Problem List

5.1 Total mass of a shell

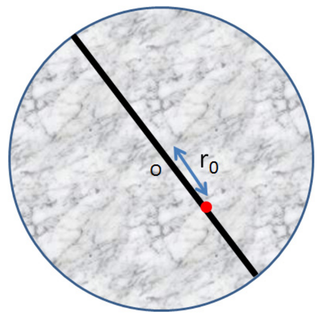
Given: Marino – Fall 2011

Consider a spherical shell that extends from $r = R$ to $r = 2R$ with a **non-uniform density** $\rho(r) = \rho_0 r$. What is the total mass of the shell?

5.2 Tunnel through the moon

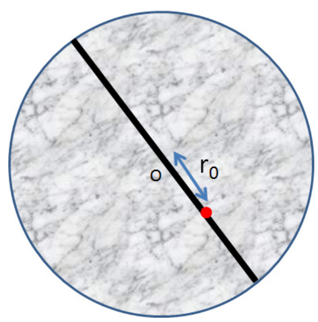
Given: Marino – Fall 2011

Imagine that NASA digs a straight tunnel through the center of the moon (see figure) to access the Moon's ^{3}He deposits. An astronaut places a rock in the tunnel at the surface of the moon, and releases it (from rest). Show that the rock obeys the force law for a mass connected to a spring. What is the spring constant? Find the oscillation period for this motion if you assume that Moon has a mass of 7.35×10^{22} kg and a radius of 1.74×10^{6} m. Assume the moon's density is uniform throughout its volume, and ignore the moon's rotation.



Given: Pollock – Spring 2011

Imagine (in a parallel universe of unlimited budgets) that NASA digs a straight tunnel through the center of the moon (see figure). A robot place a rock in the tunnel at position $r = r_0$ from the center of the moon, and releases it (from rest). Use Newton's second law to write the equation of motion of the rock and solve for r(t). Explain in words the rock's motion. Does the rock return to its initial position at any later time? If so, how long does it takes to return to it? (Give a formula, and a number.) Assume the moon's density is uniform throughout its volume, and ignore the moon's rotation.



Given: Pollock – Spring 2012

Now lets consider our (real) planet Earth, with total mass M and radius R which we will approximate as a *uniform* mass density, $\rho(r) = \rho_0$.

(a) Neglecting rotational and frictional effects, show that a particle dropped into a hole drilled straight through the center of the earth all the way to the far side will oscillate between the two endpoints. (Hint: you will need to set up, and solve, an ODE for the motion)

(b) Find the period of the oscillation of this motion. Get a number (in minutes) as a final result, using data for the earth's size and mass. (How does that compare to flying to Perth and back?!)

Extra Credit: OK, even with unlimited budgets, digging a tunnel through the center of the earth is preposterous. But, suppose instead that the tunnel is a straight-line "chord" through the earth, say directly from New York to Los Angeles. Show that your final answer for the time taken does not depend on the location of that chord! This is rather remarkable - look again at the time for a free-fall trip (no energy required, except perhaps to compensate for friction) How long would that trip take? Could this work?!

5.3 Gravitational field above the center of a thin hoop

Given: Pollock – Spring 2011, Spring 2012

Consider a very (infinitesimally!) thin but massive loop, radius R (total mass M), centered around the origin, sitting in the x-y plane. Assume it has a uniform linear mass density λ (which has units of kg/m) all around it. (So, it's like a skinny donut that is mostly hole, centered around the z-axis)

(a) What is λ in terms of M and R? What is the direction of the gravitational field generated by this mass distribution at a point in space a distance z above the center of the donut, i.e. at $(0, 0, z)$ Explain your reasoning for the direction carefully, try not to simply "wave your hands." (The answer is extremely intuitive, but can you justify that it is correct?)

(b) Compute the gravitational field, $\vec{g}$, at the point $(0, 0, z)$ by directly integrating Newton's law of gravity, summing over all infinitesimal "chunks" of mass along the loop.

