

Problem List

- 6.1 Properties of harmonic oscillator phase space plots
- 6.2 Limits of the SHO approximation
- 6.3 2D non-isotropic oscillator with commensurate natural frequencies
- 6.4 Linearly damped oscillator
- 6.5 Phase space of a damped critically oscillator
- 6.6 Damped oscillations, modeling real data
- 6.7 Modeling the quadratically damped oscillator
- 6.8 Picking out the transients
- 6.9 Period runaway – Modeling a pendulum at large angles
- 6.10 A gravitational oscillator?
- 6.11 Parallel and series springs
- 6.12 Estimating damping parameters
- 6.13 Energy considerations in a 1D simple harmonic oscillator
- 6.14 Leading and lagging in electrical oscillations
- 6.15 Resonant behavior
- 6.16 Sketching phase space diagrams
- 6.17 Adding linear damping to an undamped oscillator

These problems are licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. Please share and/or modify.

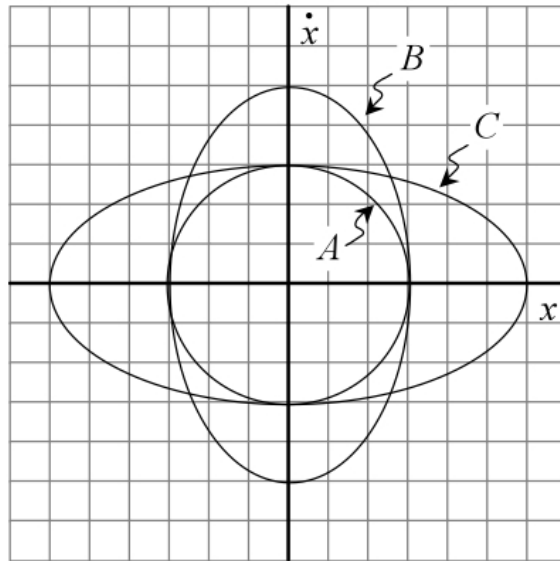
6.1 Properties of harmonic oscillator phase space plots

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/Phase-DHM/HWPhase-DHM.pdf>

Given: Marino – Fall 2011, Pollock – Spring 2011

Consider the phase space plots (A, B, and C) shown below.



- (0.5 points) Could all three plots correspond to the same simple harmonic oscillator (i.e., same mass and same spring constant)? Explain why or why not.
- (0.5 points) Which pair of plots could be used to show the effect of keeping the total energy constant but increasing the mass (while keeping the spring constant fixed)? Clearly indicate which plot would correspond to the smaller mass. Explain without performing any calculations.
- (0.5 points) Suppose that plots A and C correspond to systems with springs with the same spring constant, but different masses. Do these two systems have the same total energy? If not, which one has more total energy? Explain how you can tell.
- (0.5 points) Suppose that in the diagram each division along the position axis corresponds to 0.1 m; along the velocity axis, 0.10 m/s. What is the period of the oscillator shown by path B?

6.2 Limits of the SHO approximation

Given: Marino – Fall 2011

We often consider the simple pendulum to be an example of simple harmonic motion. In this problem we will see how accurate this is.

- (a) (0.5 points) Imagine a vertical pendulum of length l and mass m . By considering the forces on the pendulum and applying Newton's second law, obtain differential equation for the forces on the pendulum in terms of θ , the angle with respect to the vertical axis. (Hint: It might be easiest to work in polar coordinates.)
- (b) (0.5 points) Show that in the limit where θ is small that you obtain a second-order linear differential equation with constant coefficients. What is the general solution to this equation?
- (c) (0.5 points) Consider $m=1$ kg, and $l=1$ m. If the pendulum is released at rest at $t = 0$ from $\theta=0.05$ radians, what is the particular solution for $\theta(t)$ in your expression from part b? What is the period of this solution?
- (d) (0.25 points) If instead the initial condition is that $\theta=1.25$ radians at $t = 0$, what happens to the period of the motion?
- (e) (0.5 points) Now use Mathematica to solve the exact equation that you found in part a (keeping the $\sin(\theta)$ in the expression) for the case where the pendulum is released from rest at $t = 0$ from $\theta=0.05$ radians. Make sure that you figure out how to apply initial conditions. Is there an exact solution if you use `DSolve` or do you have to use `NDSolve`?
- (f) (0.5 points) Make a plot of the solution. Now use the `FindRoot` command to figure out where the solution is zero.¹ (You should have some idea of where to look for the roots from the plot that you made.) Use the locations of two roots to determine the period. How does this compare to the period of the approximate small angle solution found above in part c?
- (g) (0.5 points) Now use Mathematica to determine a numerical solution for the case where the pendulum is released from rest at $t = 0$ from $\theta=1.25$ radians. Plot the solution. Once again, use the `FindRoot` command to determine the locations of two roots to determine the period. How does this compare to the case where it was released from $\theta = 0.05$ in part f?
- (h) (0.25 points) Based on these results, what can you say about how well simple harmonic motion approximates the behavior of a pendulum?

¹For a sample Mathematica notebook that solves a second-order ODE and uses the `FindRoot` function, please look at "Another ND Solve Example" on CULearn, in the Mathematica Examples folder. You might find this to be a useful starting point.

