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Taylor Ch. 3 : Energy (See also Boas 6.4 TO 6.8)

Energy is a rather subtle concept! It's a scalar quantity associated with <u>all</u> physical systems which is <u>conserved</u> (in total) for any/all physical processes.

I typically (if a bit naively!) think of it casually as a quantity that tells you about how much <u>work</u> a system can do.*

In phys 2210 we focus on MECHANICAL ENERGY:

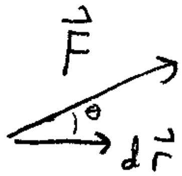
1) Kinetic Energy (associated with <u>motion</u> in a reference frame)
2) Potential Energy (" " particular, conservative <u>forces</u>)

The history of "energy" starts (mostly) well <u>after</u> Newton, Thompson ("Lord Kelvin") established the 1st Law of thermodynamics, essentially "conservation of energy"

KE is called "T" $= \frac{1}{2} m v^2$ for a point mass.

* The relation to <u>work</u> is key, that's our 1st order of business

If a force $\vec{F}$ acts on a moving pointlike object, for small movements $d\vec{r}$, the work done is $dW_{\text{by } \vec{F}} = \vec{F} \cdot d\vec{r}$



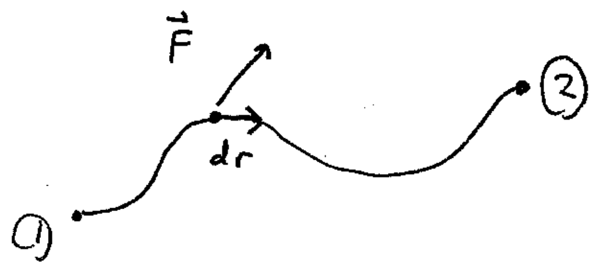
$dw = F\,dr\cos\theta.$

- Work is a signed scalar. If $\theta > 90°$, dw is negative!
- $dw = 0$ if $\vec{F} \perp d\vec{r}$. (So e.g. the ice rink does no work on the frictionless sliding puck, the earth does no work on a satellite in circular orbit!)
- Other forces may be acting too. This formula gives $dW_{\text{by this } \vec{F}}$

For larger movements, just ADD UP the little dW's:

$$W_{\text{by } \vec{F}} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

along path followed



This integral is a sum. (It's not the "area under this curve"!!)

For pointlike masses:

$$\vec{F}_{net} \cdot d\vec{r} = \left(m \frac{d\vec{v}}{dt}\right) \cdot d\vec{r} = \left(m \frac{d\vec{v}}{dt}\right) \cdot \left(\frac{d\vec{r}}{dt} dt\right) = m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$$

"cancel the dt"

$$\overset{\text{Trick, convince yourself}}{=} \frac{m}{2} \frac{d}{dt}(\vec{v} \cdot \vec{v})\,dt \overset{\text{pointlike, constant } m}{=} \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt$$

You might prefer "running that proof backwards"

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{F}_{net}\cdot\vec{v}$$ (Cancelling dt everywhere in previous page's proof)

Bottom line, with $T \equiv \frac{1}{2}mv^2 \equiv$ Kinetic energy, for point masses.

$$\boxed{dT = \vec{F}_{net}\cdot d\vec{r} = dW_{net}}$$ where dW_{net} means "work done by the net force".

or, for finite movements

$$\Delta T = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{net}\cdot d\vec{r} = W_{net}, \text{ going from } \vec{r}_1 \text{ to } \vec{r}_2$$

This is the Work - Energy theorem. For point masses,

Net work done = change in KE.

- If W_{net} is negative, object slows down!
- Theorem is only true for point masses *
- Theorem is only true for net work of all forces.

* [Consider e.g a system of 2 masses $\xrightarrow{F_{ext,1}} \overset{m_1}{\bigcirc}$ $\quad \overset{m_1}{\bigcirc} \leftarrow F_{ext,2}$

Here $F_{net,ext} = 0$, but $\Delta T > 0$ as both masses speed up.

Also, $\underbrace{\sum_i W_{by\ F_i}}_{\text{each is positive!}} \neq W_{F_{net}} = 0$, here!)

