Homework 13

(Due Date: Start of class on Thurs. Apr 19)

1. All members of your final project group should work on this problem, but only *one* group member should submit this progress report. To submit, create a .pdf document and upload it to the D2L "DISCUSSION" area we have specifically created for this purpose. (It appears as the very last discussion topic - look for it!) Note: If you have a lot of handwritten stuff which you cannot scan, you could also hand it in (separate pile) on Thursday in class, but we strongly prefer a single pdf file.

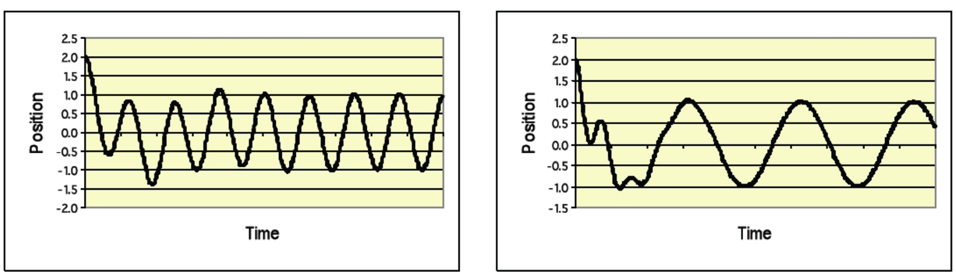
A complete progress report will include 1) a summary of your chosen problem and planned investigations,2) an outline of how you plan to complete these investigations, 3) sample calculations and/or code to demonstrate you have begun working, and 4) any resources that you are using to guide your work. Only complete reports will receive full credit for this problem, and all group members will get the same score. A sample progress report (to show you what we have in mind) is available on our website and on our D2L page.

2. *Since not all of you had a chance to look at a recent tutorial extra-problem, here it is again!*

A harmonic oscillator with a restoring force $25m\alpha^2 x$ is subject to a damping force $3m\alpha v$ and a sinusoidal driving force $F_0\cos(10\alpha t)$.

(a) Write down the differential equation that governs the motion of this oscillator. Show that if the driving force were removed the oscillator would become underdamped, and express the frequency of the oscillator in terms of the given quantities. Explain your reasoning.

(b) Each x vs. t graph below illustrates the actual motion (transient plus steady state) of a damped, driven oscillator starting at $t = 0$. For each case, is the frequency of the steady-state motion greater than, less than, or equal to that of the transient motion? Explain.



Graph 1 **Graph 2**

Figure 1:

(c) (2 pts) Identify which graph (1 or 2) would better correspond to the damped, driven oscillator described in parts a) of this problem. Explain your reasoning.

Extra credit "Reverse engineer" this problem: - write a Mathematica code which produces the plot you chose here. (Taylor's Eq's 5.68 and 5.70 will help!)

3. Do you know the precise value of the summation of the reciprocals of the squares of the natural numbers, $\sum_{n=1}^{\infty}\frac{1}{n^2}$? This is a famous problem in mathematical analysis with relevance to number theory. It was first posed by Pietro Mengoli in 1644 and solved by Euler in 1735. Since the problem had withstood the attacks of the leading mathematicians of the day, Euler's solution brought him immediate fame when he was twenty-eight. Let's solve it using Fourier series (a tool that wasn't yet available to Euler!)

(a) Define the periodic function $f(x) = x^2$ for $-\pi < x \leq \pi$ and compute the Fourier series.

(b) Evaluate the function at $x = \pi$ and use the Fourier series to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The answer is $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. *Your job is to show how this comes about!)*

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4. Any periodic function, $f(t) = f(t+T)$ can also be Fourier expanded as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega n t} \tag{1}$$

with $\omega = 2\pi/T$, and n sums over all integers.

This is very similar to our usual expansion in terms of sin's and cos's, but combines them more elegantly using complex exponentials, and is the more common form used by physicists. Let's work out the details!

(a) Show that $\frac{1}{T}\int_{-T/2}^{T/2} e^{i(m-n)t\omega}dt = 0$ if $m \neq n$ and $\frac{1}{T}\int_{-T/2}^{T/2} e^{i(m-n)t\omega}dt = 1$ if $m = n$ (both are integers)

(b) Use (a) to show that $c_m = \frac{1}{T}\int_{-T/2}^{T/2} e^{-imt\omega} f(t)dt$, where m is just a dummy index.

Hint! Look at the use of Fourier's trick from my lecture (see concept test slides), or online lecture notes on p. 11a and 11b, or Boas Ch 7.7, and figure out how to apply it here.

When expanding in complex exponentials like this, we win by having just one simple formula for the c_n's, and integrals of exponentials are easier! (The price we pay is having to sum over negative n's)

Another Extra Credit opportunity: Use Euler's formula $e^{ix} = \cos x + i \sin x$ to show that

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ a_0 & n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \end{cases}$$

Where the a's and b's are the usual Fourier coefficients we've defined in class and Taylor.

5. You've gotten a summer job interning as an audio engineer. Audio software produces periodic tones by summing pure frequencies (each with its own amplitude) - it's just summing up a Fourier series! Your boss has asked you to generate a tone with the following properties: It should have a fundamental frequency of 440 Hz (Middle A). On each cycle (i.e. 1/440th of a second), instead of being a pure cos or sin wave, your wave should start with a given amplitude A, ramp down *linearly* (rather than sinusoidally) to $-A$ in half a period, then ramp back up linearly back to $+A$, and repeat this pattern indefinitely.

(a) Sketch the function you are trying to build (over a couple of periods). Clearly label your horizontal axis. Do you expect any of the terms (cosines or sines?) in the Fourier series to vanish? Briefly, why?

(b) Compute the Fourier series for your waveform (which means coming up with a formula for the amplitude of each Fourier term.)

(c) To check your solution, use a computer to plot the sum of the first 3 non-zero terms in your series (over a range of several periods.) It should resemble the sketch that you drew in part a. If you're using Mathematica, there is a function called "Play", try it out! (Set A=1) Compare it to Play[Cos[440 * 2 Pi t], t,0,1] and describe in words how your tone is similar, and how it differs, from the pure tone.