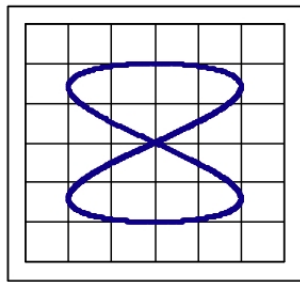
6.3 2D non-isotropic oscillator with commensurate natural frequencies

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

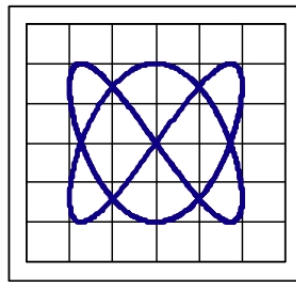
Available at: <http://perlnet.umaine.edu/imt/SHM-2D/HWH2D.pdf>

Given: Marino – Fall 2011, Pollock – Spring 2011, Spring 2012

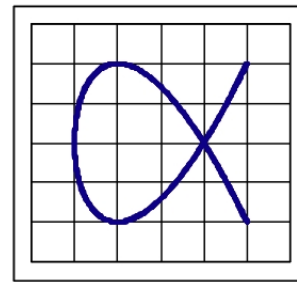
Consider the motion of a two-dimensional non-isotropic oscillator, in which the spring constant in x direction (k_1) is not equal to the spring constant in the y direction (k_2). Each trajectory below depicts the possible motion of a unique non-isotropic oscillator. All three oscillators, share the property that the angular frequencies ω_1 and ω_2 for the motions along the x- and y-axes are commensurate, i.e., that the angular frequencies satisfy the following relationship: $\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2}$, where n_1 and n_2 are integers. For each case below, (i) determine whether ω_1 is greater than, less than, or equal to ω_2 , and (ii) determine the values n_1 and n_2 that satisfy the condition. Explain.



Trajectory #1



Trajectory #2



Trajectory #3

6.4 Linearly damped oscillator

Given: Marino – Fall 2011

Consider an underdamped 1D oscillator of mass m and spring constant k that is subjected to a damping force of $\vec{F}_{drag} = -b\vec{\dot{x}}$. You may assume that the mass is on a horizontal frictionless table so you only need to consider the spring force and the drag force. At time $t = 0$ the system is released from rest at $x = A$.

- (a) (0.5 pts) What is the solution for $x(t)$? Take the derivative to find $\dot{x}(t)$.
- (b) (0.5 pt) Use part (a) to find an expression for $E(t)$. (Hint: Some trig double angle identities might help simplify the expression a bit.)
- (c) (0.75 pts) Find the rate of energy loss, dE/dt , and show that it is proportional to \dot{x}^2 .
- (d) (0.25 pts) Is the rate of energy loss a constant? Are there any times when the energy loss rate is 0? Explain why this makes sense.

6.5 Phase space of a critically damped oscillator

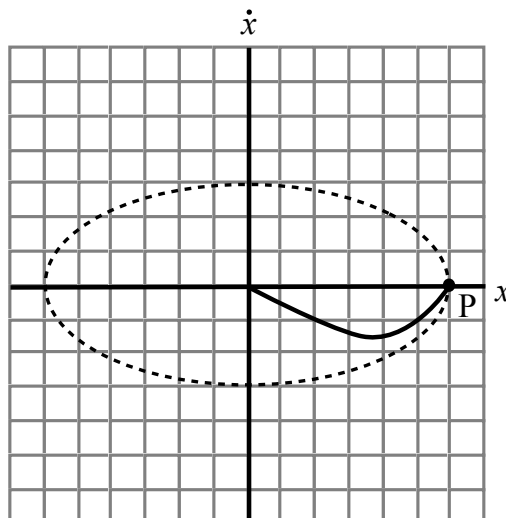
Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/Phase-DHM/HWPhase-DHM.pdf>

Given: Marino – Fall 2011

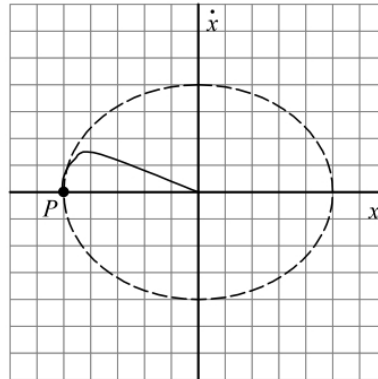
Shown below are phase space plots for (i) a simple harmonic oscillator (dashed) and (ii) the same oscillator with a retarding force applied (solid). Point P represents the initial conditions of the oscillator in both instances. In the diagram, you may assume each division along the position axis corresponds to 0.1 m; along the velocity axis, 0.10 m/s.

- Explain how you can tell that the damped oscillator is not underdamped.
- Is the damped oscillator critically damped or overdamped? Explain how you can tell.
- If you said in part b that the oscillator is {critically damped, overdamped}, then draw how the phase space plot would be different if the oscillator (starting at point P) were instead {overdamped, critically damped}. Explain your reasoning



Given: Pollock – Spring 2011, Spring 2012

The figure below shows the phase space plot for a simple harmonic oscillator (dashed) and the same oscillator with a retarding force applied (solid). Point P represents the initial conditions of the oscillator in both instances. In the diagram, you may assume each division along the position axis corresponds to 0.1 m; along the velocity axis, 0.10 m/s.



