UNIVERSITY OF COLORADO AT BOULDER **PHYS 2210**

CLASSICAL MECHANICS AND MATH METHODS, SPRING, 2011

Homework 2

(Due Date: Start of class on Thurs. Jan 20)

NOTE: Be sure to show your work and explain what you are doing. (Correct answers, for which we cannot follow the work, are worth no credit).

1. (4 pts) A particle is projected with velocity v_0 straight up a slope which makes an angle α with the horizontal. Find the time required to return to the starting point. (assume frictionless motion) Explicitly check the units of your final formula, and discuss the special cases $\alpha \to 0$ and $\alpha \to 90°$.

2. A soccer player kicks the ball with a velocity v_0 at an angle 45° (to reach maximum range). She want to kick it in such a way that it barely passes on top of two opposite team players of height h (12 pts: 4+4+4).

(a) Show that if the separation between the two opponents is $d \leq \frac{v_0}{g}\sqrt{v_0^2 - ?gh}$, the soccer player succeeds. (What is the numerical value of the ? in this formula?)

(b) Use Mathematica to plot d as a function of v_0, setting $h = 2m$. *Hint: If you want to plot a function $f[x]$ vs x with $x_0 < x < x_1$ with Mathematica you can use the instruction* Plot$[f[x], \{x, x_0, x_1\}]$. *Sqrt[] is the syntax for square roots. Note that Mathematica can't plot a function with symbolic coefficients (so you have to plug in the values for both g and h) Important: when you type a command into Mathematica, you must hit Shift-return, rather than just return, to have it evaluate what you typed!!.*

(c) Give a physical explanation of any major features in your plot.

3. Trajectory of a particle (8 pts: 4 each). A particle moves in a two-dimensional orbit defined by

$$x(t) = \rho_0[2 - \sin(\omega t)] \quad (1)$$
$$y(t) = \rho_0[1 - \cos(\omega t)] \quad (2)$$

(a) Sketch the trajectory. Find the velocity and acceleration (as vectors, and also their magnitudes), and draw the corresponding velocity and acceleration vectors along various points of your trajectory. Discuss the results physically - can you relate your finding to what you know from previous courses? Finally: what would you have to change if you want the motion go the other way around?

(b) Prove (in general, not just for the above situation) that if velocity, $\vec{v}(t)$, of any particle has constant magnitude, then its acceleration is orthogonal to $\vec{v}(t)$. Is this result valid/relevant for the trajectory discussed in part a?
Hint There's a nice trick here - consider the time derivative of $|\vec{v}(t)|^2 = \vec{v} \cdot \vec{v}$.

Note: What you have proven in part b is quite general, and very useful. It explains why, e.g., $d\hat{\vec{r}}/dt$ must point in the $\hat{\vec{\phi}}$ direction in polar coordinates. Do you see why?

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4. In this problem we want to generalize the analysis that you did in class for the motion of a particle in polar coordinates to spherical coordinates (4 pts each lettered part). The three unit vectors: $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ which describe spherical coordinates can be written as:

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}, \quad (3)$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}, \quad (4)$$
$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}. \quad (5)$$

(a) Let's investigate that the definitions given in Eqs.(**??**-**??**) make sense. First, define in your own words what orthonormal vectors are, and then check to see if these vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are orthonormal. Then, convince yourself (and the grader) that e.g. at least the x-component of $\hat{r}$ is correct, using a simple geometric picture. (Taylor's Fig 4.16, or Boas Fig 4.5 should help)

(b) If the particle is constrained to move with $\phi = 0$, state in simple words what this means in terms of the particle motion. Sketch the three spherical unit vectors at some point $\phi = 0$ and $r = R$ for some particular (nonzero) angle θ of your choice.

(c) Show that the velocity of any particle in spherical coordinates is given by:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \quad (6)$$

(d) Use the previous part to compute the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ in terms of $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$, and then show that $L_z = mr^2\dot{\phi}\sin^2\theta$.

Hint! Just think about these three unit vectors - can you very quickly and intuitively argue what the cross product of $\hat{r}$ is with each of the spherical unit vectors without doing any calculations at all? Just be careful of signs.

Extra credit (up to 4 bonus points, but won't count off if you don't do it) If a particle is restricted to move on the surface of a sphere, $r(t) = R$, and if angular momentum (in particular L_z) is conserved (meaning, L_z doesn't change with time) *and assuming the particle starts its motion somewhere on the z axis*, what can you say about $\dot{\phi}$ at later times?

5. The pair of coupled ODEs

$$\frac{dx(t)}{dt} = Ax(t) - Bx(t)y(t) \quad (7)$$
$$\frac{dy(t)}{dt} = -Cy(t) + Dx(t)y(t) \quad (8)$$

is referred to as the Lotka-Volterra equation and is supposed to represent the evolution of the populations of a predator and its prey as a function of time, A, B, C, D are positive constants.(12 pts: 4 each)

(a) Which of the variables, $x(t)$ or $y(t)$, represents the predator? Which represents the prey? What reasons do you have for your choice?

(b) What do the parameters A, B, C, and D represent? Why do you say so?

(c) Are there values for the populations that would lead to a stable solution for both groups?

You might be wondering why we are asking you to think about an ecology question in Phys 2210. This question is really just about developing intuitions about ODEs. Physics is full of ODEs! You will need such intuitions in almost every branch of physics.