(c) Compute the gravitational potential at the point $(0, 0, z)$ by directly integrating $-Gdm/r$, summing over all infinitesimal "chunks" dm along the loop. Then, take the z-component of the gradient of this potential to check that you agree with your result from the previous part.

(d) In the two separate limits $z << R$ and $z >> R$, Taylor expand your g-field (in the z-direction)out only to the first non-zero term, and convince us that both limits make good physical sense.

(e) Can you use Gauss' law to figure out the gravitational potential at the point $(0, 0, z)$? (If so, do it and check your previous answers. If not, why not?)

Extra credit: If you place a small mass a small distance z away from the center, use your Taylor limit for $z << R$ above to write a simple ODE for the equation of motion. Solve it, and discuss the motion

5.4 Gravitational force near a metal-cored planet surrounded by a gaseous cloud

Given: Pollock – Spring 2011

Jupiter is composed of a dense spherical core (of liquid metallic hydrogen!) of radius R_c. It is surrounded by a spherical cloud of gaseous hydrogen of radius R_g, where $R_g > R_c$. Let's assume that the core is of uniform density ρ_c and the gaseous cloud is also of uniform density ρ_g. What is the gravitational force on an object of mass m that is located at a radius r from the center of Jupiter? Note that you must consider the cases where the object is inside the core, within the gas layer, and outside of the planet.

5.5 Sphere with linearly increasing mass density

Given: Pollock – Spring 2011

A planet of mass M and radius R has a nonuniform density that varies with r, the distance from the center according to $\rho = Ar$ for $0 \leq r \leq R$.

(a) What is the constant A in terms of M and R? Does this density profile strike you as physically plausible, or is just designed as a mathematical exercise? (Briefly, explain)

(b) Determine the gravitational force on a satellite of mass m orbiting this planet. In words, please outline the method you plan to use for your solution. (Use the easiest method you can come up with!) In your calculation, you will need to argue that the magnitude of $\vec{g}(r, \theta, \phi)$ depends only on r. Be very explicit about this - how do you know that it doesn't, in fact, depend on θ or ϕ?

(c) Determine the gravitational force felt by a rock of mass m inside the planet, located at radius $r < R$. (If the method you use is *different* than in part b, explain why you switched. If not, just proceed!)

Explicitly check your result for this part by considering the limits $r \to 0$ and $r \to R$.

5.6 Jumping off Vesta

Given: Pollock – Spring 2011

You are stranded on the surface of the asteroid Vesta. If the mass of the asteroid is M and its radius is R, how fast would you have to jump off its surface to be able to escape from its gravitational field? (Your estimate should be based on parameters that characterize the asteroid, not parameters that describe your jumping ability.) Given your formula, look up the approximate mass and radius of the asteroid Vesta 3 and determine a numerical value of the escape velocity. Could you escape in this way? (Briefly, explain) If so, roughly how big in radius is the maximum the asteroid could be, for you to still escape this way? If not, estimate how much smaller an asteroid you would need, to escape from it in this way?

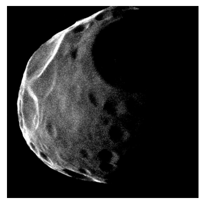


Figure 1:

5.7 Gravitational force between two massive rods

Given: Pollock – Spring 2011

Consider two identical uniform rods of length L and mass m lying along the same line and having their closest points separated by a distance d as shown in the figure

(a) Calculate the mutual force between these rods, both its direction and magnitude.

(b) Now do several checks. First, make sure the units worked out (!) The, find the magnitude of the force in the limit $L \to 0$. What do you expect? Briefly, discuss. Lastly, find the magnitude of the force in the limit $d \to \infty$? Again, is it what you expect? Briefly, discuss.