- Explain how you can tell that the damped oscillator is not underdamped.
- Is the damped oscillator critically damped or overdamped? Explain how you can tell.
- If you said in part b that the oscillator is {critically damped, overdamped}, then draw how the phase space plot would be different if the oscillator (starting at point P) were instead {overdamped, critically damped}. Explain your reasoning

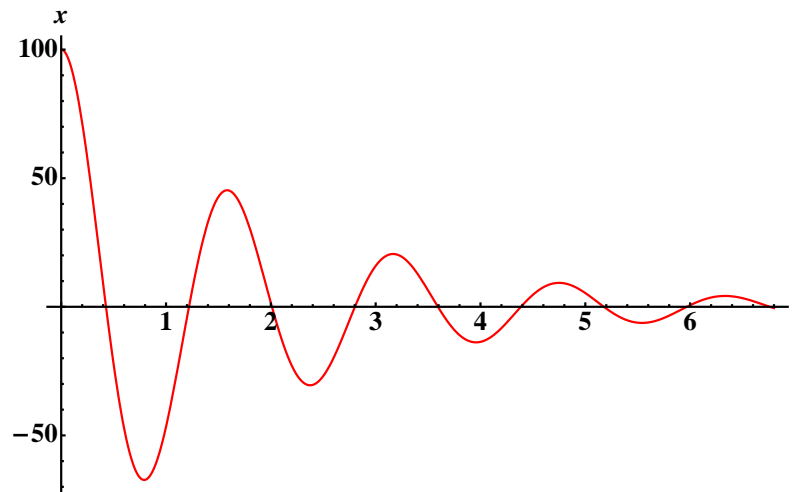
6.6 Damped oscillations, modeling real data

Given: Marino – Fall 2011, Pollock – Spring 2012

In the following pair of problems you will compare our model for a damped harmonic oscillator to real data collected from a video of a spring-mass system immersed in viscous oil. The video from which this position vs time data is collected is here: <http://www.youtube.com/watch?v=ZYFVKZPut9w>. Data was obtained using free video tracking software (Tracker) and saved in a comma-separated variables (CSV) file.

First, you will describe a few properties of a damped oscillator to assist you in your analysis of the data.

- (a) Write down the differential equation for a damped harmonic oscillator with undamped natural frequency, ω_0 , and damping parameter, β . Write down the general solution for this differential equation for the underdamped case ($\beta < \omega_0$). Then, consider the following sketch of a solution to this differential equation for a particular choice of β and ω_0 when the oscillator was displaced and let go.



You can obtain an estimate for $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ using the time between successive zero crossing. Estimate ω_1 in the figure above and explain how you obtained your result.

- (b) You can also obtain an estimate for β using successive maxima or minima from the plot. Estimate β and ω_0 . Explain how you obtained your result.
- (c) If ω_0 were kept constant but β were increased (but still below ω_0), sketch how the plot above would change. What physical change to a spring-mass immersed in oil can be done to increase β ? Answer both questions for the case where ω_0 is kept constant but β is decreased. What property of the sketch does β appear to control?
- (d) If β were kept constant but ω_0 were decreased (but still above β), sketch how the plot above would change. What physical change to a spring-mass immersed in oil can be done to decrease ω_0 ? Answer both questions for the case where β is kept constant but ω_0 is increased. What property of the sketch does ω_0 appear to control?

You will now analyze a real spring-mass system immersed in viscous oil for which position versus time data was collected for the oscillator.

- (a) The position versus time data appears in a comma-separated variables (CSV) file on our course home page and also our D2L page. Importing this data is straight-forward in Mathematica using the `Import` command. You'll need to store the data in a variable, so you can plot it.

For example, `data = Import["/file/to/datafile.csv"]` will import the values in the CSV file into the variable `data`. (You will use Insert → Filepath to generate the right file path on your computer - if you need a little more help, a video describing how to import data into Mathematica is located here: <http://youtu.be/MmS3JNk7JE4> .)

Plot this data using the `ListPlot` command,

```
experiment = ListPlot[data[[All, {1, 2}]], PlotRange -> All].
```

You are storing this plot (as `experiment`) for later, so you can display the experimental data and the results from your model on the same plot.

- (b) Using the plot of the data, estimate values for ω_1 , β , and ω_0 . Determine the initial position and initial velocity for the oscillator. This oscillator was displaced and released from rest.
- (c) Plot the particular solution to damped harmonic oscillator for your model parameters (ω_1 , β , and ω_0) and initial conditions. You should store this plot (e.g., `model = Plot[...]`).
- (d) Plot the result of your model and the experimental data on the same axes using the `Show` command. This is why you stored the plots earlier (e.g., `Show[experiment, model]`). How does your model match the experimental data? Can you tweak your model parameters to make the fit better?
- (e) Why did we do this? Experimental physicists collect data and often attempt to fit a model of that system to their data. It's likely that you got pretty good but not great agreement with the data that was collected. Can you identify at least 3 aspects of the physical system (spring mass immersed in oil) that might have improved the model? Can you identify at least 2 aspects of the data collection procedure (video tracking software) that might have helped you better estimate your model parameters (ω_1 , β , and ω_0)?

