Problem List

7.1 Fourier series expansion of absolute value function

Given: Marino – Fall 2011

For this problem, consider the periodic function $f(x)$: (one period of the function is specified here)

$$\begin{aligned} f(x) &= -x, \text{ for } -\frac{\pi}{2} < x < 0 \\ &= x, \text{ for } 0 < x < \frac{\pi}{2} \end{aligned}$$

(a) (2 pt) Find the Fourier Series expansion for $f(x)$. Be sure to include a sketch of the function.

(b) (1 pt) To check that your solution is correct, use a computer, plot the sum of the **first 3 non-zero terms** in this series from $-\pi$ to π. It should resemble the sketch that you drew in part a.

7.2 Fourier series expansion of step function

Given: Marino – Fall 2011

For this problem, consider the periodic function $g(x)$: (one period of the function is specified here)

$$\begin{aligned} g(x) &= 1, \text{ for } -1 < x < 1 \\ &= 0, \text{ for } 1 < x < 5 \end{aligned}$$

(a) (2 pt) Find the Fourier Series expansion for $g(x)$. Be sure to include a sketch of the function.

(b) (1 pt) To check that your solution is correct, use a computer, plot the sum of the first **first 4 non-zero terms** in this series from -10 to 10. It should resemble the sketch that you drew in part a.

7.3 Orthogonality of functions

Given: Marino – Fall 2011

Show that the functions x^2 and $\sin x$ are orthogonal on the interval (-1,1). (Hint: You should not need to work out the integral. What do you know about even and odd functions?)

7.4 Fourier transform of sinusoidal pulse

Given: Marino – Fall 2011

Consider the following function:

$$\begin{aligned} f(x) &= \sin x, \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ &= 0, \text{ for } |x| > \frac{\pi}{2} \end{aligned}$$

(a) (1 pt) Demonstrate that the Fourier transform of $f(x)$ is $g(\alpha) = \dfrac{-i\alpha \cos\left(\frac{\alpha\pi}{2}\right)}{\pi(1-\alpha^2)}$. Hint: It might help to express $e^{-i\alpha x}$ in terms of sines and cosines. Recalling the properties of integrals of odd and even functions can also save you some work.

(b) (0.5 pt) Using your answer for $g(\alpha)$, express $f(x)$ as a Fourier integral (i.e. substitute your result for $g(\alpha)$ into the Fourier integral for $f(x)$). Leave the answer as an integral. Do not evaluate the integral.

7.5 Introducing the Legendre polynomials

Given: Marino – Fall 2011

We mentioned in class that the first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \end{aligned}$$

(a) (0.5pt) Use the recursion relation to find $P_3(x)$. You must show all work to receive any credit.

(b) (1 pt) Use the built-in Mathematica function `LegendreP` to find $P_3(x)$ and $P_2(x)$. Plot $P_3(x)$ and $P_2(x)$ from $x = -1$ to $x = 1$.

(c) (0.5 pt) What kind of symmetry do $P_3(x)$ and $P_2(x)$ have? Are $P_2(x)$ and $P_3(x)$ orthogonal over the interval (-1,1)? Explain how you can tell that without having to evaluate any integrals.

7.6 Using Fourier series to compute the infinite sum of reciprocal squares

Given: Pollock – Spring 2011, Spring 2012

Do you know the precise value of the summation of the reciprocals of the squares of the natural numbers, $\sum_{n=1}^{\infty} \frac{1}{n^2}$? This is a famous problem in mathematical analysis with relevance to number theory. It was first posed by Pietro Mengoli in 1644 and solved by Euler in 1735. Since the problem had withstood the attacks of the leading mathematicians of the day, Euler's solution brought him immediate fame when he was twenty-eight. Let's solve it using Fourier series (a tool that wasn't yet available to Euler!)

(a) Define the periodic function $f(x) = x^2$ for $-\pi < x \leq \pi$ and compute the Fourier series.

(b) Evaluate the function at $x = \pi$ and use the Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The answer is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Your job is to show how this comes about!)