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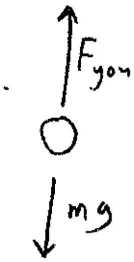
Example: Lift a mass at constant speed, so $\Delta T = 0$.

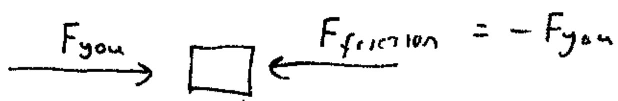


$W_{\text{by you}} = \vec{F}_{you} \cdot \Delta\vec{r} = (+mg)\hat{y} \cdot (h\hat{y}) = +mgh$

$W_{\text{by gravity}} = \vec{F}_{grav} \cdot \Delta\vec{r} = (-mg)\hat{y} \cdot (h\hat{y}) = -mgh$

$F_{net} = 0$, $W_{net} = 0$, $\Delta T = 0$. Good!

Example: Slide a box across frictionfull floor @ constant speed.



$W_{\text{by you}} = \vec{F} \cdot \Delta\vec{r} = (+F_{you}\hat{x})(+\Delta x)\hat{x} = +F\Delta X$

$W_{\text{by friction}} = (-F_{you}\hat{x})(+\Delta x)\hat{x} = -F\Delta X$

$F_{net} = 0$, $W_{net} = 0$, $\Delta T = 0$. Good!

Example: Apple falls h, from rest.

$W_{\text{by gravity}} = (-mg)\hat{y} \cdot (-h\hat{y}) = +mgh$

$\Delta T = \frac{1}{2}m(v_f^2 - 0) = \frac{1}{2}m(2g\Delta y) = mgh$

equal, good!

Example: Sliding box slows due to friction, from $v_0 \to 0$.

$W_{\text{by friction}} = (-\mu_K N)\hat{x} \cdot (\Delta x\hat{x}) = -\mu_K mg\Delta X$

so here $\Delta T = W_{net} < 0$. Makes sense, slows down, (+this formula relates Δx to v_0^2 and μ_K.)

Note: No exceptions! Friction, gravity, human force, combo's; $\Delta T = W_{net}$, always!

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Computing $W = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$ = A line integral (or "path" along a)

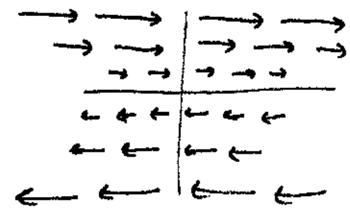
Must know the path, in general, to evaluate!

Many tricks! (See Boas 6.8) Best is often to "parametrize" path.

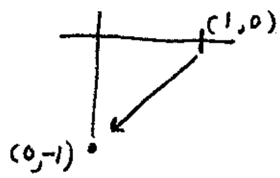
See Taylor's Ex 4.1!

My Example

- Suppose $\vec{F} = y\,\hat{i}$ Can you picture this?



- Particle moves from $(1,0)$ to $(0,-1)$



How much work did $\vec{F}$ do?

The Procedure (it's pretty universal!)

0) Pick your coordinate system! Here, Cartesian: $\vec{r} = x\hat{i} + y\hat{j}$.

1) In your coord system, generic $d\vec{r}$ is always $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

2) Compute $\vec{F} \cdot d\vec{r} = (y\,\hat{i}) \cdot (dx\,\hat{i} + dy\,\hat{j}) = y\,dx + 0\,dy$

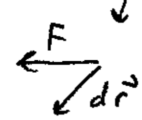
3) Write out $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{x=1}^{0} y\,dx + \int_{y=0}^{-1} 0\,dy$ ← end ← start

④ Parametrize this path! Here, look at picture, $y = x - 1$ (slope is 1; intercept)

then substitute in: $\int_{x=1}^{0} y\,dx = \int_{x=1}^{0} (x-1)\,dx$

⑤ Integrate: $\int_1^0 (x-1)\,dx = \frac{x^2}{2}\Big|_1^0 - x\Big|_1^0 = -\frac{1}{2} + 1 = + \frac{1}{2}$.

Check sign: Look at picture, I agree!



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Parametrization step is not unique!

