Solving eq'n of motion (N-2) generally means solving an "ORDINARY DIFFERENTIAL EQ'N", or ODE (Boas 8.1)

(Ordinary means no partial derivatives - later!)

Notation: $y' = y'(x) = \frac{dy(x)}{dx} = \frac{dy}{dx}$; $y'' = \frac{d^2y}{dx^2}$, etc

$$\dot{y} = \dot{y}(t) = \frac{dy(t)}{dt}$$

Notation: "Order" of ODE = highest derivative

so e.g $m\ddot{r} = F(r)$ is a "2nd order ODE"

Notation: "Linear" ODE means y and all its deriv's appear without any power (or other functions)

Ex Linear ODEs:

$$a\,y''(x) + c\,y(x) = 0$$

$$a(x)\,y''(x) + b(x)\,y'(x) = c(x)$$

$$\sqrt{x}\,y''(x) + \frac{\tan x}{x^3}\,y(x) = \ln x$$

↑ Linear in y, fn(x) doesn't matter!

Ex: not Linear ODEs:

$$(y''(x))^2 + y(x) = 0$$

$$\underbrace{y' \cdot y''}_{} + y = 0$$

not linear!

$$y'' = \overline{-y^2}$$

Notation "Homogeneous" ODE means every term contains y and/or derivs of y

Ex Homogeneous	Not Homogeneous
$a y''(x) + b y'(x) + c y = 0$	$a y'' + b y' + c y = d$
$\sqrt{x}\, y''(x) - (x^2+1) y(x) = 0$	even just a constant!

Key ODE fact: Any linear ODE of order n has a general sol'n $y(x)$ involving n independent constants which can only be determined by "boundary conditions" or "initial conditions" (like, $y(x=0)$ or $y'(x=0)$, etc)

---

this fact is not always true for non-linear ODE's!

Example: $y''(t) = -\omega^2 y(t)$, we said the sol'n was

$$y = A \sin \omega t + B \cos \omega t$$

Two (2^{nd} order!) constants

Notation: "Separable" ODE is when you can rearrange (treating "dx" as an algebraic object") so that

(functions of y) dy = (functions of x) dx

we like separable ODE's because you solve them by simply integrating both sides.

Example: $\frac{dy}{dx} = f(x)$ is separable 1st order ODE

$$dy = f(x)\,dx \Rightarrow \int dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx + C$$ → our one constant!

Example: $m\,dv/dt = -cv^2$ ← Newton with drag, see soon!

$\therefore \frac{dv}{v^2} = -\frac{c}{m}\,dt$ separated! fn(v) on left, fn(t) on right.

Integrate: $-\frac{1}{v} = -\frac{c}{m}t + C_0$ ← our one constant, for our 1st order ODE!

You can find C_0 if you have an "initial condition", e.g if $v(t=0) = v_0$ is given, $-1/v_0 = 0 + C_0$, + thus

$$v(t) = \frac{1}{\frac{c}{m}t - C_0} = \frac{1}{\frac{c}{m}t + \frac{1}{v_0}} = \frac{v_0}{1 + \frac{cv_0}{m}t}$$

<u>Example</u> ("Math-y", not so physics-y". (Linear, 1st order ODE)

Solve $xy' - y = xy$, Given "boundary condition" $y(x=1)=1$

$x\frac{dy}{dx} - y = xy$. Let's <u>separate</u> x's + y's. Divide by x:

$\frac{dy}{dx} - \frac{y}{x} = y$. Still not there! <u>Group</u> the y's:

$\frac{dy}{dx} = y/x + y = y\left(1+\frac{1}{x}\right)$ Ahhh! Now I see it!

$dy/y = \left(1+\frac{1}{x}\right)dx$. Separated! So, now just integrate

$\ln y = x + \ln x + c$ ← our <u>one</u> constant!

<u>General</u> sol'n: $y = e^{(x+\ln x + c)} = xe^{x}e^{c} = c'xe^{x}$

just rewrite the constant

[Boundary condition ("BC") tell us c':

$y(x=1)=1 \Rightarrow 1 = c'\cdot 1\cdot e^{1}$, so $y(x) = \frac{1}{e}xe^{x} = xe^{x-1}$.]

2210 (18) (Boas 8.3)

There are many tricks to solve ODE's.

Boas has a cool one for any linear, 1^{st} order ODE,

$$y'(x) + P(x)\, y(x) = Q(x).$$

P: any fn; Q: any fn! (This makes it inhomogeneous!)

Main idea:

1) Solve the homogeneous eq'n $y_c'(x) + P(x)\, y_c(x) = 0$, generally it's separable, so easy! This is the "complementary" eq'n.

2) Find some particular sol'n y_p to our diff eq, "$y_{particular}$"

Then $y_c + y_p$ = "$y_{general} + y_{particular}$" IS the full general sol'n.

Boas finds a trick to find y_p, there's a formula, it's pretty elegant

→ Boas p.401, eq 3.9.

Use it if you ever need it!

Back to physics! (We disc

$F = m\ddot{x}$ is a 2^{nd} order

classical mechanics!)

= Bring Laptop

— Ivan Anich, Jeffrey Ballard, Anthony Barbarino (T

In phys 1110: $m\ddot{x} = $ consta... , gravity)

$m\ddot{x} = -kx$ ...gs, SHM)

We can do more interesting & realistic problems now!

