can you think of a clever use of symmetry and limits of integration to deal with that unpleasant absolute value?).*

(b) Consider a flat disk of radius R, with an areal mass density given in polar coordinates (it's flat, so we don't have any third "z" dimension) given by $\sigma(r, \theta) = Ar^2 \sin^2(\theta)$, where $A$ is a given constant. Describe (in words, and a simple sketch) this mass density, what does it look like physically? Now, determine the total mass of the thin disk in terms of A and $R$.

## 0.12 The Joy of Taylor Expansions

**Given: Pollock – Spring 2011**

*Taylor series are the single most important and common approximation technique used throughout physics. We need to keep practicing with them!*

(a) Suppose $f(x) = 1/(1-x)$. Taylor expand to find an approximate expression for $f(x)$, going out to second order. Use it to estimate f(0.1) and f(5). In both cases, compare the exact result for $f(x)$ with your series approximation, going to first order (terms proportional to x) and also second order ($x^2$). Does your estimate improve at second order in both cases? Comment on the general criterion about x you would guess tells you when the series approach seems to be a fruitful one here.

(b) Now let $f(x) = (1+x)/(1-x)$. Taylor expand to approximate $f(x)$ out to second order. There are many ways to do this. First , use the usual formal formula for Taylor expansion. Another way is to take your series answer in part a, and multiply it by (1+x). Try it this way too, and use it to check yourself, both methods should agree to all orders! Which method do you prefer?

(c) Now let $f(x) = (a+x)/(a-x)$, where a is a constant with units, like $a = 5m$. Taylor expand to second order. In this case, do NOT start "from scratch" blindly using the Taylor formula. Instead, factor out "a", so that it looks *exactly* like what you had in part b, except the "thing you're expanding in" isn't x. What is it? For this part, what is now the general criterion on x that tells you when this series approach seems to be a fruitful one? Why? (And please do not say "x must be small" or "$x << 1$". Small compared to what? You cannot compare meters to numbers, or apples to oranges!)

*(Once you understand this trick you will be spared having to ever painfully Taylor expand much of anything - the front flyleaf of your text will tell you how to expand almost anything you'll ever encounter)*

(d) One astronaut is in a circular orbit around the Earth a distance $R$ from the center of the Earth. Another astronaut is in orbit a bit closer to the Earth, at a distance $R - d$. The difference in the magnitude of the gravitational field between their locations is $\Delta g = \frac{GM_E}{(R-d)^2} - \frac{GM_E}{R^2}$ Use an appropriate series expansion to obtain an approximate expression for $\Delta g$. What relationship between $d$ and $R$ are you assuming? Your answer must contain one non-zero term. Does the sign of your answer seem sensible? Lastly - use your result to find the difference in g between the bottom and top of Gamow tower. Is your answer consistent with the common claim that $g = 9.8 m/s^2$ anywhere near earth? Comment.

**Given: Pollock – Spring 2012**

*Taylor series are the single most important and common approximation technique used throughout physics. We need to keep practicing with them!*

(a) Suppose $f(x) = 1/(1+x)$. Find an approximate expression for $f(x)$, going out to second order. Use it to estimate f(0.1) and f(5). In both cases, compare the exact result for $f(x)$ with your series approximation, going to first order (terms proportional to x) and also second order ($x^2$). Does your estimate improve at second order in both cases? Comment on the general criterion about x you would guess tells you when the series approach seems to be a fruitful one here.

(b) Now let $f(x) = (1-x)/(1+x)$. Taylor expand to approximate $f(x)$ out to second order. There are many ways to do this. First, use the usual formal formula for Taylor expansion. Another way is to take your series answer in part a, and multiply it by (1-x). Try it this way too, and use it to check yourself, both methods should agree to all orders! Which method do you prefer?