6.7 Modeling the quadratically damped oscillator

Given: Pollock – Spring 2012

- (a) Solve Boas Problem 8.6.12 (That’s chapter 8, section 6, problem 12, on page 423). We just want the general solution here.
Please also tell us *explicitly* if the associated (complementary) homogenous problem is (choose one) {over-damped, under-damped, undamped, critically damped} How do you tell?
- (b) Now, given that the system starts with $y(0) = 0$, $y'(0) = 0$, solve that problem completely. There should be no undetermined coefficients left!
- (c) Sketch the homogeneous *piece* of the solution in part b) above (by itself). Also sketch the particular solution *piece* (by itself). Finally add them to sketch the full solution - noting any important aspects of the sketch. (Don’t worry about precision, this is a sketch, not a plot. We want to know the *features*.) Then, use a computer to plot the actual solution. Print it out and discuss - in particular, discuss if it did not exactly match with your sketch. (The differences may or may not be subtle, depending on the care of your initial sketch)
- (d) Let’s change the problem slightly. Suppose the middle term (the “4D” term) is *quadratic* rather than *linear* drag. Write down the new ODE (don’t use the “D” notation, just write it out as an ODE) This is no longer analytically solvable! But that needn’t stop us - keeping all numerical coefficients the same as Boas had, just replacing linear with quadratic drag, use your favorite computational environment to solve and plot the new solution (with the same initial conditions). Did changing linear for quadratic drag change any qualitative aspects? Briefly, comment.
- (e) Invent two *physics* problems, for which the ODEs are the ones used in parts a and d respectively. Be explicit - what are the values of all relevant physical parameters, in SI units? What physics would induce you to choose the ODE of part d instead of part a?

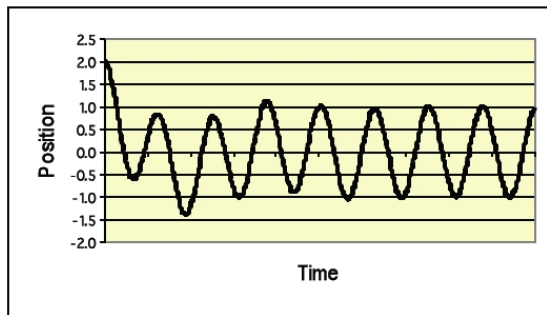
6.8 Picking out the transients

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*
Available at: <http://perlnet.umaine.edu/imt/FHM/HWFHM.doc>

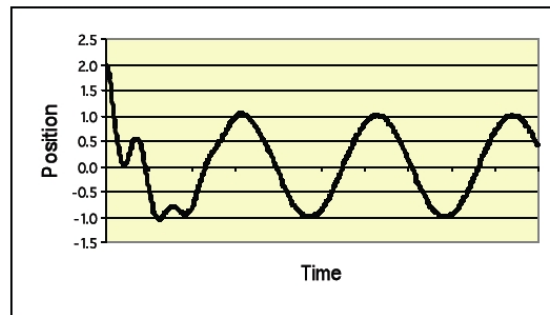
Given: Marino – Fall 2011, Pollock – Spring 2011, Spring 2012

A harmonic oscillator with a restoring force $25m\alpha^2x$ is subject to a damping force $4m\alpha\dot{x}$ and a sinusoidal driving force $F_0\cos(2\alpha t)$.

- Write down the differential equation that governs the motion of this oscillator.
- Show that if the driving force were removed the oscillator would become underdamped, and express the frequency of the oscillator in terms of the given quantities. Explain your reasoning.
- For any damped oscillator that is driven by a sinusoidal external force, we know that the eventual (steady-state) motion is sinusoidal in nature. However, before the oscillator reaches steady state, its motion can be thought of as the algebraic sum of the steady-state motion plus a transient oscillatory motion whose amplitude dies exponentially with time. The transient component of the motion (considered by itself) could accurately describe the motion of the same oscillator with the driving force turned off (but with the damping still present).
 - Each x vs. t graph below illustrates the actual motion (transient plus steady-state) of a damped, driven oscillator starting at $t = 0$. For each case, is the frequency of the steady-state motion greater than, less than, or equal to that of the transient motion? Explain.



Graph 1



Graph 2

- Identify which graph (1 or 2) would better correspond to the damped, driven oscillator described in parts a-b of this problem. Explain your reasoning.

