CLASSICAL MECHANICS AND MATH METHODS, SPRING, 2011

Homework 11

(Due Date: Start of class on Thurs. March 31)

1. Consider a simple pendulum of length $L = 10m$.

(a) In an ideal world (assuming no damping, and making the small angle approximation) determine the period of oscillation.

Now imagine we take into account air friction and determine that it causes the period to change by 0.1%. Using Taylor's notation introduced in Eq. 5.28, what is the damping factor β? By what factor will the amplitude of oscillation decrease after 10 cycles?

(b) Which effect of damping would be more noticeable - the change of the period or the decrease of the amplitude? Explain.

2. Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small *frictionless* object of radius ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. At time $t = 0$ the puck is released at height h with a purely angular initial velocity ω_0.

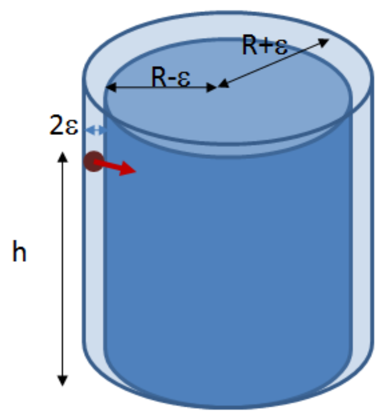

Figure 1:

(a) Write down Newton's second law first in terms of cartesian coordinates and then in terms of cylindrical polar coordinates. Which form would be easiest to use to solve for the motion of the ball described above? Why? (*If you need a reminder about what cylindrical polar coordinates are, problem 1.47 on p. 40 of Taylor might help. If you need a reminder about Newton's law in other coordinate systems, you might look back to Taylor's discussion that led up to Eq 1.48*)

(b) Describe and/or sketch a prediction of what the motion of the ball will be for $t > 0$. You can assume that h and ω are positive, and that the ball is subject to the force of gravity (on earth) which points in the $-\hat{z}$ direction.

(c) Use the equations you wrote in part (a) to solve for the motion of the ball. Does it match what you predicted? If not, explain how to reconcile any differences.

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3. Consider the motion of a 2D non-isotropic oscillator, in which the spring constant in the x direction (k_x) is not equal to the spring constant in the y direction (k_y). Each trajectory below depicts the possible motion of a unique non-isotropic oscillator. All three oscillators, share the property that the angular frequencies ω_x and ω_y for the motions along the x- and y-axes are commensurate, i.e., that the angular frequencies satisfy the following relationship: $\frac{\omega_x}{n_1} = \frac{\omega_y}{n_2}$ where n_1 and n_2 are integers. For each case below,

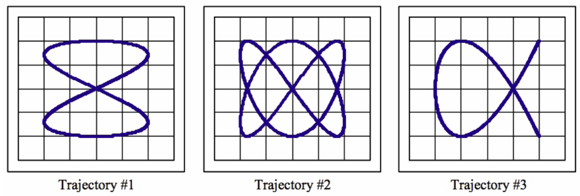

Figure 2:

(a) determine whether ω_x is greater than, less than, or equal to ω_y, and determine the values n_1 and n_2 that satisfy the condition. Explain.

(b) Using Mathematica generate the three Lissajous plots shown above (You can download the Mathematica code from the course webpage). What relative phase angle (the difference between x and y phases) is needed for the three trajectories (and what flexibility is there in this choice)?

4. Fig 2 shows the phase space plot for a simple harmonic oscillator (dashed) and the same oscillator with a retarding force applied (solid). Point P represents the initial conditions of the oscillator in both instances. In the diagram, you may assume each division along the position axis corresponds to 0.1 m; along the velocity axis, 0.10 m/s.

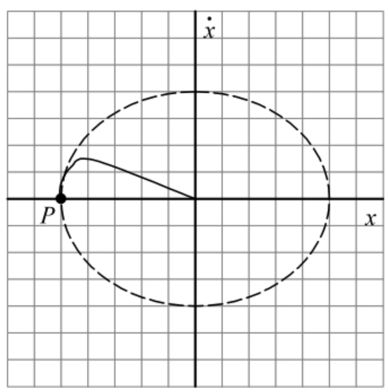

Figure 3:

(a) Explain how you can tell that the damped oscillator is not underdamped.

(b) Is the damped oscillator critically damped or overdamped? Explain how you can tell.

(c) If you said in part b that the oscillator is {critically damped, overdamped}, then draw how the phase space plot would be different if the oscillator (starting at point P) were instead {overdamped, critically damped}. Explain your reasoning

5. Energy considerations in a 1D simple harmonic oscillator:

(a) Consider a simple harmonic oscillator with period τ. Let $\langle f \rangle$ denote the average value of a function $f(t)$ averaged over one complete cycle:

$$\langle f \rangle = \frac{1}{\tau} \int_0^\tau f(t) dt. \tag{1}$$

Prove that $\langle T \rangle = \langle U \rangle = E/2$, where E is the total energy of the oscillator, T the kinetic energy and U the potential energy.

Hint Start by proving the more general, and extremely useful, result that $\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = 1/2$. Explain why these two results are almost obvious, then prove them by using trig identities to rewrite $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos(2\theta)$.

(b) If an additional damping force $F_{damp} = -b\dot{x}$ is added, find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ by straightforward differentiation (getting a simple result in terms of x, $\dot{x}$, and/or $\ddot{x}$) and then with the help of Taylor 5.24, show that it is exactly minus the rate at which energy is dissipated by F_{damp}.

The rest is pure extra credit, worth up to 6 points!

The position of an overdamped oscillator is given by Eq. 5.40 in Taylor's text. Find the constants C_1 and C_2 in terms of the initial position x_0 and velocity v_0. Then, sketch the behavior of $x(t)$ for the two separate cases $x_0 = 0$, and $v_0 = 0$. Finally, show that if you let $\beta = 0$, your solution for x(t) matches the correct solution for undamped motion.

(I find this rather remarkable, since the solution you started from is for the overdamped case, it wasn't supposed to work for the undamped case?!) As Taylor puts it, the math is sometimes cleverer than we are!