Problem List

4.1 Drawing equipotentials

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*
Available at: `http://perlnet.umaine.edu/imt/CFP/HWCFP.pdf`

Given: Marino – Fall 2011

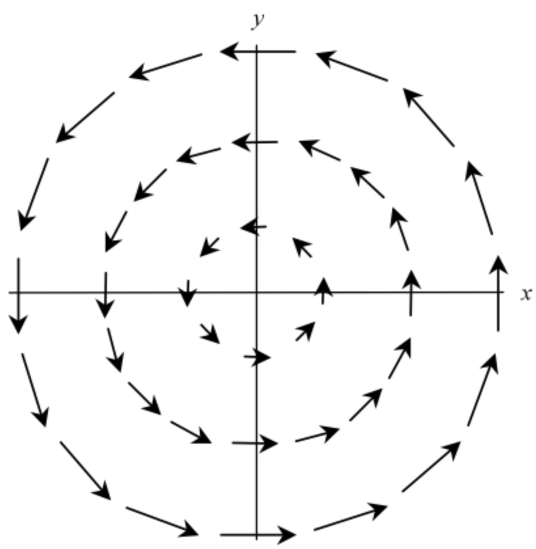
Two diagrams, each representing a force field in the x-y plane, are shown below.

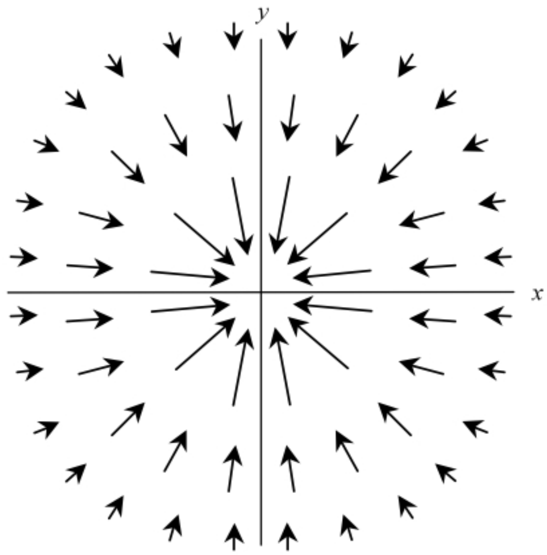
(a) For each case, is it possible to draw a self-consistent set of equipotential contours for that situation?

If so: Draw a representative set of equipotential contours for that situation. Each drawing should clearly show (1) the correct shape of the contour lines, (2) the correct relative spacing of the contours, and (3) label the regions that correspond to highest and lowest potential energy. (You can draw right on the HW printout and staple it to your HW submission.)

If not: Explain why, on the basis of the force field diagram, drawing such contours is impossible.

(b) Are these forces conservative? How can you tell?





4.2 Particle in a 2D well

Given: Marino – Fall 2011

A particle is in a 2D potential of $U(\vec{r}) = x^2 + 3y^2$.

(a) (0.5 points) What is the force on the particle?

(b) (0.5 points) Using the `ContourPlot` function in Mathematica, make an equipotential plot of $U(\vec{r})$ over the range $x = -1 \to 1$ and $y = -1 \to 1$. (Be sure to attach your plot to this assignment.)

(c) (0.5 points) On your printed contour plot, draw an arrow at the point (0.5,0.5) in the direction that is approximately perpendicular to the equipotential contours at that location and points toward increasing potential. Then evaluate your answer for part b at (0.5,0.5). How does the direction of the force compare to the arrow that you just drew?

4.3 Computing line integrals of the magnetic field

Given: Pollock – Spring 2011

The magnetic field inside a particular very long current carrying wire is given by $\vec{B} = -(\frac{B_0}{R})r\hat{\phi}$, where R is the radius of the wire, and r is the distance from the center of the wire. Ampere's law, discovered by Ampere in 1826, relates the integrated magnetic field around any closed loop to the total electric current passing through the loop, $\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{through}$. If we want to determine the current passing through the loops shown in Fig.1, we need to evaluate the line integral of $\vec{B} \cdot d\vec{r}$.

