UNIVERSITY OF COLORADO AT BOULDER **PHYS 2210**

CLASSICAL MECHANICS AND MATH METHODS, SPRING, 2011

Homework 13

(Due Date: Start of class on Thurs. Apr 14)

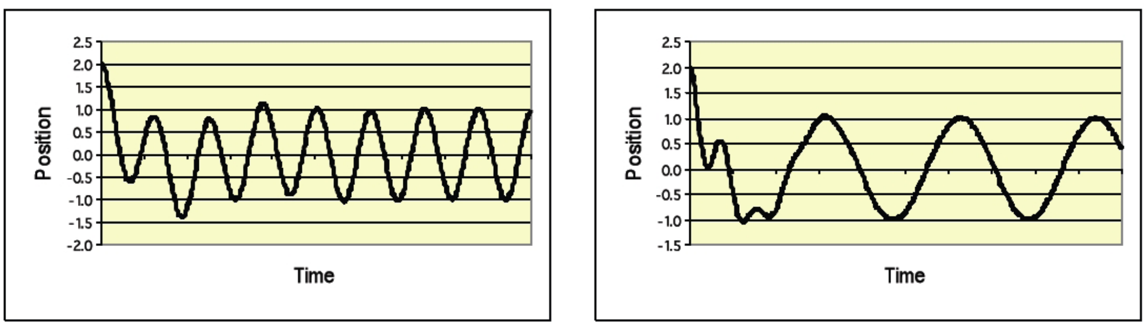
As always, lettered parts are worth 4 points unless otherwise stated

1. *Since not all of you had a chance to look at the recent tutorial extra-problem, we decided to assign it as a homework problem!*

A harmonic oscillator with a restoring force $25m\alpha^2x$ is subject to a damping force $3m\alpha v$ and a sinusoidal driving force $F_0\cos(10\alpha t)$.

(a) Write down the differential equation that governs the motion of this oscillator. Show that if the driving force were removed the oscillator would become underdamped, and express the frequency of the oscillator in terms of the given quantities. Explain your reasoning.

(b) Each x vs. t graph below illustrates the actual motion (transient plus steady state) of a damped, driven oscillator starting at $t = 0$. For each case, is the frequency of the steady-state motion greater than, less than, or equal to that of the transient motion? Explain.



Graph 1 **Graph 2**

Figure 1:

(c) (2 pts) Identify which graph (1 or 2) would better correspond to the damped, driven oscillator described in parts a) of this problem. Explain your reasoning.

2. Do you know the precise value of the summation of the reciprocals of the squares of the natural numbers, $\sum_{n=1}^{\infty}\frac{1}{n^2}$? This is a famous problem in mathematical analysis with relevance to number theory. It was first posed by Pietro Mengoli in 1644 and solved by Euler in 1735. Since the problem had withstood the attacks of the leading mathematicians of the day, Euler's solution brought him immediate fame when he was twenty-eight. Let's solve it using Fourier series (a tool that wasn't yet available to Euler!)

(a) Define the periodic function $f(x) = x^2$ for $-\pi < x \leq \pi$ and compute the Fourier series.

(b) Evaluate the function at $x = \pi$ and use the Fourier series to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The answer is $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. *Your job is to show how this comes about!)*

CONTINUED

3. Any periodic function, $f(t) = f(t+T)$ can also be Fourier expanded as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega n t} \tag{1}$$

with $\omega = 2\pi/T$.

This is very similar to our usual expansion in terms of sin's and cos's, but combines them more elegantly using complex exponentials, and is the more common form used by physicists. Let's work out the details!

(a) Show that $\frac{1}{T}\int_{-T/2}^{T/2} e^{i(m-n)t\omega} dt = 0$ if $m \neq n$ and $\frac{1}{T}\int_{-T/2}^{T/2} e^{i(m-n)t\omega} dt = 1$ if $m = n$. Here m, n are integers.

(b) Use (a) to show that $c_m = \frac{1}{T}\int_{-T/2}^{T/2} e^{-imt\omega} f(t) dt$, where m is just a dummy index. *Hint! Look at the use of Fourier's trick from my lecture on Apr 7 (see concept test slides), or online lecture notes on p. 11a and 11b, or Boas Ch 7.7, and figure out how to apply it here.*

(c) Use Euler's formula $e^{ix} = \cos x + i \sin x$ to show that

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ a_0 & n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \end{cases}$$

Where the a's and b's are the usual Fourier coefficients we've defined in class and Taylor.

Note that, when expanding in complex exponentials like this, we "win" by having just one simple formula for the c_n's (and integrals of exponentials are often easier).

The price we pay is that we have to run our sum over negative n's

4. An inclined plane is tilted at an angle α above the horizontal. A ball is launched (initial velocity v_0) into the air directly up the inclined plane, launched at an *additional* angle β above the plane's surface.

(a) Draw the situation here, and roughly sketch a predicted path of the ball for some initial conditions that you choose.

(b) (*Hint: before you start to work on this question, read the whole thing and then think carefully about what coordinate system would be easiest here.*) Find the ball's position as a a function of time (until it hits the plane). Show that the ball lands a distance $R = 2v_0^2 sin\beta cos(\beta + \alpha)/(gcos^2\alpha)$ from its launch point. Show that for given v_0 and α, the maximum possible range up the inclined plane (i.e. the maximum value of the distance between landing and launch point) is $R_{max} = v_0^2/[g(1 + sin\alpha)]$.

(c) Think of two specific combinations of α, β, and v_0 for which you can easily predict the outcome (either the ball's position as a function of time, or the range/maximum range) and use these combinations to check your calculations in the previous step (or to check that the final formulas we gave you are OK)

5. Evaluate the following integrals: (2 pts each)

(a) $\int_0^\pi dx \sin(x)\delta(x - \pi/2)$

(b) $\int_0^3 (5t - 2)\delta(2 - t)dt$

(c) $\int_0^5 (t^2 + 1)\delta(t + 3)dt$

(d) $\int_{-\infty}^{\infty} e^x \delta(3x)$

(e) If a particle feels a force $F(t)$ of the form $F = A\delta(t)$, with t the time, what are the units of A? Give a brief physical interpretation to this formula - what sort of physical force are we trying to represent here?

Extra credit Go back to problem 1, and "reverse engineer it" - write a Mathematica code which produces the plot which corresponds to the oscillator described in part a of the problem. (Taylor's Eq's 5.68 and 5.70 will help!)