6.9 Period runaway – Modeling a pendulum at large angles

Given: Pollock – Spring 2011, Spring 2012

Consider again the simple pendulum problem. In homework 7 you showed that the pendulum’s potential energy (measured from the equilibrium level) is $U(\phi) = mgL(1 - \cos \phi)$ (where L is the pendulum length) and that under the small angle assumption the motion is periodic with period $\tau_0 = 2\pi\sqrt{l/g}$. In this problem we will use Mathematica to find the period for large oscillations as well:

- Using conservation of energy, determine $\dot{\phi}$ as a function of ϕ , given that the pendulum starts at rest at a starting angle of Φ_0 . Now use this ODE to find an expression for the time the pendulum takes to travel from $\phi = 0$ to its maximum value Φ_0 . (You will have a formal integral to do that you cannot solve analytically! That’s ok, just write down the integral, being very explicit about the limits of integration) Because this time is a quarter of the period, write an expression for the full period, as a multiple of τ_0 .
- Use MMA to evaluate the integral and plot τ/τ_0 for $0 \leq \Phi_0 \leq 3$ rad. For small Φ_0 , does your graph look like that you expect? (Discuss - what value DO you expect?) What is τ/τ_0 for $\Phi_0 = \pi/8$ rad? How about $\pi/2$ rad? What happens to τ as the amplitude of the oscillation approaches π ? Explain.

Note: The integral has some curious numerical pathologies. If your plot has little gaps in it, it’s because MMA is generating tiny imaginary terms added to the result. You might just try plotting the real part of the integral, using the command `Re[]`. Similarly, if it gives you symbolic results, you might fix that by using the command `N[]`, which forces MMA to evaluate the expression as a number, if it can! By the way, this integral is called the “complete elliptic integral of the first kind”.

- A real grandfather clock has a length of about half a meter, and you pull the pendulum, oh, about 10 cm to the side. When designing the clock for practical use, would the clockmaker be safe in making the “small angle” approximation? (What we mean here is to justify your answer with a calculation that tells you whether this clock would be annoying or useful to have in your house! Can you be concrete about what you would consider “annoying” about a clock?)

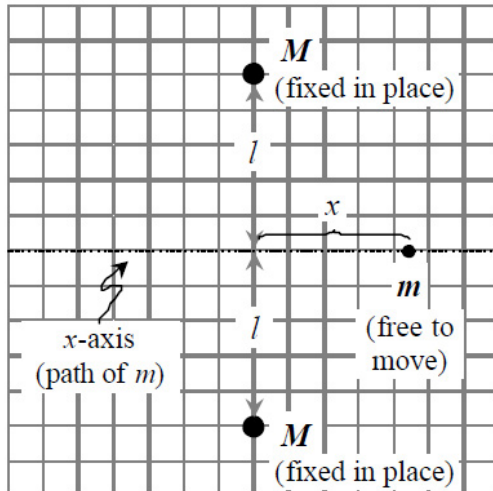
6.10 A gravitational oscillator?

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/SHM/HWSHM.pdf>

Given: Pollock – Spring 2011, Spring 2012

Two large solid spheres, each with mass M , are fixed in place a distance $2l$ apart, as shown. A small ball of mass m is constrained to move along the x -axis, shown as a dashed line. (Let $x = 0$ represent the point on the axis directly between the spheres, with $+x$ to the right.)



- Compute the net force on the ball by both large spheres, and write down a differential equation that governs the motion of the small ball.
- Is the net force exerted on the small ball a restoring force? Explain! Under what limiting conditions for x can we say the motion of m can be *approximated* as simple harmonic motion? (For large x ? For small x ? Large or small compared to what?) In this limiting case, find the period of motion in terms of the given parameters. Explain your reasoning.

6.11 Parallel and series springs

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/SHM/HWSHM.pdf>

Given: Pollock – Spring 2011

An ideal (massless) spring with force constant k is used to hang a crate from the ceiling, as shown (left, below). Suppose that the spring were treated as two separate springs (1 and 2) connected end-to-end. (See Figure) Treat each spring as having its own spring constant (k_1 or k_2) given by Hooke's law, $|F_i| = k_i x_i$, where $|F_i|$ is the magnitude of the force exerted on (or by) spring i when the length of that spring *changes* by an amount x_i .

- (a) Relate the magnitudes of the forces F_1 and F_2 exerted by each spring individually, and the magnitude of the force F exerted on the crate. Explain. (Hint: Sketch separate free body diagrams for the springs and the crate.) Then, find the relationship between the stretches x_1 and x_2 of the individual springs and the stretch x of the original (single) spring. Explain your reasoning. Based on these results, determine an expression for the spring constant of the original spring (k) in terms of the spring constants (k_1 and k_2) of the two individual “spring pieces”. Based on this problem, if you chop an ideal spring in half, what happens to its spring constant? Show all work.

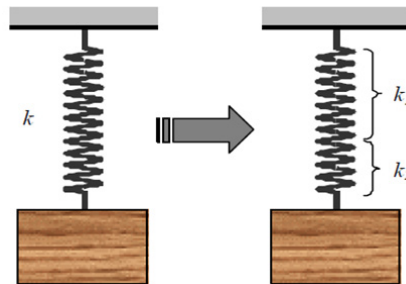


Figure 1:

- (b) Now consider the case in which you replace the original spring with two new springs (3 and 4) that connect directly to the crate and to the ceiling. (See Figure.) Assume that the crate ends up at the same equilibrium height in this new situation as when there was just a single spring (k). Following the same logic as you did above, determine an expression for the spring constant k of the original spring in terms of the spring constants k_3 and k_4 of the new springs. Explain your reasoning.