Including $F = F(x)$, or $F(t)$, or $F(v)$!

(often, writing as $m\dot{v} = F$ will help)

1^{st} class of new problems is PROJECTILES with DRAG

In real life, $\vec{F} = \vec{F}_{grav} + \vec{F}_{drag, \text{ or resistance}}$ ← Taylor calls it $\vec{f}$.

$\vec{f}_{drag}$ depends (usually) on velocity, + points opposite $\vec{v}$,

$$\vec{f} = -f(v)\,\underbrace{\hat{v}}_{\text{Unit vector in } \vec{v} \text{ direction}}$$

[Magnetism is $\perp$ to $\vec{v}$, so this is a separate case]

$f(v) = |f|$ depends on speed $v = |\vec{v}|$.

$f(v)$ could be anything! Taylor's theorem says (soon!)

$f(v) \simeq a + bv + cv^2 + \cdots$ (Let's look at leading terms)

2210 - 20

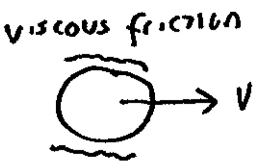
Phys 1110 considered $F_{friction} = \mu N =$ constant, indep. of $|\vec{v}|$, but this is not considered "drag". So, let's set $f(v=0)=0$,

$$f(v) = bv + cv^2$$

Linear ← bv ; quadratic drag ← cv^2.

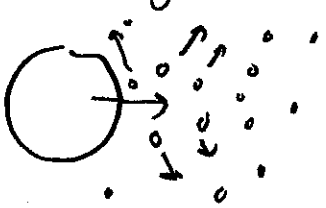
Low speeds, dominates. ; High speeds, this dominates

- The physics of f_{linear} is viscosity. (Move through mud, or molasses, or even air)



So "b" depends on size of object and some property (viscosity) of medium

- The physics of $f_{quadratic}$ is from "moving the medium out of the way"



you must accelerate medium particles away. the faster you go, the faster the recoiling particles must bounce to get away

and you encounter more particles each second

Both of these involve a factor of "v", hence v^2.

So "c" depends on area of object (how much it "sweeps out") and also density, ρ, of the medium (more molecules to sweep)

For spherical objects, then, of Diameter D

$f_{linear} = bv = (\beta D)\, v$. this is "Stokes Law"

D → Diameter

β → measure of viscosity

$f_{quad} = cv^2 = (\gamma D^2)\, v^2$

D^2 → Area goes like D^2.

γ → depends on density ρ, also shape/streamlining

$\equiv \left(\frac{1}{2} C_D \rho A\right) v^2$ This defines C_0

A → cross sectional area $= \pi D^2/4$

ρ → Density of medium

C_D → "the drag coefficient"

C_D is dimensionless (convince yourself!) → $\left(\rho \cdot A \cdot v^2 = \frac{kg}{m^3} \cdot m^2 \cdot \frac{m^2}{s^2} = \frac{kg\, m}{s^2} \checkmark\right)$

Depends on shape, mostly

→ (Air has $\rho \simeq 1.29\ \mathrm{Km/m^3}$)

C_D (Hummer) $\approx .57$

C_D (sphere) $= .5$

C_D (Prius) $= .26$

C_D (747) $= .03$

$$\frac{f_{quad}}{f_{lin}} = \frac{cv^2}{bv} = \frac{\gamma D_{iam}}{\beta} v = \underbrace{\left(\frac{1}{2} \frac{C_D \rho}{\beta} \frac{\text{Area}}{\text{Diam}}\right)}_{} v \propto \text{"Reynolds \#"}$$

For Big v, f_{quad} dominates; this sets the scale!

$\boxed{\text{Terminal velocity}}$: As $v\uparrow$, f also $\uparrow$, so as objects fall, eventually f balances weight, + acceleration $\to 0$. That means constant V = "$V_{TERMINAL}$".

~120 mi/hr ≈ 55 m/s for skydiver in freefall, (but depends on body orientation, clothes, etc)

Linear drag: $\vec{f}(v) = -b\underbrace{v\,\hat{v}} = -b\vec{v}$

(so if $F_{net}=0 \Rightarrow$) $\underline{b V_{term} = mg}$

$$\text{or } \boxed{V_{term} = \frac{mg}{b}} = \frac{mg}{\beta D}$$

Checks: Lower $\beta \Rightarrow$ higher V_T ✓
Bigger objects ⇒ lower V_t, ✓
elephants die, ants survive!
Denser objects of given size ✓
⇒ higher V_t, I believe it

Quadratic drag: $\vec{f}(v) = -cv^2\,\hat{v} = -c|v|\,\vec{v}$

So if $F_{net}=0 \Rightarrow$ $cV_T^2 = mg$

$$\text{or } \boxed{V_{term} = \sqrt{\frac{mg}{c}}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{mg}{\frac{1}{2}C_D\rho A}}$$

(But, this is just "steady state" ...)