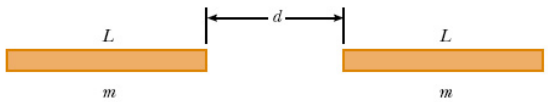
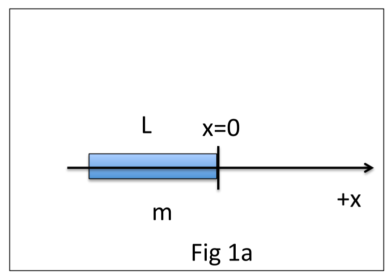


Figure 2:

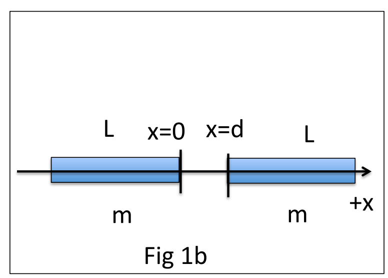
Given: Pollock – Spring 2012

Determining the gravitational force between two rods:

(a) Consider a thin, uniform rod of mass m and length L (and negligible other dimensions) lying on the x axis (from x=-L to 0), as shown in fig 1a. Derive a formula for the gravitational field "g" at any arbitrary point x to the right of the origin (but still on the x-axis!) due to this rod.



(b) Now suppose a second rod of length L and mass m sits on the x axis as shown in fig 1b, with the left edge a distance "d" away. Calculate the mutual gravitational force between these rods.



(c) Let's do some checks! Show that the units work out in parts a and b.

Find the magnitude of the force in part a, in the limit $x >> L$. What do you expect? Briefly, discuss!

Finally, verify that your answer to part b gives what you expect in the limit $d >> L$. *(Hint: This is a bit harder! You need to consistently expand everything to second order, not just first, because of some interesting cancellations)*

5.8 Potential energy – Check your answer!

Given: Pollock – Spring 2011

On the last exam, we had a problem with a flat ring, uniform mass per unit area of σ, inner radius of R, outer radius of $2R$. A satellite (mass m) sat a distance z above the center of the ring. We asked for the gravitational potential energy, and the answer was

$$U(z) = -2\pi G\sigma m(\sqrt{4R^2 + z^2} - \sqrt{R^2 + z^2}) \tag{1}$$

(a) If you are far from the disk (on the z axis), what do you *expect* for the formula for U(z)? (Don't say "0" - as usual, we want the functional form of $U(z)$ as you move far away. Also, explicitly state what we mean by "far away". (Please don't compare something with units to something without units!)

(b) Show explicitly that the formula above does indeed give precisely the functional dependence you expect.

5.9 Ways of solving gravitational problems

Given: Pollock – Spring 2011, Spring 2012

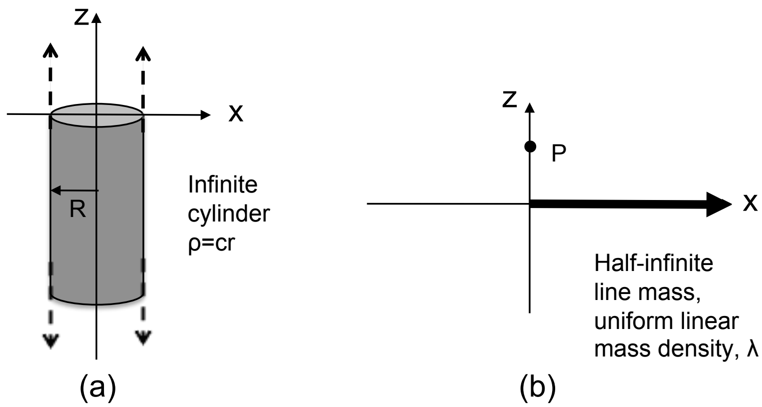


Figure 3: (a) An infinite cylinder of radius R centered on the z-axis, with *non-uniform* volume mass density $\rho = cr$, where r is the radius in cylindrical coordinates. (b) A half-infinite line of mass on the x-axis extending from $x = 0$ to $x = +\infty$, with uniform linear mass density λ.