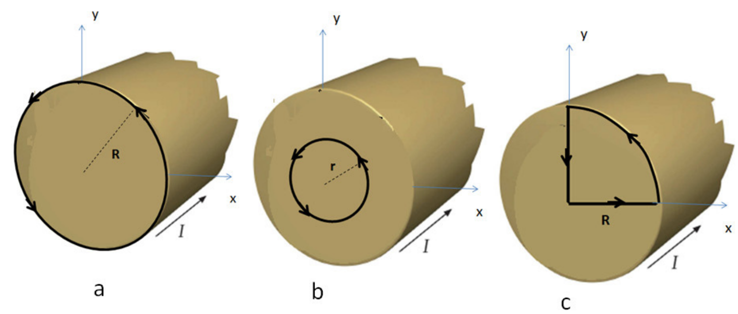


Figure 1:

(a) Explicitly compute $\oint \vec{B} \cdot d\vec{r}$ along the full circle path of radius R, shown in Fig.1a. Use this to find $I_{through}$. (Briefly, discuss the physical meaning of the sign of your answer). (2 pts)

(b) Then, compute $\oint \vec{B} \cdot d\vec{r}$ along the path of radius r_0 in Fig 1b. How does I_{thru} compare with part a? (2 pts)

(c) Compute $\oint \vec{B} \cdot d\vec{r}$ along the quarter circle path in Fig 1c. Compare your answers to the above three parts, and discuss. (What do you conclude about how the current is distributed through the wire?) (2 pts)

(d) Sketch a vector plot of $\vec{B} = -(\frac{B_0}{R})r\hat{\phi}$. Rewrite $\vec{B}$ entirely in Cartesian coordinates, and then, use the command VectorPlot in Mathematica to generate a plot of $\vec{B}$ to check your hand-drawn sketch. (4 pts)

Given: Pollock – Spring 2012

The magnetic field inside a long current carrying wire is $\vec{B} = (\frac{B_0}{R})r\hat{\phi}$, where R is the radius of the wire, and r is the distance from the center of the wire. Ampere's law relates the integrated magnetic field around any closed loop to the total current passing through the loop, $\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{through}$. If we want to determine the current passing through the loops shown in Fig.1, we need to evaluate the line integral of $\vec{B} \cdot d\vec{r}$.

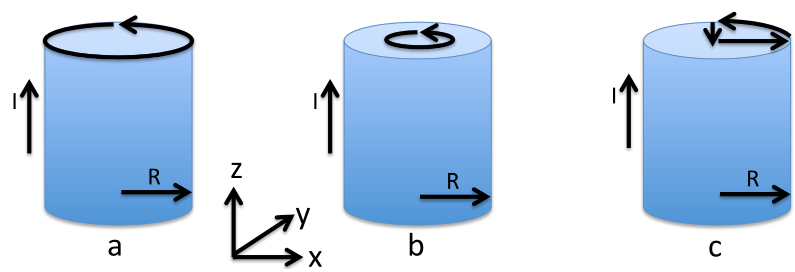


Figure 2:

(a) Explicitly compute $\oint \vec{B} \cdot d\vec{r}$ along the full circle path of radius R, shown in Fig.1a. Use this to find $I_{through}$. Briefly, discuss the physical meaning of the sign of your answer. If I had asked you to integrate the other way around the circle, what sign(s) in your solution would have changed? Would your final conclusion regarding the direction of "I" have changed? Be clear about this - what does the sign of your integral tell you?

(b) Compute $\oint \vec{B} \cdot d\vec{r}$ along the path of radius r_0 (less than R) in Fig 1b. How does I_{thru} compare with part a? Compute $\oint \vec{B} \cdot d\vec{r}$ along the quarter circle path in Fig 1c. Compare all your answers for question, and discuss - what do you conclude about how the current is distributed through the wire?

(c) Line integral practice! Do Boas Ch 6, section 8, problem #6 (p. 307) **Just pick one part - a, b,** *or* **c**. How can you prove that the answer will come out the same for the other parts *without* doing them?