(Tutorial)

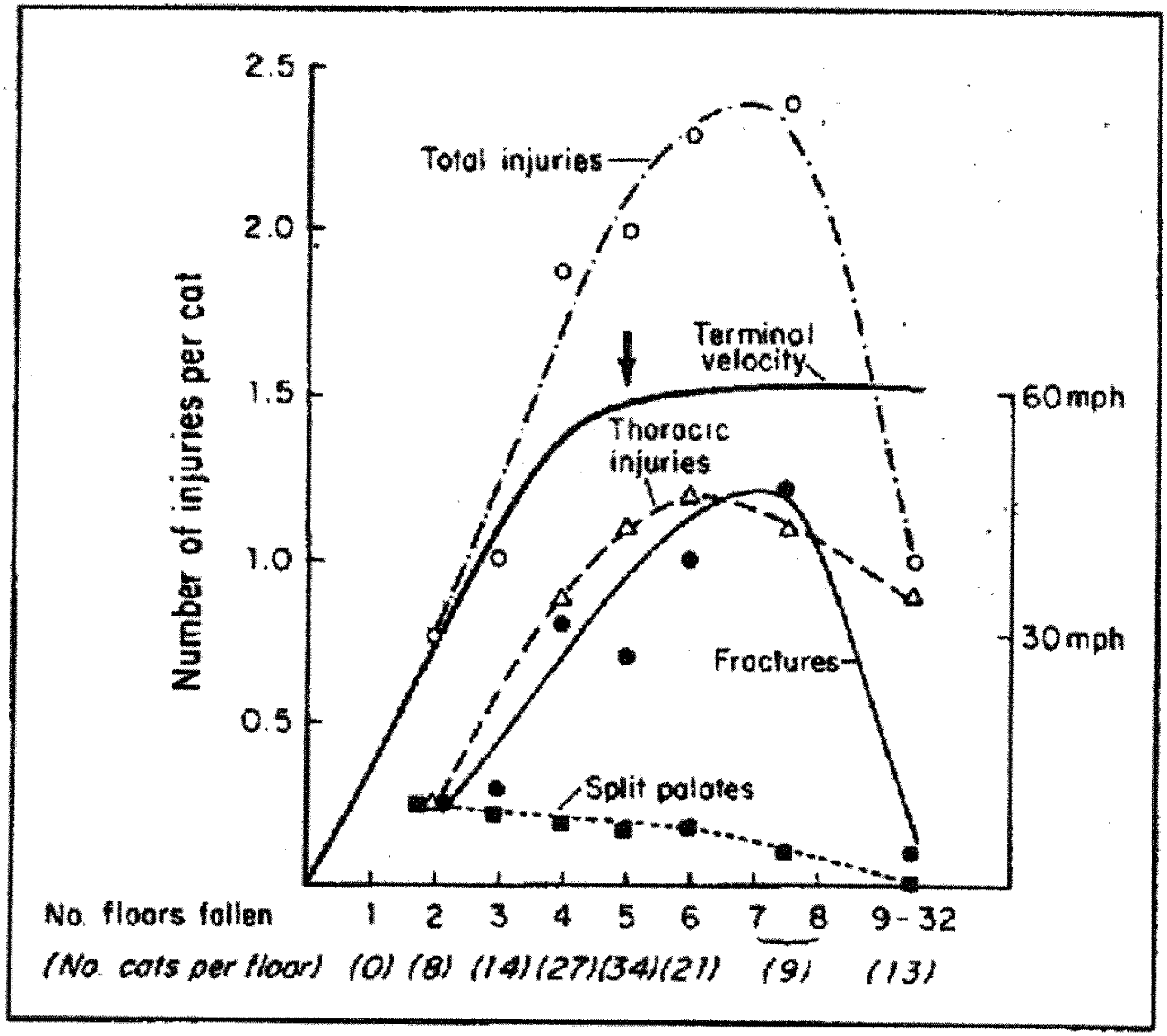


Figure 2—Relationship of injuries to distance fallen and velocity in 132 cats with high-rise syndrome: ↓ points to terminal velocity (—); total number of injuries/cat (○, -·-·-); number of thoracic injuries (pulmonary contusions +pneumothorax)/cat (△, – – –); number of fractures/cat (●, —); number of split palates/cat (■, – – – – –).

Motion with drag in 1-D.

$\vec{F}_{net} = m\vec{a}$ says $m\dot{\vec{v}} = -b\vec{v}$, so $m\dot{v}_x = -bv_x$

thus, $\dot{v}_x = -\frac{b}{m} v_x$. True if moving right ($v_x > 0$) or left ($v_x < 0$), convince yourself!

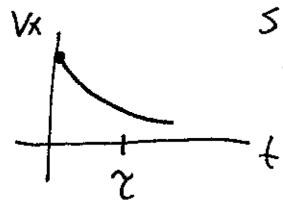
Separable! $\frac{dv_x}{v_x} = -\frac{b}{m}\,dt$. Integrate $\Rightarrow \ln v_x = -\frac{b}{m}t + C$

If $v_x(t=0) \equiv v_{x0}$ is given, then $C = \ln v_{x0}$, $v_x = v_{x0}\,e^{-bt/m}$

Note: $[b/m] = \left[\frac{N/(m/s)}{kg}\right] = \frac{\frac{kg\,m}{s^2}\cdot\frac{s}{m}}{kg} = \sec^{-1}$. So, we define

a natural time $\tau \equiv m/b$!