7.7 Using complex Fourier series notation

Given: Pollock – Spring 2011

Any periodic function, $f(t) = f(t+T)$ can also be Fourier expanded as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega nt} \tag{1}$$

with $\omega = 2\pi/T$.

This is very similar to our usual expansion in terms of sin's and cos's, but combines them more elegantly using complex exponentials, and is the more common form used by physicists. Let's work out the details!

(a) Show that $\frac{1}{T}\int_{-T/2}^{T/2} e^{i(m-n)t\omega} dt = 0$ if $m \neq n$ and $\frac{1}{T}\int_{-T/2}^{T/2} e^{i(m-n)t\omega} dt = 1$ if $m = n$.
Here m, n are integers.

(b) Use (a) to show that $c_m = \frac{1}{T}\int_{-T/2}^{T/2} e^{-imt\omega} f(t)dt$, where m is just a dummy index.

Hint! Look at the use of Fourier's trick from my lecture on Apr 7 (see concept test slides), or online lecture notes on p. 11a and 11b, or Boas Ch 7.7, and figure out how to apply it here.

(c) Use Euler's formula $e^{ix} = \cos x + i \sin x$ to show that

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ a_0 & n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \end{cases}$$

Where the a's and b's are the usual Fourier coefficients we've defined in class and Taylor.

Note that, when expanding in complex exponentials like this, we "win" by having just one simple formula for the c_n's (and integrals of exponentials are often easier).

The price we pay is that we have to run our sum over negative n's

7.8 Fourier series expansion of a periodic polynomial function

Given: Pollock – Spring 2011

For the periodic function $f(t) = At$ for $-1 < t \leq 1$ (with A a given constant)

(a) Compute the Fourier coefficients a_n and b_n

(b) Suppose this force drives a weakly damped oscillator with damping parameter $\beta = 0.05$ and a natural period $T = 2$. Find the long time motion $x(t)$ of the oscillator. Discuss your answer. Use Mathematica to plot $x(t)$ using the first four non-zero terms of the series, for $0 < t \leq 10$. (Set A=1 for this)

(c) Repeat part b) for the same damping parameter, but with natural period $T = 1$. Briefly discuss any major differences between this result and what you had in the previous part.

(d) For most of the possible natural frequencies the response of this particular driven oscillator is going to be weak. However, there are some particular natural frequencies at which you are going to see an enhanced response - determine which those are. (Be careful, think a bit about this!)

Given: Pollock – Spring 2012

For the periodic function $f(t) = A(t^2 - 1/3)$ for $-1 < t \leq 1$ (with A a given constant)

(a) Compute Fourier coefficients a_n and b_n. (Some vanish - predict which ones *before* doing any integrals)

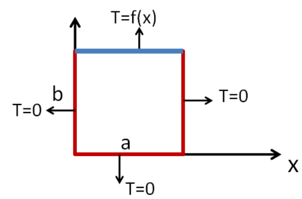
(b) Suppose this force drives a weakly damped oscillator with damping parameter $\beta = 0.05$ and a natural period $T = 2$. Find the long time motion $x(t)$ of the oscillator. Discuss your answer. Use Mathematica to plot $x(t)$ using the first four non-zero terms of the series, for $0 < t \leq 10$. (Set A=1 for this)

(c) Repeat part b) for the same damping parameter, but with natural period $T = 1$. Briefly discuss any major differences between this result and what you had in the previous part.

(d) For most of the possible natural frequencies the response of this particular driven oscillator is going to be weak. However, there are some particular natural frequencies at which you are going to see an enhanced response - determine which those are.

7.9 Finding the temperature field on a square plate

Given: Pollock – Spring 2011



For the rectangular plate problem shown above calculate the steady state temperature $T(x, y)$ everywhere inside the plate if the boundary condition on the top surface is given by

$$f(x) = \begin{cases} T_0 & 0 < x \leq a/2 \\ 0 & a/2 < x \leq a \end{cases} \tag{2}$$

For the situation above:

(a) Code up your formula in Mathematica, adding up 30 terms in your sum to find T(x,y). For concreteness, please set a=1, b=2, and T_0=100.

