Homework 6

(Due Date: Start of class on Thurs. Feb 23 )

**1.** You have a flat plate (of uniform mass density, which is mass per unit *area* in this case) of total mass M. It is in the shape of a right triangle, with legs a and b (c is the hypotenuse).

(a) Draw your triangle, choosing a coordinate system origin and orientation to make it as easy as possible for you to find the center of mass. Based on your physical and mathematical intuitions, without calculating, where do you predict the center of mass is, and why?

(b) Now mathematically determine its center of mass coordinates. Does your answer match your intuitions from part a? Briefly, explain (or reconcile!)

**2. Rocket science!**

(a) A rocket having two or more engines, stacked one on top of another and firing in succession is called a multi-stage rocket. Normally each stage is jettisoned after completing its firing. The reason rocketeers stage models is to increase the final sped (and thus, altitude) of the uppermost stage. They do this by dropping unneeded mass throughout the burn so the top stage can be very light and coast a long way upward. Let's examine the advantages of a multi-stage rocket. Suppose the rocket carries 80% of its initial mass as fuel (i.e. the mass of all the fuel is $0.8m_0$) What is the rocket final speed accelerating from rest in free space, if it burns its fuel in a single stage? Express your answer in terms of $v_{ex}$.

(b) Now suppose instead that it burns the fuel in two stages like this: In the first stage it burns a mass $0.4m_0$ of fuel. It then jettisons the (empty) first stage fuel tank. Assume this empty tank has a mass of $0.1m_0$. It then burns the remaining $0.40m_0$ of fuel. (So, we've burned the same total amount of fuel as part a, right? We simply jettisoned an empty fuel-stage in the middle) Find the final speed in this case, assuming the same value of $v_{ex}$ as in part a. Compare and discuss!

(c) Taylor worked out the rocket equation in deep space. But at launch, you can't neglect gravity - the net external force, dP/dt, is no longer zero! Follow Taylor's derivation on p. 86, and fix it up, getting to Eq 3.6 and find the "correction" term you need to add to include gravity. Assuming that $v_{exh}$ is a fixed (constant) number, and assuming that you want the rocket to simply "hover" above the ground (rather than really launching), solve the ODE you get to find rocket mass as a function of time. (Does $dm/dt$ turn out to be a constant? Explain physically why your answer to that question makes sense)

(d) If your payload (the mass that is left over after all the fuel is gone) is roughly $e^{-2} = .135$ of the initial mass, how long can you hover? Given the (very optimistic!) value of $v_{exh} = 2000m/s$, comment on why we don't all commute around with jetpacks.

**The rest of question 2 is pure EXTRA CREDIT** (Not required, worth a bonus 4 points)

So far we have considered the ideal case of a rocket without drag. In real life, however, drag can be an important limitation. Imagine the situation of a linear drag $\vec{f} = -b\vec{v}$ acting on the rocket body only (with no other external forces, so we're back to the "gravity free" case of deep space.) Once again, the net external force, dP/dt, is not zero. Follow Taylor's derivation on page 86, and fix it up, getting to Eq 3.6 and find the "correction" term you need to add, caused by drag. Solve your ODE, to show that if the rocket starts from rest and ejects a mass at constant rate $\dot{m} = -k$ (with k a given constant), its speed is given by $v = \frac{k}{b} v_{ex} \left[1 - \left(\frac{m}{m_0}\right)^A\right]$ What is $A$ in terms of $k, b, v_{ex}$ and $m_0$? Alternatively, if you'd rather not do the integral analytically, put your ODE into Mathematica, pick some reasonable numbers (see my lecture notes) and simply plot v as function of m!

*HINT: since dm/dt=-k, you can eliminate any stray "dt" terms that appear in your ODE.*

**If you want yet another 2 points of extra credit** and more practice with Taylor expansions: what is the corresponding speed if we ignore drag? Show that the equation you got reproduces the speed for the drag free case if $b \to 0$ (see hint below). Calculate the first non-vanishing correction introduced by a finite drag to the speed. Does the sign of your correction make physical sense? Briefly, discuss.