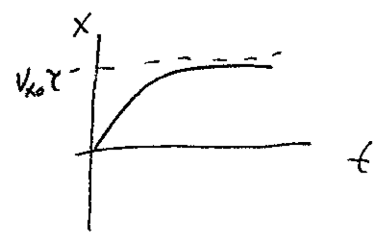
$v_x(t) = v_{x0}\,e^{-t/\tau}$



Slows down in "typical time" τ.

To get $x(t)$, $\frac{dx}{dt} = v_x$, so $x(t) - x(0) = \int_{t=0}^{t} v_x\,dt = v_{x0}(-\tau)e^{-t/\tau}\Big|_0^t$

so $\underline{x(t) - x(0)} = \underline{-v_{x0}\tau\left(e^{-t/\tau} - 1\right)}$



Note: As $t \to \infty$, $x(t) - x(0) \simeq v_{x0}\tau$ is finite

It "grinds to a halt", (the same distance it would have drifted without drag in the natural time τ.)

Vertical

Motion, with drag, arbitrary initial conditions

Start in 1-D (e.g. dropping) My convention, not Taylor's (up = "plus") $+y \uparrow$ $\quad \uparrow bv \quad \downarrow mg$

Falling object:

$$\vec{F}_{net} = m\vec{a} \Rightarrow mg(-\hat{y}) - b\vec{v} = m\dot{\vec{v}}$$

In 1-D, $m\dot{v}_y = -mg - bv_y$

Separable! (or, use Boas trick, try that yourself) Boas p.40)

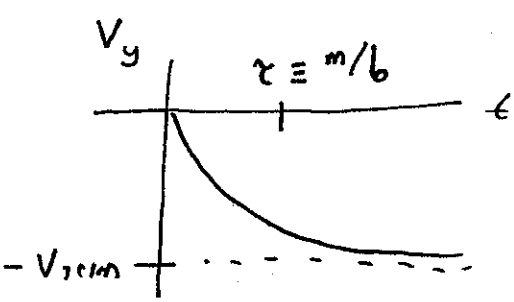
$$\frac{dv_y}{g + \frac{b}{m}v_y} = -dt \Rightarrow \frac{m}{b}\ln\left(g + \frac{bv_y}{m}\right) = -t + C$$

so $g + \frac{bv_y}{m} = Ae^{-bt/m}$

<u>Note</u>: as $t \to \infty$, $v_y \to -g\frac{m}{b} = -v_t$, (See previous page! Sign is correct with my convention.)

If start from rest, $v_y(t=0) = 0 \Rightarrow A = g$, and so

$$v_y(t) = \frac{mg}{b}\left(e^{-bt/m} - 1\right) = -v_{term}\left(1 - e^{-bt/m}\right)$$



Note: small $t \Rightarrow e^{-bt/m} \approx 1 - bt/m + \dots$

$v_y \approx -v_{term}\, bt/m \approx -gt$

Ah, same as <u>no</u> drag, for small t

If we start with non-zero V_{0y}, then

$$g + \frac{b V_y(t=0)}{m} = A \text{ , so } \quad \left(\text{using } V_t \equiv \frac{mg}{b}, \text{ and } \tau \equiv \frac{m}{b}\right)$$

$$V_y(t) = \frac{m}{b}\left(-g + Ae^{-bt/m}\right) = -V_t\left(1 - e^{-t/\tau}\right) + \underbrace{V_{0y} e^{-t/\tau}}_{\text{extra!}}$$

If want motion, integrate $V_y = dy/dt$,

$$\text{so} \quad y(t) - y(0) = \int_0^t V_y \cdot dt = \int_0^t \left[-V_t + (V_t + V_{0y}) e^{-t/\tau}\right] dt$$

$$= -V_t \cdot t - \tau\,(V_t + V_{0y})\left(e^{-t/\tau} - 1\right)$$

(See Taylor for sketches...)

Comment: Note that $\tau = m/b$ is a "natural time constant" here.

It has units of $\mathrm{kg}/(\mathrm{N/m/s}) = \dfrac{\mathrm{kg}}{\frac{\mathrm{kg\,m}}{s^2}\cdot\frac{s}{m}} = \sec$ ✓

(It tells you the natural "decay time" for this motion, the timescale for speed to approach terminal velocity)

Comment: If $V_y(0)$ is +, my sol'ns + eq'ns all still work. So, ball could initial be thrown up (or released, or thrown down) The above is still all fine.

2210 24 b.

What about Quadratic drag, $f_D \sim cv^2$?

1st consider pure 1-D motion with drag (only)

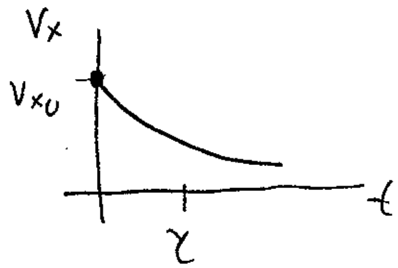
$\vec{F} = m\vec{a}$ says $m\dot{\vec{V}} = -c|V|\vec{V}$, so $m\dot{V}_x = -c|V_x|V_x$

If $V_{x0} > 0$, (then $|V_x| = V_x > 0$), $\dot{V}_x = -\frac{c}{m}V_x^2$. Separable!

$$\frac{dV_x}{V_x^2} = -\frac{c}{m}dt \Rightarrow -\frac{1}{V_x} = -\frac{c}{m}t + C_1$$

As usual, if V_{x0} is given at $t=0$, $V_x(t) = \dfrac{1}{\frac{ct}{m} + \frac{1}{V_{x0}}} = \dfrac{V_{x0}}{1 + \frac{cV_{x0}t}{m}}$

Here, we have a new "natural time", $\tau \equiv \dfrac{m}{cV_{x0}}$. (Check units yourself!)