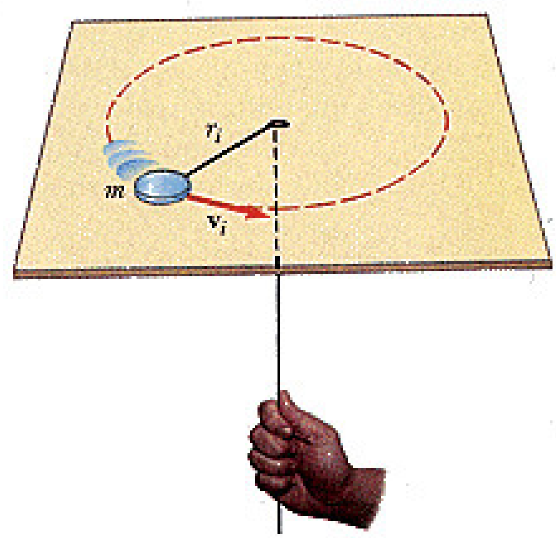
4.4 Shrinking an orbit

Given: Pollock – Spring 2011

A puck (mass m) on a frictionless air hockey table is attached to a cord passing through a hole in the surface as in the figure. The puck is moving in a circle of radius r_i with angular velocity ω_i. The cord is then slowly pulled from below, shortening the radius to r_f (r-final)

(a) What is angular velocity of the puck when the radius is r_f? (2 pts)

(b) Assuming that the string is pulled so slowly that we can approximate the puck's path by a circle of slowly shrinking radius, calculate the work done in moving the puck from r_0 to r. (Look back at Taylor Eq 1.48. "Slowly" means that $\dot{r}$ is tiny, as is the angular acceleration, so only the centripetal force will be important). Compare your answer with the puck's gain in kinetic energy, and comment briefly.



4.5 You're a collision investigator!

Given: Pollock – Spring 2011

A driver traveling downhill in a 1200 kg SUV on a road with a 4.5 degree slope slams on his brakes and skids 30 m before hitting a parked car. An insurance investigator hires an expert who measures the coefficient of kinetic friction between the tires and road to be $\mu_k = 0.45$. Is the investigator correct to accuse the driver of traveling faster than the 25 MPH speed limit? Explain. (While there are many ways to solve this problem, please solve it using work and energy.)

Given: Pollock – Spring 2012

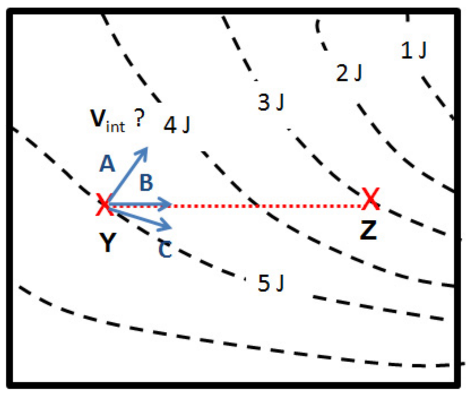
A driver traveling *downhill* slams on the brakes and skids 40 m on before hitting a parked car. You have been hired as a physics expert to help the insurance investigators decide if the driver had been traveling faster than the 25 MPH speed limit at the start of this event. The slope of the hill is 5 degrees. Assuming braking friction has the usual form μN, what is the "critical value" of μ for which you would conclude the driver was speeding? Can you convince the investigators this driver was speeding, or do you need more information? (Do you need to know the mass of either vehicle? Road conditions?) While there are multiple ways to solve this problem, *please solve it using work and energy*

4.6 Choose an initial velocity

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*
Available at: `http://perlnet.umaine.edu/imt/CFP/ExamsCFP.doc`

Given: Pollock – Spring 2011, Spring 2012

The diagram below shows a region of space. The dashed curves indicate positions of equal potential and are labeled with the value of the potential at that curve. Three vectors originating from point Y are also shown in blue. The vector B points directly from point Y to Z. In order to a particle to be launch from Y and reach Z, which vector represents best the initial velocity? Explain your reasoning. Then, make a (very crude, no calculations required!) sketch/guess of the *path* you expect the particle to follow if launched from point Y with initial **v** given by vector A. (Explain your reasoning, briefly, so we know what you were thinking about)



4.7 Properties of conservative fields

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*
Available at: `http://perlnet.umaine.edu/imt/CFP/ExamsCFP.doc`

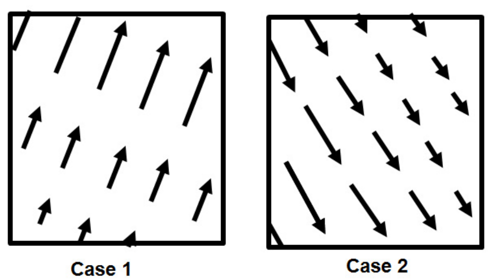
Given: Pollock – Spring 2011, Spring 2012

Each diagram in the figure below depicts a force field in a region of space.