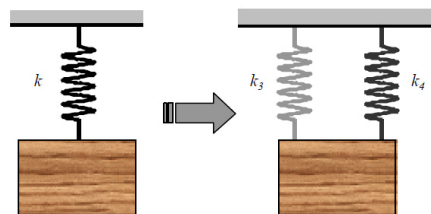


Figure 2:

6.12 Estimating damping parameters

Given: Pollock – Spring 2011

Consider a simple pendulum of length $L = 10m$.

- (a) In an ideal world (assuming no damping, and making the small angle approximation) determine the period of oscillation.

Now imagine we take into account air friction and determine that it causes the period to change by 0.1%. Using Taylor's notation introduced in Eq. 5.28, what is the damping factor β ? By what factor will the amplitude of oscillation decrease after 10 cycles?

- (b) Which effect of damping would be more noticeable - the change of the period or the decrease of the amplitude? Explain.

6.13 Energy considerations in a 1D simple harmonic oscillator

Given: Pollock – Spring 2011

- (a) Consider a simple harmonic oscillator with period τ . Let $\langle f \rangle$ denote the average value of a function $f(t)$ averaged over one complete cycle:

$$\langle f \rangle = \frac{1}{\tau} \int_0^\tau f(t) dt. \quad (1)$$

Prove that $\langle T \rangle = \langle U \rangle = E/2$, where E is the total energy of the oscillator, T the kinetic energy and U the potential energy.

Hint Start by proving the more general, and extremely useful, result that $\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = 1/2$. Explain why these two results are almost obvious, then prove them by using trig identities to rewrite $\sin^2 \theta$ and $\cos^2 \theta$ in terms of $\cos(2\theta)$.

- (b) If an additional damping force $F_{damp} = -b\dot{x}$ is added, find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ by straightforward differentiation (getting a simple result in terms of x , \dot{x} , and/or \ddot{x}) and then with the help of Taylor 5.24, show that it is exactly minus the rate at which energy is dissipated by F_{damp} .

The rest is pure extra credit.

The position of an overdamped oscillator is given by Eq. 5.40 in Taylor's text. Find the constants C_1 and C_2 in terms of the initial position x_0 and velocity v_0 . Then, sketch the behavior of $x(t)$ for the two separate cases $x_0 = 0$, and $v_0 = 0$. Finally, show that if you let $\beta = 0$, your solution for $x(t)$ matches the correct solution for undamped motion.

(I find this rather remarkable, since the solution you started from is for the overdamped case, it wasn't supposed to work for the undamped case?!) As Taylor puts it, the math is sometimes cleverer than we are!

- (a) Solve Boas Problem 8.6.12 (That's chapter 8, section 6, problem 12, on page 423). We just want the general solution.
Please tell us explicitly if the associated (complementary) homogenous problem is (choose one) {over-damped, under-damped, undamped, critically damped} How do you tell?
- (b) Going back to the full Boas problem - invent a physics problem for which that ODE is the solution. Be explicit - what are the values of all relevant physical parameters, in SI units?
Now, given that the system starts with $y(0) = 0$, $y'(0) = 0$, solve that problem completely. There should be no undetermined coefficients left!
- (c) Sketch your solution in part b) above, noting any important aspects of the sketch. (Don't worry about precision, this is a sketch, not a plot. We want to know the *features*.) Then, use MMA to plot the actual solution. Print it out and discuss - in particular, discuss if it did not exactly match with your sketch. (The differences may or may not be subtle, depending on the care of your initial sketch)

Given: Pollock – Spring 2012

- (a) Consider a simple harmonic oscillator with period τ . Let $\langle f \rangle$ denote the average value of a function $f(t)$ averaged over one complete cycle:

$$\langle f \rangle = \frac{1}{\tau} \int_0^\tau f(t) dt. \quad (2)$$

Prove that $\langle T \rangle = \langle U \rangle = E/2$, where E is the total energy of the oscillator, T the kinetic energy and U the potential energy.

Hint Start by proving the more general, and extremely useful, result that $\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = 1/2$. Explain why these two results are almost obvious, then prove them by using trig identities to rewrite $\sin^2 \theta$ and $\cos^2 \theta$ in terms of $\cos(2\theta)$.

- (b) If we add a damping force $F_{damp} = -b\dot{x}$, find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ by straightforward differentiation (getting a simple result in terms of x , \dot{x} , and/or \ddot{x}) and then with the help of Taylor 5.24, show that it is exactly the rate at which work is done by F_{damp} .
- (c) *The rest is pure extra credit*

The position of an overdamped oscillator is given by Eq. 5.40 in Taylor's text. Find the constants C_1 and C_2 in terms of the initial position x_0 and velocity v_0 . Then, sketch the behavior of $x(t)$ for the two separate cases $x_0 = 0$, and $v_0 = 0$. Finally, show that if you let $\beta = 0$, your solution for $x(t)$ matches the correct solution for undamped motion.

I find this rather remarkable, since the solution you started from is overdamped, it wasn't supposed to work for the undamped case?! As Taylor puts it, the math is sometimes cleverer than we are!