*HINT: A helpful bit of math: you can always rewrite the function* $f(x) = c^x$ *as* $f(x) = e^{ln(c^x)} = e^{x\,ln(c)}$.

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**3.** A particle (in 2D) has a potential energy function given by $U(\vec{r}) = -e^{-(x^2+2y^2)}$. (Assume all the numerical constants have implicit correct SI metric units so that x and y are in meters, and U in Joules)

(a) Using your favorite computational tool, make an equipotential plot ("ContourPlot[ ]" in Mathematica. Use the built in help to find out the syntax and arguments, it's pretty simple) and a 3D plot ("Plot3D" in Mathematica). Include a copy of both plots with your homework. Now stare at these images. Describe in words how you would interpret this potential physically (For instance - is this attractive, repulsive, both, neither? Can you invent some physical system for which this might be a crude model? Would a particle be "bound" here?) Which is more informative for you - the countour plot or the 3D plot? Why? If you put a particle at, say, $\vec{r} = (1, 1)$, use your intuition and the plots to describe very clearly in words (without calculation) the approximate direction of the force there.

(b) Analytically compute the force on the particle. Then, use a computational tool to plot the force field ("VectorPlot[ ]" in Mathematica. If you name your plots "p1 = ContourPlot[]", p2=VectorPlot[]" and make sure the x and y range is the same, then you can plot TWO different plots on top of each other using "Show[p1,p2]".) Produce a single plot that shows the contours and the force field together. Now, what is the magnitude and direction of the force at (1,1)? Discuss whether your plots agree with your expectation in part a (and resolve any discrepancies)

(c) What changes could you make to $U(\vec{r})$ to make the potential well i) centered at the point (1,1) instead of (0,0) ii) stronger? iii) repulsive? (You might want to use your computational tool to visualize your claims here.)

**4.** The magnetic field inside a long current carrying wire is $\vec{B} = (\frac{B_0}{R})r\hat{\phi}$, where R is the radius of the wire, and r is the distance from the center of the wire. Ampere's law relates the integrated magnetic field around any closed loop to the total current passing through the loop, $\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{through}$. If we want to determine the current passing through the loops shown in Fig.1, we need to evaluate the line integral of $\vec{B} \cdot d\vec{r}$.

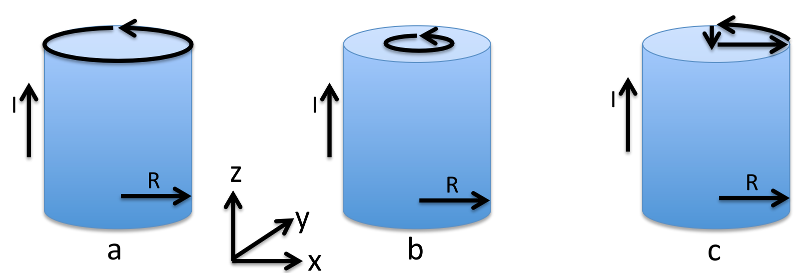


Figure 1:

(a) Explicitly compute $\oint \vec{B} \cdot d\vec{r}$ along the full circle path of radius R, shown in Fig.1a. Use this to find $I_{through}$. Briefly, discuss the physical meaning of the sign of your answer. If I had asked you to integrate the other way around the circle, what sign(s) in your solution would have changed? Would your final conclusion regarding the direction of "I" have changed? Be clear about this - what does the sign of your integral tell you?

(b) Compute $\oint \vec{B} \cdot d\vec{r}$ along the path of radius $r_0$ (less than R) in Fig 1b. How does $I_{thru}$ compare with part a? Compute $\oint \vec{B} \cdot d\vec{r}$ along the quarter circle path in Fig 1c. Compare all your answers for question, and discuss - what do you conclude about how the current is distributed through the wire?

(c) Line integral practice! Do Boas Ch 6, section 8, problem #6 (p. 307) **Just pick one part - a, b,** *or* **c**. How can you prove that the answer will come out the same for the other parts *without* doing them?