(a) For which force fields can you identify a closed path over which $\oint \vec{F} \cdot d\vec{l} \neq 0$? For each such case, clearly indicate an appropriate path on the diagram.

(b) For each case indicate if $\nabla \times \vec{F} = 0$ everywhere in the box? Also, for each case, could the force depicted in the diagram be conservative? Briefly, explain.

(c) For each case, is it possible to draw a self-consistent set of equipotential contours for that situation? If so: Draw them! (Each drawing should clearly show the correct shape of the contour lines, the correct relative spacing of the contours, and label the regions that correspond to highest and lowest potential energy) If not: Explain why drawing such contours is impossible.



4.8 Fictitious potential energy of a satellite

Given: Pollock – Spring 2011

Consider the following expression for the potential energy of a satellite in a far away solar system.

$$U(x, y, z) = 2x^2 - y^2 - xy + 3z \tag{1}$$

(a) Compare the magnitude of the force on the satellite at point 1 (0,0,1) and point 2 (1,2,0).

(b) Is this a conservative force? Clearly justify your answer.

(c) For a arbitrary scalar function f(x, y, z), evaluate the components of $\vec{\nabla} \times \vec{\nabla} f$ in Cartesian coordinates and show that the result is 0. Does this formal result relate to part b? (If not, why not? If so, briefly comment)

Extra credit! Explicitly compute the work done by this force as you follow a path from the origin (0,0,0) to point 2, along a straight line. Then, check your answer very simply using the potential function given. Explain the idea of your check.

4.9 Time derivative of the total energy of a pendulum

Given: Pollock – Spring 2011

Imagine a simple pendulum consisting on a point mass m fixed to the end of a massless rod (length l), whose other end is pivoted from the ceiling to let it swing freely in the vertical plane. The pendulum's position can be identified by its angle ϕ from the equilibrium position.

(a) Write the equation of motion for ϕ using Newton's second law

(b) Show that the pendulum's potential energy (measured from the equilibrium level) is $U(\phi) = A(1 - \cos\phi)$. Find A in terms of m, g and l.

(c) Write down the total energy as a function of ϕ and $\dot{\phi}$.

(d) Show that by differentiating the energy respect to t you can recover the equation of motion you found in (a). (What basic principle of physics are you using, here?)

(e) Assuming that the angle ϕ remains small throughout the motion, solve for $\phi(t)$ and show that the motion is periodic. What is the period of oscillation?

Given: Pollock – Spring 2012

A simple pendulum consists of a point mass m fixed to the end of a massless rod (length l), whose other end is pivoted from the ceiling to let it swing freely in the vertical plane. The pendulum's position can be identified simply by its angle ϕ from the equilibrium position.

(a) Write the equation of motion for ϕ using Newton's second law. Assuming that the angle ϕ remains small throughout the motion, solve for $\phi(t)$ and show that the motion is periodic. What is the period of oscillation?

(b) Show that the pendulum's potential energy (measured from the equilibrium level) is $U(\phi) = A(1 - \cos\phi)$. Find A in terms of m, g and l. Then, write down a formula for the *total energy* as a function of ϕ and $\dot{\phi}$.

(c) Show that by differentiating the energy with respect to t you can recover the equation of motion you found in (a). (What basic principle of physics are you using, here?)

(d) Picking simple values for parameters (e.g, $m = l = 1, g = 10$), use your favorite computational environment (e.g., NDSolve in Mathematica) to solve for $\phi(t)$ given a fairly small starting angle ϕ_0. Provide output that clearly shows that the period of oscillation is very close to the theoretical prediction from part d. Repeat with $\phi_0 = 2$ rad (which is quite large), and show that the period now deviates from ideal. (Does the period get *larger* or *smaller* as ϕ_0 increases?)

(e) Using your numerical solution to part (d) and your formulas for energy from part (b), generate a single plot which graphs KE, U, and total E as a function of time, for one period, in the case where $\phi_0 = 2$ rad is NOT small. Briefly, comment! (Do we still conserve energy when we cannot use the small angle approximation any more?)