There are two general methods we use to solve gravitational problems (i.e. find $\vec{g}$ given some distribution of mass).

(a) Describe these two methods. We claim one of these methods is easiest to solve for $\vec{g}$ of mass distribution (a) above, and the other method is easiest to solve for $\vec{g}$ of the mass distribution (b) above. Which method goes with which mass distribution? Please justify your answer.

(b) Find $\vec{g}$ of the mass distribution (a) above for any arbitrary point outside the cylinder.

(c) Find the x component of the gravitational acceleration, g_x, generated by the mass distribution labeled (b) above, at a point P a given distance z up the positive z-axis (as shown).

5.10 Rod with linearly increasing mass density

Given: Pollock – Spring 2012

Consider a very (infinitesimally!) thin but massive rod, length L (total mass M), centered around the origin, sitting along the x-axis. (So the left end is at (-L/2, 0,0) and the right end is at (+L/2,0,0) Assume the mass density λ (which has units of kg/m)is *not* uniform, but instead varies linearly with distance from the origin, $\lambda(x) = c|x|$.

(a) What is that constant "c" in terms of M and L? What is the direction of the gravitational field generated by this mass distribution at a point in space a distance z above the center of the rod, i.e. at $(0, 0, z)$ Explain your reasoning for the direction carefully, try not to simply "wave your hands." (The answer is extremely intuitive, but can you justify that it is correct?)

(b) Compute the gravitational field, $\vec{g}$, at the point $(0, 0, z)$ by directly integrating Newton's law of gravity, summing over all infinitesimal "chunks" of mass along the rod.

(c) Compute the gravitational potential at the point $(0, 0, z)$ by directly integrating $-Gdm/r$, summing over all infinitesimal "chunks" dm along the rod. Then, take the z-component of the gradient of this potential to check that you agree with your result from the previous part.

(d) In the limit of large z what do you *expect* for the functional form for gravitational potential? (Hint: Don't just say it goes to zero! It's a rod of mass M, when you're far away what does it look like? *How* does it go to zero?) What does "large z" mean here? Use the binomial (or Taylor) expansion to verify that your formula does indeed give exactly what you expect. (Hint: you cannot Taylor expand in something BIG, you have to Taylor expand in something small.)

(e) Can you use Gauss' law to figure out the gravitational potential at the point $(0, 0, z)$? (If so, do it and check your previous answers. If not, why not?)

5.11 Sphere with constant internal gravitational field

Given: Pollock – Spring 2012

(a) Imagine a planet of total mass M and radius R which has a nonuniform mass density that varies just with r, the distance from the center. For this (admittedly very unusual!) planet, suppose the gravitational field strength *inside the planet* turns out to be **independent** of the radial distance within the sphere. Find the function describing the mass density $\rho = \rho(r)$ of this planet. (Your final answer should be written in terms of the given constants.)

(b) Now, determine the gravitational force on a satellite of mass m orbiting this planet at distance $r > R$. (Use the easiest method you can come up with!) Explain your work in words as well as formulas. For instance, in your calculation, you will need to argue that the magnitude of $\vec{g}(r, \theta, \phi)$ depends only on r. Be explicit about this - how do you know that it doesn't, in fact, depend on θ or ϕ?

(c) As a final check, explicitly show that your solutions inside and outside the planet (parts a and b) are *consistent* when $r = R$. Please also comment on whether this density profile strikes you as physically plausible, or is it just designed as a mathematical exercise? Defend your reasoning.

5.12 Throwing a rock off the moon

Given: Pollock – Spring 2012

Assuming that asteroids have roughly the same mass density as the moon, make an estimate of the largest asteroid that an astronaut could be standing on, and still have a chance of throwing a small object (with their arms, no machinery!) so that it completely escapes the asteroid's gravitational field. (This minimum speed is called "escape velocity") Is the size you computed typical for asteroids in our solar system?