Sample Progress Report for Final Project

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Summary

Our group has decided to model a pair of coupled differential equations represent a simple mathematical model for a laser. The model is non-linear, but can be investigated both using analytic and numerical methods. We will describe this model in detail connecting each variable in the model to the physics it controls/describes. We have found fixed points (i.e., critical points) for this system and have begun to determine the stability of this system near those critical points. This system has a number of variables which appear to control the stability and we are starting to characterize these controls and their affect on the stability of the system. For the numerics, we plan to plot interesting results from the analytics and model particular trajectories of the system. In each trajectory, we plan to discuss if and how the model produces lasing.

Outline

1. Describe the physics of lasers and the mathematical model
2. Find solutions to special cases where at least some of our variables are not changing with time any more ($d/dt = 0$, called "fixed points")
3. Find fixed (critical) points and characterize their stability
4. Discuss simplifications (i.e., the quasi-static approximation) of the model
5. Qualitatively discuss lasing given by the stability of certain points
6. Model the system numerical and discuss how lasing occurs in this model

Set up/Progress

How do lasers work?

Most lasers work on the principle of an "inverted population". That is, atoms in the laser system a "pumped" into an excited state. The resulting decay of those atoms to a lower state produces photons at a particular wavelength. The atoms in a laser system are typically immersed in a "gain medium". The photons that are emitted bounce between two mirrors on either end of the gain medium, causing further stimulated emission which amplifies (increases power) the output. One of the mirrors is partially transparent so that some photons escape. These photons are of a single wavelength and coherent (roughly, the same phase).

Making sense of the laser model

The laser model [?] that we are interested in is given by:

$$\frac{dn}{dt} = GnN - kn, \tag{1}$$

$$\frac{dN}{dt} = -GnN - fN + p. \tag{2}$$

In this model, there are N excited atoms in the system which produce the n laser photons. The number of laser photons (n) increases when there are more excited atoms (N) and more laser photons (n) available to excite those atoms. Moreover, the gain coefficient, G, for the medium controls how effective these photons are in exciting more atoms and thus creating more photons. This is the first term in Eq. 1, GnN. The number of photons (n) decreases as photons leave the system at a rate k; this is proportional to the number of laser photons in the system (n). This is the second term in Eq. 1, $-kn$. As more photons are emitted by atoms, fewer atoms are excited; the rate at which this happens is proportional to the number of excited atoms (N) and the efficiency of the medium (G). This is the first term in Eq. 2, $-GnN$. The number of excited atoms (N) also drops as atoms emit photons at a rate f; again, this is also proportional to the number of excited atoms (N). This is the second term in Eq. 2, $-fN$. Finally, the number of excited atoms (N) increases as the pumping of energy into the system increases, p. This is the final term in Eq. 2, p.

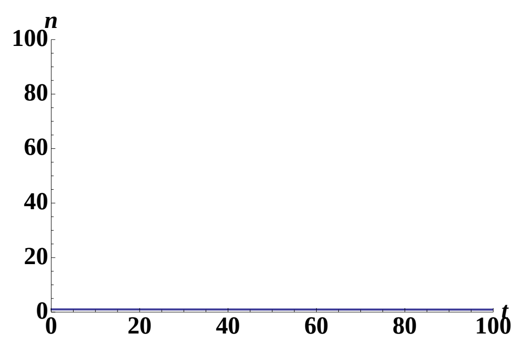
Predictions of the laser model

In the quasi-static limit, when $\dot{N}$ is small compared to $\dot{n}$, the model reduces to a single non-linear differential equation,

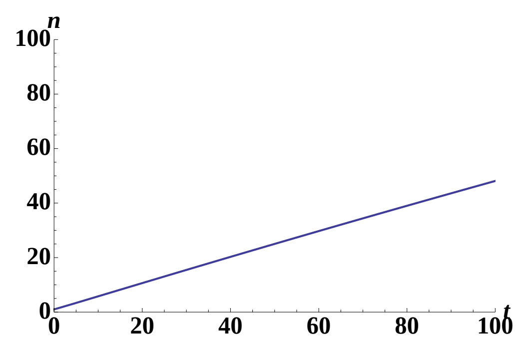
$$\frac{dN}{dt} = 0 = -GnN - fN + p \longrightarrow N = \tag{3}$$

$$\frac{dn}{dt} = Gn\left(\frac{p}{Gn+f}\right) - kn \longrightarrow \frac{dn}{dt} = k\left(\frac{p}{k} - \frac{f}{G} - n\right)\frac{Gn}{f+Gn}. \tag{4}$$

In this limit, we have investigated the effects of low pumping (low energy input) and high pumping (high energy input). In these limits, we observe no lasing in the low pumping scenario and lasing in the high pumping scenario. Both systems began with one laser photon in the model.



(a) Low pumping produces no lasing (number of laser photons does not increase)



(b) High pumping produces lasing (number of laser photons increases)

Code for Figure (a)

```
g = 1; k = 0.001; f = 0.1; p = 0.0001;
sol = NDSolve[{n'[t] == k (p/k - f/g - n[t]) g n[t]/(f + g n[t]), n[0] == 1}, n, {t, 0, 100}]
Plot[n[t] /. sol, {t, 0, 100}, PlotRange -> {{0, 100}, {0, 100}},
 AxesLabel -> {Style[t, Large, Bold], Style[n, Large, Bold]},
 LabelStyle -> Directive[Large, Bold], PlotStyle -> Thick]
```

Code for Figure (b)

```
g = 1; k = 0.001; f = 0.1; p = 0.5;
sol = NDSolve[{n'[t] == k (p/k - f/g - n[t]) g n[t]/(f + g n[t]), n[0] == 1}, n, {t, 0, 100}]
Plot[n[t] /. sol, {t, 0, 100}, PlotRange -> {{0, 100}, {0, 100}},
```

```
AxesLabel -> {Style[t, Large, Bold], Style[n, Large, Bold]},
LabelStyle -> Directive[Large, Bold], PlotStyle -> Thick]
```

References

[1] S.H. Strogatz, *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering*. Westview Pr, 1st Edition, 1994.

[2] J.R. Taylor, *Classical mechanics*. University Science Books, 1st Edition, 2005.