(c) Now let $f(x) = (a-x)/(a+x)$, where a is a constant with units, like $a = 5m$. Expand to second order. In this case, do NOT start "from scratch" blindly using the math formula. Instead, factor out "a", so that it looks *exactly* like what you had in part b, except the "thing you're expanding in" isn't x. What is it? For this part, what is now the general criterion on x that tells you when this series approach seems to be a fruitful one? Why? (And please do not say "x must be small" or "$x << 1$". Small compared to what? You cannot compare meters to numbers, or apples to oranges!) *(Once you understand this trick you will be spared having to ever painfully Taylor expand much of anything - the front flyleaf of your text will tell you how to expand almost anything you'll ever encounter)*

(d) Taylor's eqn 2.33 gives the velocity of a ball that has been dropped from rest with linear drag. Obtain an approximate expression for $v_y$(t) in the case of small drag. Include terms to 2nd order (Feel free to use the front flyleaf to save some grief!) What is the physical meaning of the leading nonzero term in your approximate solution? What is your interpretation of the *sign* of the next term? As in the previous question, what exactly do we mean by "small drag" here - what exactly is small, compared to what, here?

## 0.13 The Joy of Complex Numbers

**Given: Pollock – Spring 2011**

*Practicing with complex numbers.*

(a) If $z_1 = -\sqrt{3} + i$, draw $z_1$ in the complex plane. Compute its real and imaginary parts and magnitude. Write $z_1$ as $Ae^{i\theta}$ and determine $A$ and $\theta$.

(b) If $z_2 = 1/(1+i)$, draw $z$ in the complex plane. Compute its real and imaginary parts and magnitude. Write $z$ as $Ae^{i\theta}$ and determine $A$ and $\theta$.

(c) What are the real and imaginary parts of $1/z_3^2$ with $z_3 = 0.5e^{-i\pi/3}$.

(d) Compute $Z = z_1 * z_3$ with $z_1$ and $z_3$ given in parts (a) and (c). Draw $Z$ in the complex plane. Compute its real and imaginary parts and magnitude. Write $Z$ as $Ae^{i\theta}$ and determine $A$ and $\theta$.

**Given: Pollock – Spring 2012**

*Practicing with complex numbers. (2 pt each). See Boas Ch 2.4-12 for more review, or further practice!)*

(a) If $z_1 = 1 - \sqrt{3}i$, draw $z_1$ in the complex plane. Compute its real and imaginary parts and magnitude. Write $z_1$ as $Ae^{i\theta}$ and determine $A$ and $\theta$.

(b) If $z_2 = 1/(1-i)$, draw $z_2$ in the complex plane. Compute its real and imaginary parts and magnitude. Write $z$ as $Ae^{i\theta}$ and determine $A$ and $\theta$.

(c) If $z_3 = 0.5e^{-i\pi/6}$, what are the real and imaginary parts of $1/z_3^2$? *(Note that this is just a simple square, not an absolute value squared, so the answer does not need to come out purely real!)*

(d) Compute $Z = z_1 * z_3$ with $z_1$ and $z_3$ given in parts (a) and (c). Draw $Z$ in the complex plane. Compute its real and imaginary parts and magnitude. Write $Z$ as $Ae^{i\theta}$ and determine $A$ and $\theta$.

(e) Pick either one of the following two items to prove - whichever you prefer!

- Use Euler's theorem to rewrite $\sin x$ in terms of complex exponentials, and then use that to show that $\int_0^{2\pi} \sin(n\theta)\sin(m\theta)d\theta$ vanishes if n and m are unequal integers. *(We'll make good use of this, soon!)*

- Euler's theorem says that $e^{i\theta} = \cos\theta + i\sin\theta$. Call $z = e^{i\theta}$ and evaluate $z^2$ two ways to show the (very handy!) trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

## 0.14 The Joy of Delta Functions

**Given: Marino – Fall 2011**

Evaluate the following integrals:

(a) $\int_0^\pi \sin(x)\delta(x - \frac{\pi}{2})dx$

(b) $\int_{-\infty}^\infty e^x \delta(3x)dx$

(c) $\int_0^{2\pi} \cos(x)\delta(x + \pi)dx$

(d) $\int_{-10}^{10} x^2 \delta(1 - x)dx$

**Given: Pollock – Spring 2011, Spring 2012**

Evaluate the following integrals:

(a) $\int_0^\pi dx \sin(x)\delta(x - \pi/2)$

(b) $\int_0^3 (5t - 2)\delta(2 - t)dt$

(c) $\int_0^5 (t^2 + 1)\delta(t + 3)dt$

(d) $\int_{-\infty}^\infty e^x \delta(3x)$

(e) If a particle feels a force $F(t)$ of the form $F = A\delta(t)$, with $t$ the time, what are the units of $A$? Give a brief physical interpretation to this formula - what sort of physical force are we trying to represent here?