Note that to define a function in Mathematica you need to be careful about syntax. If, for instance, $T(x, y) = \Sigma_n n\ sin(n\pi x)e^{-n\pi y}$, *the Mathematica syntax for that might be*

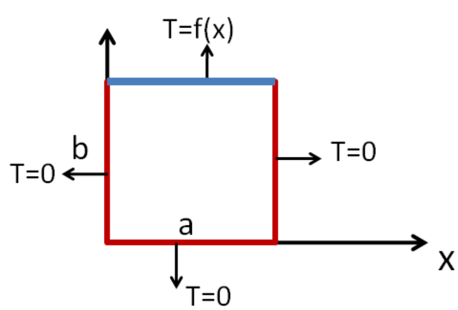
t[x_,y_] := Sum[n Sin[n Pi x] Exp[-n Pi y],{n,1,30}]

Watch out for the underscores for variables x and y on the left side (they do NOT appear again on the right side), and the colon before the equals sign. Don't capitalize functions that you invent, caps are reserved for MMA built-in functions.

(b) As a check, set y=2, and just plot your T[x,2] (from x=0 to 1) to make sure you did that Fourier sum right. (That's the boundary condition we gave you, so you know what it should look like!) Comment on any interesting or surprising features you observe about this plot, is it exactly what you expected?

(c) Plot your full T(x,y) using the function Plot3D. See the documentation center for the basic syntax of Plot3D. Briefly, discuss/document the key features of your plot. Please tell us explicitly what does the height of this graph represent, physically? *Plot3D is very cool. You can rotate the resulting curve with your mouse to really get a good look at your result.*

Given: Pollock – Spring 2012



You are a summer engineering-physics intern, working on a modified Rayleigh-Bernard convection (`http://goo.gl/Itbi2`) experiment, and the edges of the rectangular bottom plate of the system are held at the temperatures as shown in the figure above.

(a) Your advisor has asked you to determine the steady state temperature distribution T(x,y) for the plate, so another student can put that boundary information into a computational model for the convective flow she's constructed. The boundary condition on the top surface is given by

$$f(x) = \begin{cases} 0 & 0 < x \leq a/2 \\ 2T_0 & a/2 < x \leq a \end{cases} \tag{3}$$

(b) You want to check that your solution makes sense. Code up your formula, adding up 30 terms in your sum to find T(x,y). For concreteness, please set a=1, b=2, and T_0=100. The very first thing I would want to do to check this code would be to set y=2, and do a simple plot of T[x,2] (from x=0 to 1), to make sure that the Fourier sum is giving what you expect. (What DO you expect? Sketch it first, then plot it. Comment on any interesting or surprising features you observe about this plot, is it exactly what you expected?)

Some Mathematica reminders: if, for instance, $T(x, y) = \Sigma_n 2n\ sin(n\pi x)e^{-n\pi y}$*, the Mathematica syntax for that might be*

t[x_,y_] := Sum[2n Sin[n Pi x] Exp[-n Pi y],{n,1,30}]

Watch out for the underscores for variables x and y on the left side (they do NOT appear again on the right side), and the colon before the equals sign. Don't capitalize functions that you invent, caps are reserved for MMA built-in functions.

(c) Now that you are more confident your (nasty!) Fourier work is probably ok, plot your full T(x,y) using a 3D plot function (MMA users will find Plot3D useful. See the documentation center for the basic syntax of Plot3D.) Briefly, discuss/document the key features of your plot. Please tell us explicitly what does the height of this graph represent, physically? Given these results, and the goal of your summer internship project, do you have any comments for your advisor? *Plot3D is very cool. You can rotate the resulting curve with your mouse to really get a good look at your result.*

Extra credit, let's just play around with Fourier series a little!

