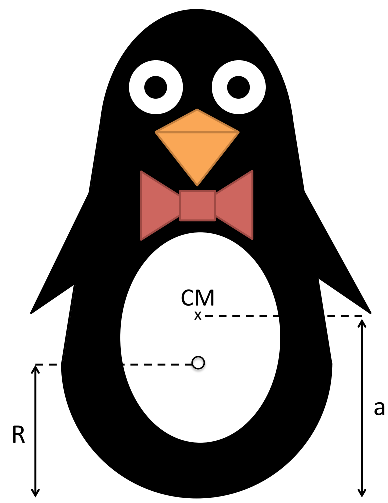
Homework 8

(Due Date: Start of class on Thurs. March 8)

1. Look at the figure below, which shows a possible design for a child's toy. The designer (you!) wants to build a "weeble" (which wobbles, but doesn't fall down). It is a single, solid object, the base shape is a perfect hemisphere (radius R) on the bottom, connected to the rest of the body (here, a penguin). The CM is shown, it is "a" above the base. You don't need to try to compute "a", assume it is a given quantity! (As the designer, how might you control/change the position of a?)

(a) Write down the gravitational potential energy when the weeble is tipped an angle θ from the vertical, as a function of given quantities (m, a, R, g, and of course θ).

(b) Assuming the toy is released from rest, determine the condition(s) for any equilibrium point(s). Then consider stability: what values of (or relations between) "a" or "R" ensure that the weeble doesn't fall down? Explain. Can this toy work as desired?



2. A particle is under the influence of a force $\vec{F} = (-ax + bx^3)\hat{x}$, where a and b are constants.

(a) What are the units of a and b? Assuming a and b are positive, find $U(x)$ (assuming $U(0) = 0$) and sketch it. (Sketch means hand-drawn, not plotted with Mathematica. Include all features you consider interesting in your sketch, including e.g. zero crossings, behavior at large x, "scales" of your axes, etc)

(b) Find all equilibrium points and determine if they are stable or unstable.

(c) Qualitatively describe the motion of an object in this force field released from rest at $x = (1/2)\sqrt{a/b}$. Then, qualitatively describe the motion of an object in this force field released from rest at $x = \sqrt{2a/b}$.

(d) Now suppose a and b are both negative. Repeat parts b and c, commenting on what has changed.

(e) Setting a=b=-1 (with appropriate units), Taylor expand the potential of part d *around the outermost stable equilibrium point* (not around x=0!!) Truncate your series after the first "nontrivial" term. Plot the resulting approximate potential that a particle near that point feels, on top of the "real" potential. (Note this time I ask you to "plot", not "sketch", so use a computer) Use your plot to help you discuss the range of x for which you think this approximation would be reasonable.

Note that this potential, or small variants of it, occurs in various physics situations. This is a more accurate form of the true force on a pendulum than our usual approximation, can you see why? Part d is the (famous!) case of the "Higgs potential" in particle physics, which is being actively investigated at CERN's LHC collider.

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3. Consider a very (infinitesimally!) thin but massive rod, length L (total mass M), centered around the origin, sitting along the x-axis. (So the left end is at (-L/2, 0,0) and the right end is at (+L/2,0,0) Assume the mass density λ (which has units of kg/m)is *not* uniform, but instead varies linearly with distance from the origin, $\lambda(x) = c|x|$.

(a) What is that constant "c" in terms of M and L? What is the direction of the gravitational field generated by this mass distribution at a point in space a distance z above the center of the rod, i.e. at $(0, 0, z)$ Explain your reasoning for the direction carefully, try not to simply "wave your hands." (The answer is extremely intuitive, but can you justify that it is correct?)

(b) Compute the gravitational field, $\vec{g}$, at the point $(0, 0, z)$ by directly integrating Newton's law of gravity, summing over all infinitesimal "chunks" of mass along the rod.

(c) Compute the gravitational potential at the point $(0, 0, z)$ by directly integrating $-Gdm/r$, summing over all infinitesimal "chunks" dm along the rod. Then, take the z-component of the gradient of this potential to check that you agree with your result from the previous part.

(d) In the limit of large z what do you *expect* for the functional form for gravitational potential? (Hint: Don't just say it goes to zero! It's a rod of mass M, when you're far away what does it look like? *How* does it go to zero?) What does "large z" mean here? Use the binomial (or Taylor) expansion to verify that your formula does indeed give exactly what you expect. (Hint: you cannot Taylor expand in something BIG, you have to Taylor expand in something small.)

(e) Can you use Gauss' law to figure out the gravitational potential at the point $(0, 0, z)$? (If so, do it and check your previous answers. If not, why not?)

4. (a) Imagine a planet of total mass M and radius R which has a nonuniform mass density that varies just with r, the distance from the center. For this (admittedly very unusual!) planet, suppose the gravitational field strength *inside the planet* turns out to be **independent** of the radial distance within the sphere. Find the function describing the mass density $\rho = \rho(r)$ of this planet. (Your final answer should be written in terms of the given constants.)

(b) Now, determine the gravitational force on a satellite of mass m orbiting this planet at distance $r > R$. (Use the easiest method you can come up with!) Explain your work in words as well as formulas. For instance, in your calculation, you will need to argue that the magnitude of $\vec{g}(r, \theta, \phi)$ depends only on r. Be explicit about this - how do you know that it doesn't, in fact, depend on θ or ϕ?

(c) As a final check, explicitly show that your solutions inside and outside the planet (parts a and b) are *consistent* when $r = R$. Please also comment on whether this density profile strikes you as physically plausible, or is it just designed as a mathematical exercise? Defend your reasoning.

5. Now lets consider our (real) planet Earth, with total mass M and radius R which we will approximate as a *uniform* mass density, $\rho(r) = \rho_0$.

(a) Neglecting rotational and frictional effects, show that a particle dropped into a hole drilled straight through the center of the earth all the way to the far side will oscillate between the two endpoints. (Hint: you will need to set up, and solve, an ODE for the motion)

(b) Find the period of the oscillation of this motion. Get a number (in minutes) as a final result, using data for the earth's size and mass. (How does that compare to flying to Perth and back?!)

Extra Credit: OK, even with unlimited budgets, digging a tunnel through the center of the earth is preposterous. But, suppose instead that the tunnel is a straight-line "chord" through the earth, say directly from New York to Los Angeles. Show that your final answer for the time taken does not depend on the location of that chord! This is rather remarkable - look again at the time for a free-fall trip (no energy required, except perhaps to compensate for friction) How long would that trip take? Could this work?!