Similar to linear drag in shape, but inverse time rather than exponential decay.

For $X(t)$, use $\dfrac{dX}{dt} = V_x(t)$ and integrate:

$$X(t) - X(0) = \int_{t=0}^{t} \frac{V_{x0}}{1 + \frac{cV_{x0}}{m}t}\,dt = V_{x0}\,\tau \ln(1 + t/\tau)\Big|_0^t$$

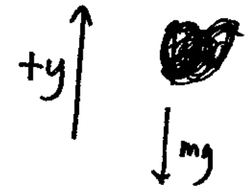
$$= \frac{m}{c}\ln(1 + t/\tau)$$

Note: At $t \to \infty$, x does continue to grow, but $\ln(1 + t/\tau)$ is a <u>very</u> slow "blowup". (ln is a "boring function"?)

This is different from linear drag, where $X(\infty)$ was <u>finite</u>.

In vertical motion (with gravity), what about QUADRATIC DRAG? (Recall $V_t^2 = \frac{mg}{c}$)

$$F = ma \Rightarrow m\dot{V}_y = -mg - c|V_y|V_y.$$



$$\text{so} \quad \dot{V}_y = -g - \frac{c}{m}|V_y|V_y = -g\left(1 + \frac{|V_y|V_y}{V_t^2}\right)$$

If drop from rest, V_y is always negative, $|V_y| = -V_y$, and

$$\text{falling:} \quad \dot{V}_y = -g\left(1 - V_y^2/V_t^2\right)$$

1st order, non-linear, separable.

$$\frac{dV_y}{1 - V_y^2/V_t^2} = -g\,dt \qquad \text{Can integrate!}$$

Look it up, or just do it by integration by parts:

$$\text{Trick:} \quad \frac{1}{(1-x)(1+x)} = \frac{1/2}{1-x} + \frac{1/2}{1+x} \qquad \text{Convince yourself!}$$

$$\text{Let} \quad x = V_y/V_t \qquad dx = dV_y/V_t$$

$$\int \frac{dV}{1 - V_y^2/V_t^2} = \int -g\,dt = -gt + C$$

$$V_t \int \frac{dx}{(1-x)(1+x)} = -\frac{1}{2}V_t \ln(1-x) + \frac{1}{2}V_t \ln(1+x)$$

$$\text{So,} \quad \frac{V_t}{2}\ln\left(\frac{1 + V_y/V_t}{1 - V_y/V_t}\right) = -gt + C$$

(If start at $V_y = 0$, $C = 0$, convince yourself)

So falling from rest, in quadratic drag case:

$$\frac{1 + V_y/V_t}{1 - V_y/V_t} = e^{-2gt/V_t}, \text{ so (convince yourself!)}$$

$$\frac{V_y}{V_t} = \frac{e^{-2gt/V_t} - 1}{e^{-2gt/V_t} + 1}.$$

[Note, $t \to \infty$, $\frac{V_y}{V_t} \to -1$, that's my sign convention!]

Review of Hyperbolic functions (Boas Ch 2.12)

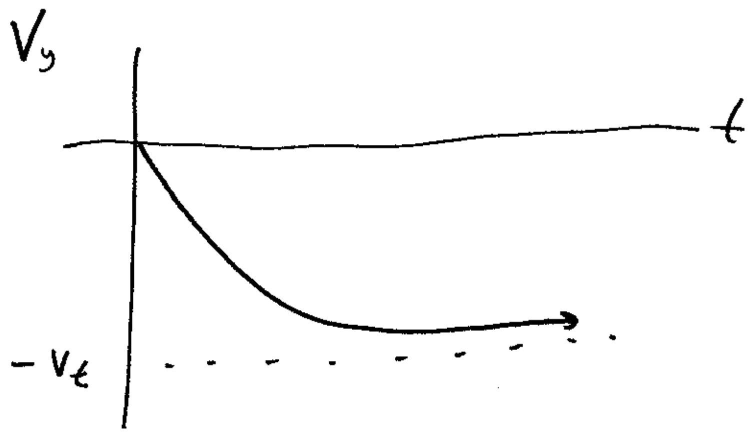
$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\text{So } \frac{V_y}{V_t} = \tanh\left(-\frac{gt}{V_t}\right) = -\tanh\frac{gt}{V_t}$$

Because of my sign convention



Qualitatively similar to linear drag case.

