Homework 14

(Due Date: Start of class on Thurs. Apr 26)

1. Evaluate the following integrals: (2 pts each)

(a) $\int_0^\pi dx \sin(x)\delta(x-\pi/2)$

(b) $\int_0^3 (5t-2)\delta(2-t)dt$

(c) $\int_0^5 (t^2+1)\delta(t+3)dt$

(d) $\int_{-\infty}^{\infty} e^x \delta(3x)dx$ (Hint: Can you try a u-sub?)

(e) If a particle feels a force $F(t)$ of the form $F = A\delta(t)$, with t the time, what are the units of A? Give a physical interpretation to this formula - what sort of force are we trying to represent here?

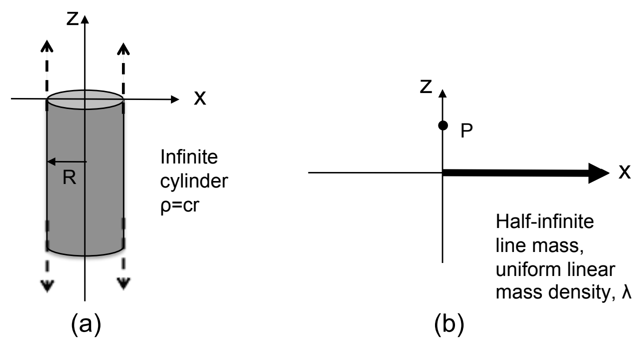


Figure 1: (a) An infinite cylinder of radius R centered on the z-axis, with *non-uniform* volume mass density $\rho = cr$, where r is the radius in cylindrical coordinates. (b) A half-infinite line of mass on the x-axis extending from $x = 0$ to $x = +\infty$, with uniform linear mass density λ.

2. There are (at least) two general methods we use to solve gravitational problems (i.e. find $\vec{g}$ given some distribution of mass).

(a) Describe these two methods. We claim one of these methods is easiest to solve for $\vec{g}$ of mass distribution (a) above, and the other method is easiest to solve for $\vec{g}$ of the mass distribution (b) above. Which method goes with which mass distribution? Please justify your answer.

(b) Find $\vec{g}$ of the mass distribution (a) above for any arbitrary point outside the cylinder.

(c) Find the x component of the gravitational acceleration, g_x, generated by the mass distribution labeled (b) above, at a point P a given distance z up the positive z-axis (as shown).

3. For the periodic function $f(t) = A(t^2 - 1/3)$ for $-1 < t \leq 1$ (with A a given constant)

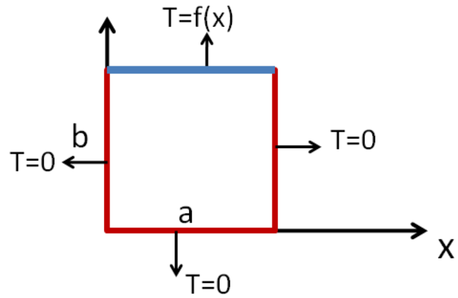
(a) Compute Fourier coefficients a_n and b_n. (Some vanish - predict which ones *before* doing any integrals)

(b) Suppose this force drives a weakly damped oscillator with damping parameter $\beta = 0.05$ and a natural period $T = 2$. Find the long time motion $x(t)$ of the oscillator. Discuss your answer. Use Mathematica to plot $x(t)$ using the first four non-zero terms of the series, for $0 < t \leq 10$. (Set A=1 for this)

(c) Repeat part b) for the same damping parameter, but with natural period $T = 1$. Briefly discuss any major differences between this result and what you had in the previous part.

(d) For most of the possible natural frequencies the response of this particular driven oscillator is going to be weak. However, there are some particular natural frequencies at which you are going to see an enhanced response - determine which those are.

CONTINUED



4. You are a summer engineering-physics intern, working on a modified Rayleigh-Bernard convection (`http://goo.gl/Itbi2`) experiment, and the edges of the rectangular bottom plate of the system are held at the temperatures as shown in the figure above.

(a) Your advisor has asked you to determine the steady state temperature distribution T(x,y) for the plate, so another student can put that boundary information into a computational model for the convective flow she's constructed. The boundary condition on the top surface is given by

$$f(x) = \begin{cases} 0 & 0 < x \le a/2 \\ 2T_0 & a/2 < x \le a \end{cases} \tag{1}$$

(b) You want to check that your solution makes sense. Code up your formula, adding up 30 terms in your sum to find T(x,y). For concreteness, please set a=1, b=2, and T_0=100. The very first thing I would want to do to check this code would be to set y=2, and do a simple plot of T[x,2] (from x=0 to 1), to make sure that the Fourier sum is giving what you expect. (What DO you expect? Sketch it first, then plot it. Comment on any interesting or surprising features you observe about this plot, is it exactly what you expected?)

Some Mathematica reminders: if, for instance, $T(x,y) = \Sigma_n 2n\ sin(n\pi x)e^{-n\pi y}$, the Mathematica syntax for that might be

t[x_,y_] := Sum[2n Sin[n Pi x] Exp[-n Pi y],{n,1,30}]

Watch out for the underscores for variables x and y on the left side (they do NOT appear again on the right side), and the colon before the equals sign. Don't capitalize functions that you invent, caps are reserved for MMA built-in functions.

(c) Now that you are more confident your (nasty!) Fourier work is probably ok, plot your full T(x,y) using a 3D plot function (MMA users will find Plot3D useful. See the documentation center for the basic syntax of Plot3D.) Briefly, discuss/document the key features of your plot. Please tell us explicitly what does the height of this graph represent, physically? Given these results, and the goal of your summer internship project, do you have any comments for your advisor? *Plot3D is very cool. You can rotate the resulting curve with your mouse to really get a good look at your result.*

Extra credit, let's just play around with Fourier series a little!

(i) For the rectangular plate problem shown above, calculate the steady state temperature $T(x,y)$ everywhere inside the plate if the boundary condition on the top surface was $f(x) = \sin(3\pi x/a)$. (Note - this is much *easier* than the problem above, if you think about it! In fact, if you look back at what you did there, you shouldn't need to do any new calculations at all.)

(ii) Keep the boundary condition the same as part i on the bottom and left edges (T=0), and keep the boundary condition the same at the top edge too, i.e. T(x,y=b)= $\sin(3\pi x/a)$. But let's change the right edge: Let T(x=a,y) = $\sin(3\pi y/b)$. Find T(x,y) everywhere else. *Hint: Once again, there is no need to do any real calculations here - you can pretty much just write down the answer if you think about it right!*