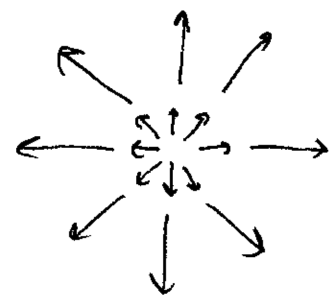
E.g., $\left.\begin{array}{l} x = 1-u \\ y = -u \end{array}\right\}$ $u = 0$ to 1 <u>also</u> generates that line!

So $u = 1-x$, $du = -dx$

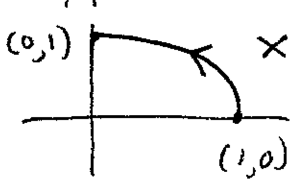
So then, $\displaystyle\int_{x=1}^{0} y\,dx = \int_{u=0}^{1} (-u)(-du) = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = +\frac{1}{2}.$

<u>Another</u> <u>Example</u>

- Suppose $\vec{F} = \vec{r}$ Can you picture this?



- Suppose particle moves along sideways parabola



$x = 1 - y^2$ Work done by $\vec{F}$?

0) Pick coord system. Cartesian is fine, $\vec{r} = x\hat{i} + y\hat{j}$

1) So (<u>always</u>, in Cartesian) $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

2) $\vec{F}\cdot d\vec{r} = (x\hat{i} + y\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j}) = x\,dx + y\,dy$

3) $\displaystyle\int \vec{F}\cdot d\vec{r} = \int_{\substack{\text{on path} \\ x=1}}^{x=0} x\,dx + \int_{\substack{\text{on path} \\ y=0}}^{y=1} y\cdot dy$ ← end / ← start

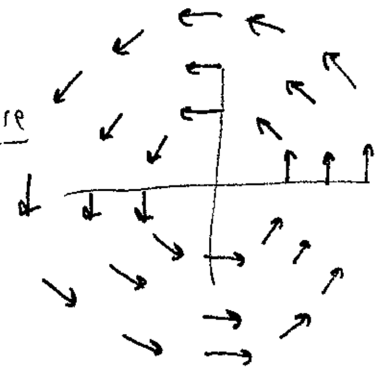
4) On path, $x = 1-y^2$, so $dx = -2y\,dy$.

$\displaystyle\therefore W = \int_{y=0}^{y=1} (1-y^2)(-2y\,dy) + \int_0^1 y\,dy = -y^2 + \frac{2y^4}{4} + \frac{y^2}{2}\Big|_0^1 = \underline{\underline{0}}$

Third Example

Suppose $\vec{F} = c\hat{\varphi}$ (in polar coords)

Can you picture this?

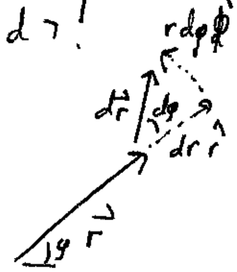


and you move along a quarter circle from (1,0) to (0,1). Work done?

0) Pick coord system: Plane polar! $\vec{r} = r\hat{r}$

1) So (Always, in polar coords!) $d\vec{r} = dr\,\hat{r} + r\,d\varphi\,\hat{\varphi}$

Look back at our early derivation of $\vec{v} = \frac{d\vec{r}}{dt}$, just cancel out the dt!



2) $\vec{F}\cdot d\vec{r} = c\hat{\varphi}\cdot d\vec{r} = c\,r\,d\varphi$

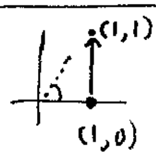
3) $\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = \int_{\varphi=0}^{\varphi=\pi/2} r\,c\,d\varphi$ along our path

4) Parametrize our path. Here, $r=1$, so it's simple, nothing to do

$$\int_{\varphi=0}^{\pi/2} (1)(c)\,d\varphi = c\,\pi/2.$$

(Makes sense, $\vec{F}\cdot d\vec{r}$ is the same all along the path)

| Suppose our path had been up a straight line. |



Given the polar form of $\vec{F}$, I would stick with polar coords, all the way to step 3) $\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = \int_{\varphi=0}^{\pi/4} r\,c\,d\varphi$ ← look @ the path, that is the final φ value

4) Along this path, $x = r\cos\varphi = 1$, so $r = 1/\cos\varphi$ "parametrizes" the path!