6.14 Leading and lagging in electrical oscillations

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/FHM/HWFHM.doc>

Given: Pollock – Spring 2011

A series LRC circuit (Taylor Fig 5.10) is connected across the terminals of an AC power supply that produces a voltage $V(t) = V_0 e^{i\omega t}$. The “equation of motion” for the charge $q(t)$ across the capacitor is as follows:

$$L\ddot{q} + R\dot{q} + q\frac{1}{C} = V_0 e^{i\omega t} \quad (3)$$

The above differential equation will have a steady-state solution of the form:

$q(t) = q_0 e^{i(\omega t - \phi)}$ [Note: The parameters q_0 and ϕ are actually functions of ω , the frequency of the AC power supply. However, in this problem you will not have to write out these functions in full.]

- (a) In terms of q_0, ω, ϕ and the relevant coefficients from the differential equation, write down the following functions:
 - i. the potential difference $\Delta V_C(t)$ across the capacitor
 - ii. the potential difference $\Delta V_R(t)$ across the resistor
 - iii. the potential difference $\Delta V_L(t)$ across the inductor
- (b) Determine the smallest positive values of α, β , and γ (in radians) that satisfy the following Euler relations: $e^{i\alpha} = i$, $e^{i\beta} = -1$, $e^{i\gamma} = -i$
- (c) Using your results from part b), rewrite the functions in part a) so that each function can be written as a positive real number times (the same) complex exponential. Use your rewritten functions to answer the following questions.
 - i. What is the phase difference between $\Delta V_C(t)$ and $\Delta V_R(t)$?
Do the peaks of $\Delta V_C(t)$ come just *before* (“leads”) or just *after* (“lags”) the peak of $\Delta V_R(t)$? (Show/explain your reasoning)
 - ii. What is the phase difference between $\Delta V_R(t)$ and $\Delta V_L(t)$?
Does ΔV_L lead or lag ΔV_R ?

Given: Pollock – Spring 2012

A series LRC circuit (Taylor Fig 5.10) is connected across the terminals of an AC power supply that produces a voltage $V(t) = V_0 e^{i\omega t}$. The “equation of motion” for the charge $q(t)$ across the capacitor is as follows:

$$L\ddot{q} + R\dot{q} + q\frac{1}{C} = V_0 e^{i\omega t} \quad (4)$$

The above differential equation has a steady-state solution which can be written as:

$q(t) = q_0 e^{i(\omega t + \delta)}$ [Note: The parameters q_0 and δ are actually functions of ω , the frequency of the AC power supply. However, in this problem you will *not* have to solve for or write out these functions in full!]

- (a) In terms of q_0, ω, δ and the relevant coefficients from the differential equation, write down the following functions:
 - i. the potential difference $\Delta V_C(t)$ across the capacitor
 - ii. the potential difference $\Delta V_R(t)$ across the resistor
 - iii. the potential difference $\Delta V_L(t)$ across the inductor

- iv. Finally, determine the smallest positive values of α , β , and γ (in radians) that satisfy the following Euler relations: $e^{i\alpha} = i$, $e^{i\beta} = -1$, $e^{i\gamma} = -i$
- (b) Using your results from part iv, rewrite the functions in part i-iii so that each function can be written as a positive real number times a complex exponential function. Use your rewritten functions to answer the following questions.
- i. What is the phase difference between $\Delta V_C(t)$ and $\Delta V_R(t)$?
Do the peaks of $\Delta V_C(t)$ come just *before* (“leads”) or just *after* (“lags”) the peak of $\Delta V_R(t)$? (Show/explain your reasoning)
- ii. What is the phase difference between $\Delta V_R(t)$ and $\Delta V_L(t)$?
Does $\Delta V_R(t)$ lead or lag $\Delta V_L(t)$?
- (c) Assuming for simplicity that $\delta = 0$, make a crude sketch on one graph of $\Delta V_R(t)$, and $\Delta V_C(t)$ being very careful to identify which is which. Does your sketch agree with your claim in part b about “leading” or “lagging”? (Briefly, comment)

6.15 Resonant behavior

Given: Pollock – Spring 2011

In section 5.6, Taylor states that the amplitude of an oscillator subject to a sinusoidal driving force is

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

- (a) Find an expression for ω such that the amplitude A is maximized. That is, derive our text's Equation 5.73. (*Hint:* Rather than differentiate and extremize the entire expression for $A(\omega)$, make your job easier by judiciously picking which part of that expression to differentiate and extremize.)

Then, use your expression to find the exact maximum amplitude, A_{max} .

Write a second expression for the amplitude A_0 which corresponds to $\omega = \omega_0$.

Briefly, comment on the difference between A_{max} and A_0 .

- (b) Open up the resonance PhET sim, found at

<http://www.colorado.edu/physics/phet/dev/resonance/>

Select the highest number version, and then click on “dev” (near the top). Play around to see what the sim can do.

