Non-Linear Differential Equations in Mathematica

PHYS 2210 Fall 2010

YOUR NAME IS:

Recall that the motion of a pendulum (ignoring air resistance) can be described by the following differential equation

$$ml\frac{d^2\theta}{dt^2} + mg\sin(\theta) = 0, \tag{1}$$

where θ is the angle of the pendulum with respect to vertical, l is the length of the pendulum and m is the mass. When θ is small, we can make the approximation that $\sin(\theta) = \theta$ and we obtain

$$ml\frac{d^2\theta}{dt^2} + mg\theta = 0. \tag{2}$$

1. Consider Equation 2 with m=1 kg, and l=1 m. If the pendulum is released at rest at $t = 0$ from θ=0.05 radians, what is the particular solution for $\theta(t)$? Don't use Mathematica here! You should be able to solve this one exactly on your own!

2. What is the period of this solution?

3. If instead the initial condition is that θ=0.75 radians at $t = 0$, does the period of the motion change?

4. Now use Mathematica to solve Equation 1 (keeping the $\sin(\theta)$ in the expression) for the case where the pendulum is released from rest at $t = 0$ from θ=0.05 radians. Make sure that you figure out how to apply initial conditions. Is there an exact solution if you use `DSolve` or do you have to use `NDSolve` ?

5. Make a plot of the solution. (The Mathematica syntax here is a bit tricky. You might need to use `/.`which is the replacement operator.)

6. Now use the `FindRoot` command to figure our where the solution is zero. (You should have some idea of where to look for the roots from the plot that you made.) Use the locations of two successive roots to determine the period. How does this compare the to the period of the approximate solution found above in Problem 2?

7. Now find the solution to Equation 1 for the case where the pendulum is released from rest at $t = 0$ from θ=0.75 radians, and plot the solution.

8. Once again, use the `FindRoot` command to determine the locations of two successive roots to determine the period. How does this compare to the case where it was released from $\theta = 0.05$ in Problem 6?

9. Based on these results, what can you say about how well simple harmonic motion approximates the behavior of a pendulum?

If you have time left try this.....

Recall that the motion of a damped harmonic oscillator can be described by the following differential equation

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0, \tag{3}$$

where ω_0 is the natural frequency the system.

10. If $\omega_0 = 1$. For what value of β does the system change from underdamped to overdamped?

11. Equation 3 assumes a linear drag force, but as we saw at the beginning of the semester, for large objects like a softball, quadratic air resistance dominates. That gives us the following *non-linear* differential equation

$$\frac{d^2x}{dt^2} + 2\beta(\frac{dx}{dt})|\frac{dx}{dt}| + \omega_0^2 x = 0. \tag{4}$$

    Why is the absolute value sign necessary?

12. Use Mathematica to solve this equation, with the initial conditions $x(0) = 10$, $x'(0) = 0$, $\omega_0 = 1$, and $\beta = 0.1$. Plot the result. Is it underdamped or overdamped?

13. Vary the value of β. Can you find a value where it changes over from underdamped to overdamped? How does it compare to the linear case?