(i) For the rectangular plate problem shown above, calculate the steady state temperature $T(x, y)$ everywhere inside the plate if the boundary condition on the top surface was $f(x) = \sin(3\pi x/a)$. (Note - this is much *easier* than the problem above, if you think about it! In fact, if you look back at what you did there, you shouldn't need to do any new calculations at all.)

(ii) Keep the boundary condition the same as part i on the bottom and left edges (T=0), and keep the boundary condition the same at the top edge too, i.e. T(x,y=b)= $\sin(3\pi x/a)$. But let's change the right edge: Let T(x=a,y) = $\sin(3\pi y/b)$. Find T(x,y) everywhere else. *Hint: Once again, there is no need to do any real calculations here - you can pretty much just write down the answer if you think about it right!*

7.10 Basic properties of Fourier transforms

Given: Pollock – Spring 2011

(a) (2 pts) Show that if a function $f(x)$ is even, then its Fourier transform $g(\alpha)$ (as defined in Boas Eq 12.2) is also an even function.

(b) (2 pts) Show that in this case f(x) and g(α) can be written as Fourier cosine transforms and obtain Boas Eq 12.15 (Assuming that f(x) is identical to Boas' $f_c(x)$, how is $g_c(\alpha)$ from Boas' Eq 12.15 related to g(α) of Boas' Eq 12.2?

Please note the typo in the top equation of Boas 12.15, the argument of g_c should be α, not x.

7.11 Fourier transform of triangular pulse

Given: Pollock – Spring 2011

For the function shown in the figure below

(a) (1 pt) Qualitatively, how do you expect $g(\alpha)$, the Fourier transform of $f(x)$, to change if you make a bigger or smaller?

(b) (4 pts) Find the Fourier transform $g(\alpha)$.

(c) (2 pts) Sketch $g(\alpha)$ (or plot it in Mathematica if you prefer). Was your prediction in part (a) correct? Explain.

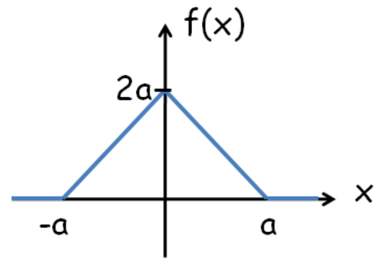


Figure 1:

7.12 Fourier transform of an important function from QM

Given: Pollock – Spring 2011

The function $e^{-c|x|}$ (with c a positive, known constant) is associated with bound states in quantum mechanics, and (pretty much as always in quantum mechanics) its Fourier transform is important to know. Find the Fourier transform of $e^{-c|x|}$, sketch it, and comment on any key features.

As a mathematical aside, use your result to (easily!) compute $\int_0^\infty \frac{\cos(\alpha x)}{\alpha^2+1} d\alpha$.

7.13 You're an audio engineer!

Given: Pollock – Spring 2012

You've gotten a summer job interning as an audio engineer. Audio software produces periodic tones by summing pure frequencies (each with its own amplitude) - it's just summing up a Fourier series! Your boss has asked you to generate a tone with the following properties: It should have a fundamental frequency of 440 Hz (Middle A). On each cycle (i.e. 1/440th of a second), instead of being a pure cos or sin wave, your wave should start with a given amplitude A, ramp down *linearly* (rather than sinusoidally) to $-A$ in half a period, then ramp back up linearly back to $+A$, and repeat this pattern indefinitely.

(a) Sketch the function you are trying to build (over a couple of periods). Clearly label your horizontal axis. Do you expect any of the terms (cosines or sines?) in the Fourier series to vanish? Briefly, why?

(b) Compute the Fourier series for your waveform (which means coming up with a formula for the amplitude of each Fourier term.)

(c) To check your solution, use a computer to plot the sum of the first 3 non-zero terms in your series (over a range of several periods.) It should resemble the sketch that you drew in part a. If you're using Mathematica, there is a function called "Play", try it out! (Set A=1) Compare it to Play[Cos[440 * 2 Pi t], t,0,1] and describe in words how your tone is similar, and how it differs, from the pure tone.