If you set the damping at a moderate value (say, 1 N m/s) and then decrease the driving frequency of the oscillator to roughly 10% below the natural frequency, what do you predict will happen? Focus first on simply position as a function of time, and predict, based on Taylor 5.68 and the formula for A above what you should see (just qualitatively is fine.) Then, watch the sim carefully. Explain how what you see is or is not consistent with what you predicted. How long does it take for the motion to settle down? (you will only be able to crudely approximate this, but if you are way off, chances are that something went wrong)

After waiting for any transients to die out in the system you have set up, focus on the relationship between the phase of the driver and the mass. What does Taylor's Fig 5.19 predict? What does it look like on the sim? (It's hard to tell exactly, just be approximate. Note that there is a “slow motion” slider, you can put rulers on the screen, and you can set up multiple oscillators with different parameters if that helps you)

- (c) Now increase the frequency of the driver to about 10% *higher* than the resonant frequency. What relationship between the phase of the driver and the mass do you expect? Does the PhET sim confirm this? How does the time it takes to reach this final value compare to what it took for the case of 10% *lower* driving frequency? (Briefly, comment)
- (d) Check out the case of driving the mass at ω_0 . What is the phase difference between driver and mass?

Building on what you have seen in the last three parts, explain Taylor Figure 5.19 in your own words.

Extra credit. Play with the sim a little more, and write down one additional (real) question that you have about something you notice!

Given: Pollock – Spring 2012

In section 5.6, Taylor states that the amplitude of an oscillator subject to a sinusoidal driving force is

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

- (a) Find an expression for ω such that the amplitude A is maximized. That is, derive our text's Equation 5.73. (*Hint:* Rather than differentiate and extremize the entire expression for $A(\omega)$, make your job easier by judiciously picking which part of that expression to differentiate and extremize.)

Then, use your expression to find the exact maximum amplitude, A_{max} .

Write a second expression for the amplitude A_0 which corresponds to $\omega = \omega_0$.

Briefly, comment on the difference between A_{max} and A_0 .

- (b) Open up the resonance PhET sim, found at <http://phet.colorado.edu/en/simulation/resonance> Play around to see what the sim can do.

By investigating the sim in a variety of ways, clearly explain Taylor’s Figure 5.19 in your own words, *referring explicitly to “experiments” or setups you made with the sim* to help you connect the figure to the behavior of the objects. (Please discuss briefly what effects “transients” have, and how you dealt with them in the sim)

Extra credit. Play with the sim a little more, and write down one additional (real, interesting) question that you have about something you notice!

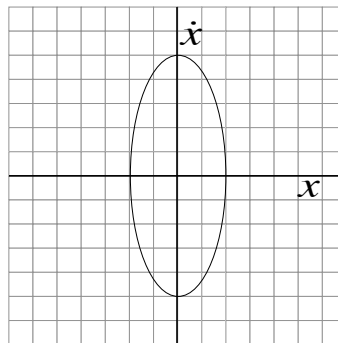
6.16 Sketching phase space diagrams

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/Phase-SHM/HWPhase-SHM.pdf>

Given: Pollock – Spring 2012

- (a) A 4 kg mass is connected to a spring with spring constant $k = 1.0 \text{ N/m}$. At $t = 0$ the mass is set into simple harmonic motion (no damping) by releasing it from rest at a point $x = -1.0 \text{ m}$ (i.e. a meter to the left of the origin). Sketch an accurate phase space plot for the oscillator. Explain your reasoning and show your work. If the direction the particle follows around the plot is ambiguous, say so, otherwise draw an arrow to show how it moves around the phase space diagram.
- (b) On the phase space trajectory you have drawn, label the point Q that represents the position and velocity of the oscillator one-quarter period after $t = 0$. Explain your reasoning. Also, on the phase space trajectory you drew, add a second trajectory (make it “dashed” so we can tell them apart) which shows the phase space plot of a system with the *same total energy* but *smaller mass*. Explain your reasoning briefly.
- (c) Now consider an elliptical phase space trajectory for a *different system*, as shown in the figure below. Assuming each unit on the horizontal axis is 10 cm, and each unit along the vertical axis is 10 cm/s, and assuming a mass $m = 1.0 \text{ kg}$, determine numerical values for the following quantities: i) angular frequency, ii) period, iii) total energy, and iv) spring constant



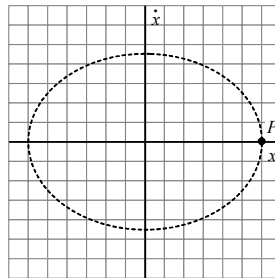
6.17 Adding linear damping to an undamped oscillator

Adapted from: Ambrose & Wittmann, *Intermediate Mechanics Tutorial*

Available at: <http://perlnet.umaine.edu/imt/Phase-DHM/HWPhase-DHM.pdf>

Given: Pollock – Spring 2012

The phase space trajectory of an undamped oscillator is shown below. In the diagram, each division along the position axis corresponds to 0.1 m; along the velocity axis, 0.10 m/s.



- First, what is the angular frequency ω_0 of the undamped oscillator? Explain how you can tell. Next - a retarding force is now applied to the oscillator for which the damping constant (using Taylor's notation, introduced in Eq. 5.28) is equal to $\beta = 0.0644\omega_0$. By what factor does the amplitude change after a single oscillation? After 10 cycles? Show all work.
- On the basis of your results above, carefully sketch the phase space plot for the first cycle of the motion of the damped oscillator, starting at point P. *There is a larger figure on the last page of this homework set that you can print out to do your sketching.*
- If this system corresponded to a real pendulum, which effect of damping would be more noticeable - the change of the period or the decrease of the amplitude? Justify your opinion.