So we need $\int_0^{\pi/4} \frac{c}{\cos\varphi}\,d\varphi = .88\,c$, from Mathematica

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Calculating line integrals is a bit of an art, but it's just a <u>sum</u>.

The main <u>physics</u> story here is, for point particles

<u>Work</u> - <u>Energy</u> <u>theorem</u>:

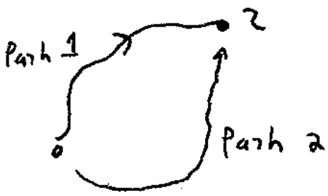
$$\Delta T = W_{net\,(1\to 2)} \quad , \quad \text{or} \quad T_2 - T_1 = \underset{\substack{\text{along your} \\ \text{particle's path}}}{\int_1^2} \vec{F}_{net} \cdot d\vec{r}$$

where $\vec{F}_{net} = \sum_i \vec{F}_i$, and $W_{net} = \sum_i W_{\substack{\text{by} \\ \text{force } i}}$

In many physics cases (Gravity, electrostatics, springs, ...)

1) $\vec{F} = \vec{F}(\vec{r})$, not (explicitly) $\vec{F}(t)$ or $\vec{F}(\vec{v})$ or $\vec{F}$ (anything <u>else</u>)

<u>and</u> 2) $W(1\to 2)$ is <u>independent</u> of the path taken.



when both are true, we say $\vec{F}$ is conservative, + lovely consequences follow!

Note: <u>friction</u> (μmg) violates condition 2, W <u>depends</u> on path!
<u>drag</u> (bv or cv^2) " " 1 <u>and</u> 2! <u>Neither</u> is conservative.

<u>Normal</u> force might satisfy condition 2 since $W(1\to 2) = 0$ (normal force is $\perp$ to $d\vec{r}$ along the path), but it is <u>not</u> just a fn of $\vec{r}$, e.g $\vec{N}$ increases if you go <u>faster</u> around a curve, it depends on v!

So, Normal force is <u>not</u> conservative.

If all forces on an object are conservative, we can define: PE or $U(\vec{r})$, the "potential energy", a function of position.

We construct it so that $E_{mech} \equiv T + U$ is conserved, $\Delta E = 0$.

I claim that $U(\vec{r}) \equiv -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$

does the trick!

- That $-$ sign is key, see below
- $\vec{r}_0$ is arbitrary, it's your choice. A different $\vec{r}_0$ gives you a slightly different function $U(\vec{r})$ (it differs by addition of a constant). So PE is arbitrary up to one undetermined constant. But, the functional form $U(\vec{r})$ is what is determined by $\vec{F}$.
- Since we're postulating $W(\vec{r}_0 \to \vec{r})$ is path independent, this $U(\vec{r})$ is perfectly unique + well-defined (given $\vec{r}_0$)

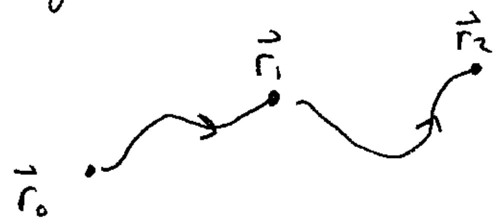
Example: $\vec{F}_{grav} = -mg\hat{y}$, so $U_{grav}(y) = -\int_0^y (-mg\hat{y})\cdot d\hat{y} = mgy$

$\leftarrow$ my choice.

The $-$ sign was key, makes no sense for PE to become smaller as $y\uparrow$, it must get bigger to conserve energy.

Let's show that this definition of $U(\vec{r})$ leads to Energy conservation.

Consider this sequence of paths:



Work $(0 \to 2)$ = Work $(0 \to 1)$ + Work $(1 \to 2)$

For conservative forces, all of these terms are path independent, so this is generically true! Subtracting to solve for Work $(1 \to 2)$:

$$W(\vec{r}_1 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_2) - W(\vec{r}_0 \to \vec{r}_1)$$

$$= -U(r_2) + U(\vec{r}_1)$$ By my def of U!

$$= -\Delta U(1 \to 2)$$ That's what Δ means

If $E = T + U$, then

$$\Delta E(1 \to 2) = \Delta T(1 \to 2) + \Delta U(1 \to 2)$$

$$= +W(1 \to 2) + (-W(1 \to 2))$$

By work-energy theorem!   From lines above

$$= 0.$$

Ahh. With our definition $U(\vec{r}) \equiv -W(\vec{r}_0 \to \vec{r})$

we proved $\Delta E(1 \to 2) = 0$. This − sign was critical in our proof!