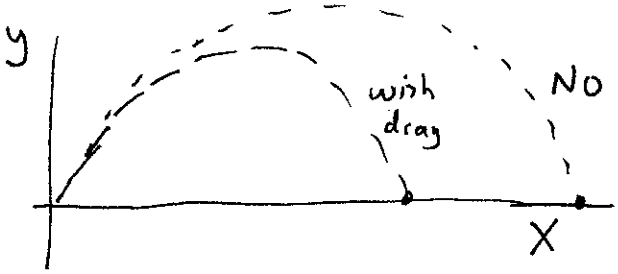
(Taylor 2.3)

"Projectile Motion"

Now, at last, 2-D motion.

Let's start w. linear drag:

$$m\vec{a} = \vec{F}$$



so
$$\begin{cases} m\dot{V}_x = -bV_x \\ m\dot{V}_y = -mg - bV_y \end{cases}$$

Recall our definitions
$$\begin{cases} \tau = m/b & \text{"natural" time} \\ V_T = mg/b & \text{"terminal" speed} \end{cases}$$

We've solved both of the eq'ns of motion

$$x(t) = \tau V_{x0}\left(1 - e^{-t/\tau}\right)$$ ← Notes p. 22b, with $x(0)=0$

$$y(t) = -V_t\, t + \tau\,(V_t + V_{y0})\left(1 - e^{-t/\tau}\right)$$ ← p. 24, with $y(0)=0$

Can find $y(x)$ by eliminating time in 1st eq'n:

$$e^{-t/\tau} = 1 - x/V_{x0}\tau \quad \text{or} \quad t = -\tau \ln\left(1 - x/V_{x0}\tau\right)$$

Thus

$$y(t) = +V_t\tau \ln\left(1 - \frac{x}{V_{x0}\tau}\right) + \frac{V_t + V_{y0}}{V_{x0}}\, x.$$

Not the prettiest of eq'ns!

For small drag (small $b \Rightarrow$ long τ) we can approximate this formula very nicely. This involves one of the most useful approximation approaches, used throughout physics. So, a brief math interlude!

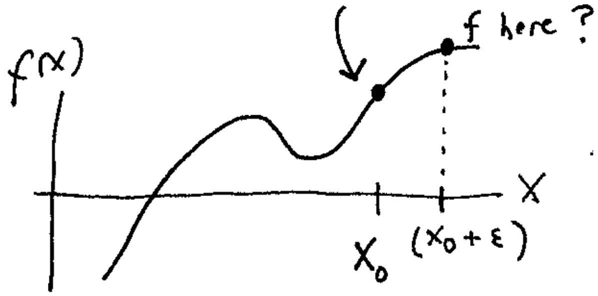
2210 -28. Boas p. 24-25

Taylor Series expansions (approximations).

we often need to know $f(x_0 + \varepsilon)$, a function near a place

small

we know a lot about.



(E.g previous page, we want $\ln\left(1 - \frac{x}{v_{x0}\tau}\right)$ for small $\frac{x}{v_{x0}\tau}$)

This is one of the premier Math tools / tricks !

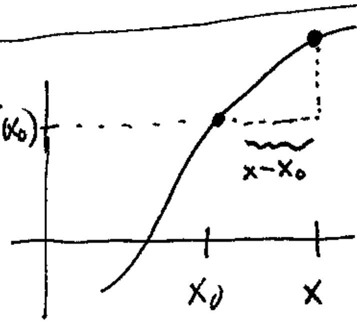
Consider $f(x)$ near x_0. Most functions can be written as a power series

$$f(x) = a_0 + a_1 \underbrace{(x - x_0)}_{\text{"linear"}} + a_2 \underbrace{(x-x_0)^2}_{\text{"quadratic"}} + \dots$$

constant constant

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Graphically



$$f(\text{here}) \approx f(x_0) + \text{slope} * (x - x_0)$$

$$= f(x_0) + f'(x_0)(x - x_0)$$

$$= a_0 + a_1 (x - x_0)$$

(This would be the 1st term in that Taylor series, treating $f(x)$ as approximately linear near x_0.)

2210 - 29.

Notice $f(x_0) = a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + \cdots$
$= a_0$

and $f'(x) = 0 + a_1 + 2 \cdot a_2 (x - x_0)^1 + 3 \cdot a_3 (x - x_0)^2 + \cdots$

so $f'(x_0) = a_1 + 0 + 0 + \cdots$

and $f''(x) = 0 + 0 + 2 \cdot 1\, a_2 + 3 \cdot 2\, a_3 (x - x_0) + \cdots$

so $f''(x_0) = 2a_2 + 0 + 0 + \cdots$

See the pattern?

$$a_0 = f(x_0)$$
$$a_1 = f'(x_0)$$
$$a_2 = f''(x_0)/2$$
$$a_3 = f'''(x_0)/3!$$
$$\ldots \quad a_n = f^{(n)}(x_0)/n!$$

So our "Taylor series around the point x_0" is

$$\boxed{f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots}$$

(Notation: If $x_0 = 0$, we call this a Maclaurin series)

If x is near x_0, so $x - x_0 \equiv \varepsilon$ is small

$$f(x_0 + \varepsilon) \approx f(x_0) + f'(x_0) \cdot \varepsilon + \frac{f''(x_0)}{2!}\varepsilon^2 + \cdots$$

(Also the Taylor series)

<u>Example</u> $f(x) = e^x$, and $x_0 = 0$.

so $f'(x) = e^x$, $f''(x) = e^x$, etc, and at $x_0 = 0$, all $= e^0 = 1$.