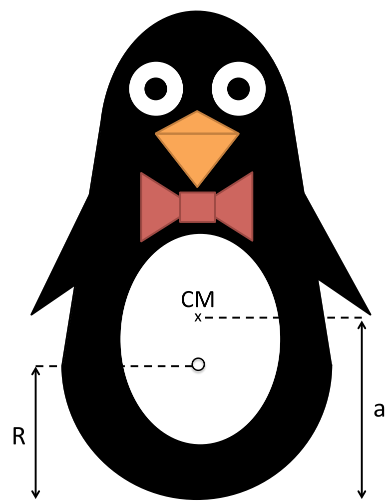
4.10 Stability of the weeble

Given: Pollock – Spring 2011, Spring 2012

Look at the figure below, which shows a possible design for a child's toy. The designer (you!) wants to build a "weeble" (which wobbles, but doesn't fall down). It is a single, solid object, the base shape is a perfect hemisphere (radius R) on the bottom, connected to the rest of the body (here, a penguin). The CM is shown, it is "a" above the base. You don't need to try to compute "a", assume it is a given quantity! (As the designer, how might you control/change the position of a?)

(a) Write down the gravitational potential energy when the weeble is tipped an angle θ from the vertical, as a function of given quantities (m, a, R, g, and of course θ).

(b) Assuming the toy is released from rest, determine the condition(s) for any equilibrium point(s). Then consider stability: what values of (or relations between) "a" or "R" ensure that the weeble doesn't fall down? Explain. Can this toy work as desired?



4.11 A simple 1D nonlinear force

Given: Pollock – Spring 2011

A particle is under the influence of a force $\vec{F} = k^2(-x + x^3/\alpha^2)\hat{x}$, where k and α are constants.

(a) What are the units of k and α? Now, determine $U(x)$ assuming $U(0) = 0$ and sketch it. (Sketch means hand-drawn, not plotted with Mathematica. Include all features you consider interesting in your sketch, including e.g. zero crossings, behavior at large x, "scales" of your axes, etc)

(b) Find the equilibrium points, if any, and determine if they are stable or unstable.

(c) Qualitatively explain the motion of an object in this force field released from rest at $x = \alpha/2$. Then, qualitatively explain the motion of an object in this force field released from rest at $x = 3\alpha/2$

This potential, or small variants of it, occurs in various physics situations. (This is a more accurate form of the true force on a pendulum than our usual approximation, can you see why?) Another (famous!) case, albeit with signs flipped, gives the "Higgs potential" in particle physics, which is being actively investigated at CERN's LHC collider.

Given: Pollock – Spring 2012

A particle is under the influence of a force $\vec{F} = (-ax + bx^3)\hat{x}$, where a and b are constants.

(a) What are the units of a and b? Assuming a and b are positive, find $U(x)$ (assuming $U(0) = 0$) and sketch it. (Sketch means hand-drawn, not plotted with Mathematica. Include all features you consider interesting in your sketch, including e.g. zero crossings, behavior at large x, "scales" of your axes, etc)

(b) Find all equilibrium points and determine if they are stable or unstable.

(c) Qualitatively describe the motion of an object in this force field released from rest at $x = (1/2)\sqrt{a/b}$.

Then, qualitatively describe the motion of an object in this force field released from rest at $x = \sqrt{2a/b}$.

(d) Now suppose a and b are both negative. Repeat parts b and c, commenting on what has changed.

(e) Setting a=b=-1 (with appropriate units), Taylor expand the potential of part d *around the outermost stable equilibrium point* (not around x=0!!) Truncate your series after the first "nontrivial" term. Plot the resulting approximate potential that a particle near that point feels, on top of the "real" potential. (Note this time I ask you to "plot", not "sketch", so use a computer) Use your plot to help you discuss the range of x for which you think this approximation would be reasonable.

Note that this potential, or small variants of it, occurs in various physics situations. This is a more accurate form of the true force on a pendulum than our usual approximation, can you see why? Part d is the (famous!) case of the "Higgs potential" in particle physics, which is being actively investigated at CERN's LHC collider.

4.12 The Morse potential

Given: Pollock – Spring 2011

The potential energy of two atoms in a molecule can sometimes be approximated by the Morse function,

$$U(r) = A\left[\left(e^{(R-r)/S} - 1\right)^2 - 1\right]$$

where r is the distance between the two atoms and A, R, and S are positive constants with $S \ll R$.

(a) (4 pts) Sketch the function for $0 < r < \infty$ (or plot in Mathematica if you prefer).