So Conservative forces means Energy is conserved.

If there is only one force (think "projectile", e.g!)

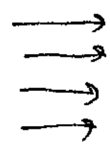
$E = T + U_F(\vec{r})$ is conserved, with U_F = "The P.E. of force F"

If you have many conservative forces (think e.g "projectile with E-field")

$$\underbrace{E = T + U_1(\vec{r}) + U_2(\vec{r}) + \dots}_{\text{Mechanical Energy.}}$$ is conserved : $\Delta E\,(1\to 2) = 0$ for any points on trajectory

Remember, this is all for point particles.

Example: $+q$ in uniform $\vec{E}$ field: $\vec{E} = E_0\hat{x}$, $\vec{F} = q\vec{E}$

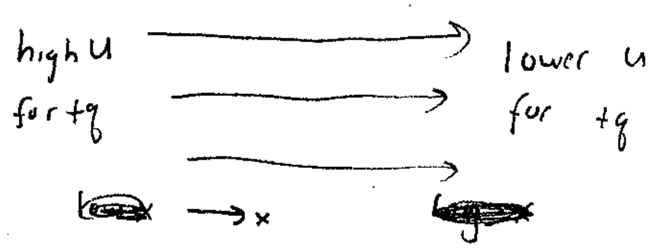


$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} (q\vec{E})\cdot d\vec{r}$$

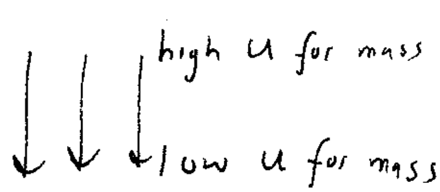
$$= -q\int_{\vec{r}_0}^{r} E\,dx = -q E_0 (x - x_0).$$

we can choose $\vec{r}_0$, let's pick the origin

$\Rightarrow U(\vec{r}) = -q E_0 x$ Bigger $x \Rightarrow$ lower U. Makes sense!



Reminds me of gravity



What about non-conservative forces?

Suppose $\vec{F}_{net} = \vec{F}_{cons} + \vec{F}_{Non\text{-}cons}$

so $\underbrace{\Delta T\,(1\to2) = W_{net}\,(1\to2)}_{\text{work-energy theorem still holds!}} = W_c\,(1\to2) + W_{NC}\,(1\to2)$

(Def of U, see p. 66)

and so

$\underbrace{\Delta E\,(1\to2) = \Delta(T+U)}_{\text{Definition of } E = T+U!} = \Delta T - W_{cons}(1\to2)$

$= W_{NC}\,(1\to2)$ by using the line above

So E_{Mech} is not conserved! And, $W_{NC}\,(1\to2)$ may depend on path.

The change of Mech. energy goes to/comes from so other form (e.g, often, thermal energy) So Total Energy is still (always!) conserved, but Mechanical " is not.

See Tayor Ex 4.3, Block sliding on incline:

$\underbrace{\Delta T}_{\text{Kinetic}} + \underbrace{\Delta U}_{\text{grav PE}} = W_{NC}(1\to2) = \underset{\text{from } F_{Normal}}{0} + \underbrace{W_{friction}}$ which is $-\mu N \times$ distance

$= -\mu\, mg\cos\theta \times$ distance

E.g start at rest + slide down hill,

$(\tfrac{1}{2}mv^2 - 0) + (0 - mgh) = -\mu mg\cos\theta \times d$. Let's you find V_f.

[If $\theta \to 0$, $\tfrac{1}{2}mv^2 = mg(h{=}0) - \mu mgd$. Sign is nonsense... but of course, the block would never slide if $\theta = 0$!!]