$\therefore$ MacLaurin series for e^x is

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots$$

$$= 1 + x + x^2/2! + x^3/3! + \cdots$$

<u>Example</u> $f(x) = \sin(x)$, and $x_0 = 0$.

$f'(x) = \cos(x)$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = +\sin x$

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3')} = -1$, $f^{(4')} = 0$, etc.

$$\sin x = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots$$

$$\sin x = x - x^3/3! + x^5/5! - \cdots$$

See Front Flyleaf of Taylor for the common ones.
Memorize 'em, you'll use them the rest of your life!

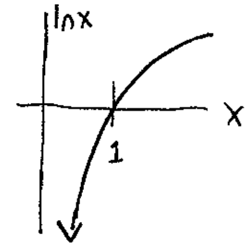
($\sin x$, $\cos x$, e^x, ...) If x small, 1st term or 2 may be all you need!!

$$\sin x \approx x - x^3/6 + \cdots \Rightarrow \sin(.1\,\text{rad}) = .1 - .001/6 = \underline{\underline{.0998}}$$

↑

"Leading term", or "first order term"

2210 - 31



Example $f(x) = \ln x$. This blows up at $x=0$, so Maclaurin series at $x_0 = 0$ is useless. But, Taylor series at $x_0 = 1$ is handy!

$$f(x) = \ln x \implies f(x_0) = \ln 1 = 0$$

$$f'(x) = 1/x = x^{-1} \implies f'(x_0) = 1^{-1} = 1$$

$$f''(x) = (-1)\, x^{-2} \implies f''(x_0) = (-1)(1)^{-2} = -1$$

$$f''' = (-1)(-2)\, x^{-3} \implies f'''(x_0) = (+2!)\,(1)^{-3} = +2!$$

so $$f(x) = f(x_0) + f'(x_0)\,(x-1) + \frac{f''(x_0)}{2!}(x-1)^2 + \cdots$$

$$= 0 + 1(x-1) - \frac{(x-1)^2}{2} + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

so $$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$ ← Not factorials!!

I prefer to rewrite this for $x - 1 = \varepsilon$, or $x = 1 + \varepsilon$

(Near $x=1$) $$\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + \varepsilon^3/3 -$$

or, (convince yourself) $$\ln(1-\varepsilon) = -\left(\varepsilon + \varepsilon^2/2 + \varepsilon^3/3 + \cdots\right)$$

(Makes sense that this is negative, look @ the graph!)

One last example, a very handy one.

$f(x) = x^a$ where a might not be an integer.

(Around $x=0$, you already have a series... with 1 term!)

But if $x_0 = 1$, you get the wonderfully useful "Binomial" Series

$$f'(x) = a x^{a-1} \longrightarrow f'(x_0=1) = a$$

$$f''(x) = a(a-1)x^{a-2} \longrightarrow f''(x_0=1) = a(a-1)$$

$\vdots$ If a is an integer, the series is finite, but in general

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots$$

$$= 1 + (x-x_0)\, a + \frac{(x-x_0)^2}{2!} a(a-1) + \ldots$$

Once again, think of $x = x_0 + \varepsilon = 1 + \varepsilon$

so $x - x_0 = \varepsilon$ is small, and

$$(1+\varepsilon)^a \approx 1 + a\varepsilon + \frac{a(a-1)}{2!}\varepsilon^2 + \frac{a(a-1)(a-2)}{3!}\varepsilon^3 + \ldots$$

e.g $\sqrt{(1+\varepsilon)} = (1+\varepsilon)^{1/2} \approx 1 + \frac{1}{2}\varepsilon + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\varepsilon^2 + \ldots$

(If you "truncate", it's approximate, but accurate if $\varepsilon \ll 1$.)

Polynomials are much simpler to deal with. It's rare in real life that physics/engineering problems can be solved exactly, but it's common they can be approximated by a series (which is as accurate as you need, if you take enough terms)

Notation: Analytic functions have an ∞ # of finite derivatives at every point, and thus the Taylor series is "exact".

Back to our 2-D projectile with ~~quadratic~~ linear drag,

$$y(x) = v_t \tau \ln\left(1 - x/v_{x0}\tau\right) + \left(\frac{v_t + v_{y0}}{v_{x0}}\right) x \quad \leftarrow \text{Exact.}$$

If drag is small (or if x is small) $\dfrac{x}{v_{x0}\tau} \equiv \varepsilon$ is small,

and $\ln(1-\varepsilon) \simeq -(\varepsilon + \varepsilon^2/2 + \varepsilon^3/3 + \cdots)$

so $$y(x) \simeq + v_t \tau (-)\left(\frac{x}{v_{x0}\tau} + \left(\frac{x}{v_{x0}\tau}\right)^2 \cdot \frac{1}{2} + \left(\frac{x}{v_{x0}\tau}\right)^3 \cdot \frac{1}{3} + \cdots\right) + \left(\frac{v_t}{v_{x0}} x + \frac{v_{y0}}{v_{x0}} x\right)$$

Cancels with very 1st term!!

$$y(x) = \frac{v_{y0}}{v_{x0}} x - \frac{1}{2}\frac{v_t\, x^2}{v_{x0}^2\,\tau} - \frac{1}{3}\frac{v_t}{v_{x0}^3\,\tau^2} x^3 + \cdots$$