(b) (2 pts) Show that the equilibrium separation r_0 at which $U(r)$ is a minimum is given by $r_0 = R$.

(c) (2 pts) Based just on your figure/sketch, what behavior do you expect for the system if it starts with no kinetic energy, but at some r close to r_0? Describe what this would mean physically for the two atoms in the molecule (i.e. tell the story of what is happening to them).

(d) (4 pts) *If our diatomic molecule has one very heavy atom, that atom can be viewed as effectively fixed at the origin, and this $U(r)$ simply describes the potential energy of the second, lighter atom (mass m, a distance $r(t)$ from the origin.)*

Approximate $U(r)$ for r close to r_0. Use this approximation of $U(r)$ to get a differential equation for $r(t)$. Can you tell from this differential equation if your prediction in part (c) was correct? Explain.

4.13 A Gaussian well

Given: Pollock – Spring 2012

A particle (in 2D) has a potential energy function given by $U(\vec{r}) = -e^{-(x^2+2y^2)}$. (Assume all the numerical constants have implicit correct SI metric units so that x and y are in meters, and U in Joules)

(a) Using your favorite computational tool, make an equipotential plot ("ContourPlot[ ]" in Mathematica. Use the built in help to find out the syntax and arguments, it's pretty simple) and a 3D plot ("Plot3D" in Mathematica). Include a copy of both plots with your homework. Now stare at these images. Describe in words how you would interpret this potential physically (For instance - is this attractive, repulsive, both, neither? Can you invent some physical system for which this might be a crude model? Would a particle be "bound" here?) Which is more informative for you - the countour plot or the 3D plot? Why? If you put a particle at, say, $\vec{r} = (1, 1)$, use your intuition and the plots to describe very clearly in words (without calculation) the approximate direction of the force there.

(b) Analytically compute the force on the particle. Then, use a computational tool to plot the force field ("VectorPlot[ ]" in Mathematica. If you name your plots "p1 = ContourPlot[]", p2=VectorPlot[]" and make sure the x and y range is the same, then you can plot TWO different plots on top of each other using "Show[p1,p2]".) Produce a single plot that shows the contours and the force field together. Now, what is the magnitude and direction of the force at (1,1)? Discuss whether your plots agree with your expectation in part a (and resolve any discrepancies)

(c) What changes could you make to $U(\vec{r})$ to make the potential well i) centered at the point (1,1) instead of (0,0) ii) stronger? iii) repulsive? (You might want to use your computational tool to visualize your claims here.)

4.14 Electric force and potential energy

Given: Pollock – Spring 2012

Electrical force is given by $\vec{F} = q\vec{E}$, where $\vec{E}$ is the electric field. Electric force is conservative in "electrostatic" situations (but is *not* conservative in all situations!)

(a) Consider a laboratory setup in which a charge q sits in a field $\vec{E} = (z^2 + 1)\hat{k}$. Prove that the resulting force is conservative, and deduce a formula for the potential energy as a function of position of this charge

(b) You release the charge (it has mass m) at the origin, starting from rest. Describe its motion qualitatively. How fast is it going when it reaches a distance h from the starting point? If the object was in the shape of a bead (*same* mass m and charge q) threaded on a curved, frictionless, rigid wire which started at the origin $(0, 0, 0)$ and ended at some point (x_0, y_0, h) would it also have that same speed you just calculated, or not? (Why?)

4.15 Electric potential and electric potential energy

Given: Pollock – Spring 2012

Electrical potential energy is given by $U = qV$, where V is the voltage. Consider a charge q moving in a region of space where the voltage is approximated by the formula $V(x, y, z) = c(2xz + x^2)$ (Assume the constant c has appropriate units so U is in Joules when x,y, z are in meters)

(a) Find the force on the charge q at the origin, and the point (1,1,1)

(b) Does this situation produce a conservative electrical force? Clearly justify your answer.

(c) For an arbitrary scalar function f(x, y, z), evaluate the components of $\vec{\nabla} \times \vec{\nabla} f$ in Cartesian coordinates and show that the result is 0. Does the formal result relate to (b)? (If not, why not? If so, briefly comment)

(d) Explicitly compute the work done by the force in part a on a charge q as you follow a path from the origin (0,0,0) to (1,1,1) along a direct, straight line path. Then, check your answer very simply using the potential function given. Explain the idea of your check.