(Let's pause for a sec + figure out $y(x)$ when there is no drag)

Oh, also note $v_t/\tau = \dfrac{mg/b}{m/b} = g$, so 2nd term is $-\dfrac{1}{2} g \dfrac{x^2}{v_{x0}^2}$

In vacuum: $x = v_{x0}\, t$ ← use this to find t

$$y = v_{y0}\, t - \tfrac{1}{2} g t^2$$

plug in here, + get

$$y(x)_{\text{no drag}} = \frac{v_{y0}}{v_{x0}} x - \frac{1}{2} g \frac{x^2}{v_{x0}^2}$$

that's it, a parabola!

$$y(x)_{\text{with drag}} = \frac{v_{y0}}{v_{x0}} x - \frac{1}{2} \frac{g x^2}{v_{x0}^2} - \frac{1}{3} \frac{v_t}{v_{x0}^3 \tau^2} x^3 + \dots$$

Sweet! The $\underline{\underline{2}}$ leading terms match, so at 1st (small x), the 2 curves look identical. It's the 3rd order term that slowly but surely "deviates" the path, and is <u>negative</u> so our drag trajectory is (of course!) be<u>low</u> the ideal path.

How <u>far</u> do we go? The <u>Range</u> R

this is x, for which $y(x=R) = 0$.

For <u>no</u> drag, $0 = \frac{v_{y0}}{v_{x0}} R_{vac} - \frac{1}{2} g \frac{R_{vac}^2}{v_{x0}^2}$. of course, $R=0$ works, but also

$$R_{vac} = \frac{2 v_{y0} v_{x0}}{g}$$ //

Phys 1110 formula!

2210 -35

With drag, $V_t \tau \ln(1 - R/V_{x0}\tau) + \frac{V_t + V_{y0}}{V_{x0}} R = 0$

If $R=0$, this is satisfied, but there's a 2nd sol'n.

In general, you can't solve this eq'n analytically for it.

But if $R/V_{x0}\tau \equiv \varepsilon \ll 1$, we can approximate it:

$$y(R)_{\text{with drag}} = \frac{V_{y0}}{V_{x0}} R - \frac{\frac{1}{2} g R^2}{V_{x0}^2} - \frac{1}{3} \frac{V_t}{V_{x0}^3 \tau^2} R^3 + \text{(terms of order } R^4 + \ldots)$$

Setting this to 0, and dividing out one power of R gives

$$0 = \frac{V_{y0}}{V_{x0}} - \frac{1}{2} \frac{g R}{V_{x0}^2} - \frac{1}{3} \frac{V_t}{V_{x0}^3 \tau^2} R^2 + \text{Order}(R^3)$$

or $$R = \frac{2 V_{y0} V_{x0}}{g} - \frac{2}{3} \frac{V_t}{V_{x0} g \tau^2} R^2 + \mathcal{O}(R^3)$$

$$R = R_{vac} - \frac{2}{3} \frac{V_t}{V_{x0} g \tau^2} R^2 + \mathcal{O}(R^3)$$

range is reduced.

Method 1: If $R \approx R_{vac}$, then plug that in to approximate R

Method 2 (better, formally): Solve this quadratic eq'n for R!

Then, if you like, use $\sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon$ to simplify, where

ε here will be some small term...

2210 -36

Quadratic drag for Projectile Motion

$$m\dot{\vec{v}} = m\vec{g} - c|v|\vec{v}, \text{ so } \quad m\dot{v}_x = -c\sqrt{v_x^2+v_y^2}\, v_x$$
$$m\dot{v}_y = -mg - c\sqrt{v_x^2+v_y^2}\, v_y$$

This is nasty - it's 2 "coupled" and non-linear equations. I can't solve it, I don't know any tricks. But, there is a practical way: If you know "initial conditions", i.e. $v_x(t_0)$ and $v_y(t_0)$

then e.g. $m\dot{v}_x = -c\sqrt{v_x^2+v_y^2}\, v_x$ means

$$m\,\frac{v_x(t_0+\Delta t) - v_x(t_0)}{\Delta t} \approx -c\sqrt{v_x^2(t_0)+v_y^2(t_0)}\; v_x(t_0)$$

you know everything but $v_x(t_0+\Delta t)$. Simple, it's linear, find it!

Similarly, find $v_y(t_0+\Delta t) \simeq v_y(t_0) - \Delta t\, g - \frac{\Delta t\, c}{m}\sqrt{v_x^2(t_0)+v_y^2(t_0)}\; v_y(t_0)$

then, iterate, find $v_x(t_0+2\Delta t)$, etc. It's a numerical approach

This is what "ND Solve" does in MMA!

you get $v_x(\text{any } t)$ and $v_y(\text{any } t)$, as numbers.

And thus $\frac{dx}{dt} = v_x(t)$ can similarly give you $x(\text{any } t)$.

This might be the engineer's approach, and it's e.g. what the PhET sim does to graph $y(x)$.

This wraps up Ch. 2:

We're punting "Motion in Uniform B"

(i.e. $\vec{F} = q\,\vec{v}\times\vec{B}$), at least for now. (You'll

cover this explicitly in Phys 3310!)