Relating U to $\vec{F}$ (and vice versa!)

we defined $U(\vec{r}) = -\int_{\vec{r}_0}^{r} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$. So if you know $\vec{F}$, you can find $U(r)$

But, can you invert this? If you know $U(r)$, can you deduce $\vec{F}$?

$$\Delta U\,(1\to 2) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

If your displacement is tiny, i.e. you go from $\vec{r}$ to $\vec{r}+d\vec{r}$,

$$dU = -\vec{F}(r)\cdot d\vec{r} = -(F_x\,dx + F_y\,dy + F_z\,dz)$$

do you see this?

these expressions are equal.

But $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$. (Chain rule).

Since dx, dy, dz can be anything you choose (e.g $dy = dz = 0$ is a choice)

I claim $\frac{\partial U}{\partial x} = -F_x$, and $\frac{\partial U}{\partial y} = -F_y$ and $\frac{\partial U}{\partial z} = -F_z$.

$$\text{so } \vec{F} = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right)$$

we write this in "shorthand" as $\vec{F} = -\vec{\nabla} U$ } Any conservative force is derivable from its potential energy

$$\boxed{\text{so } U(r) = -\int_{r_0}^{r} \vec{F}(r')\cdot d\vec{r}\,' \iff \vec{F}(\vec{r}) = -\vec{\nabla} U(\vec{r})}$$

∇ and line integrals are "inverses".

Let's talk about $\boxed{\text{Gradient}}$ $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

- $\vec{\nabla}$ is an operator. (A "vector differential operator".

It acts on fns and returns a vector function.

To me, it's basic meaning comes from

$$\boxed{df = \vec{\nabla} f \cdot d\vec{r}}$$

True for any scalar function $f(\vec{r})$
We derived this on the previous page!

↑

$$\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\vec{\nabla} f)_x\, dx + (\vec{\nabla} f)_y\, dy + \cdots \right.$$
$$\left. = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \cdots \quad \checkmark \right)$$

$df = \vec{\nabla} f \cdot d\vec{r}$ tells me how to interpret $\vec{\nabla} f$

It says how $f(r)$ changes if you move "$d\vec{r}$" away!

$\vec{\nabla} f$ is a vector, + points in the direction of max rate of change of $f(r)$

In that direction, $|\nabla f| = \frac{df}{dr}$, it's the "directional derivative"

If $\vec{\nabla} f = 0$, f is not changing in any direction, we're at a local max or min. (or inflection/saddle)

(In spherical coords, $\vec{\nabla} f \neq \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{\partial}{\partial \varphi}$! (Units ??!!)

Instead, use $\vec{\nabla} f \cdot d\vec{r} = df$ to figure out $\vec{\nabla} f$!)

Back to $\vec{F}(\vec{r}) = -\vec{\nabla} U(\vec{r})$

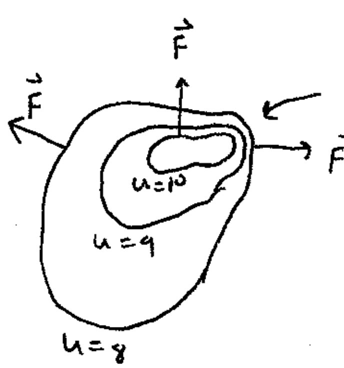
- The - sign says Force points opposite the direction of increasing U. This makes sense, $\vec{F}$ points "downhill", not "uphill"

- $\vec{F}$ is $\perp$ to "equipotential lines"

Because $dU = \vec{\nabla} U \cdot d\vec{r}$. Equipotential lines follow locations of constant U, meaning $dU = 0$, or $\vec{\nabla} U$ is $\perp$ to $d\vec{r}$, if $d\vec{r}$ follows an equipotential

So $\vec{F} \perp d\vec{r}$, if $d\vec{r}$ is following an equipotential line.

$|\vec{F}| = |\vec{\nabla} U|$, so when you draw equipotential lines, if they are closely spaced, meaning $\frac{dU}{dr}$ is big there, then $|\vec{F}|$ is big.



$\frac{\Delta U}{\Delta r}$ is big here, it's a cliff! Large Force.

$\vec{\nabla}$ is a funny beast. It's <u>sort</u> of a vector (x component is $\frac{\partial}{\partial x}$, etc)
and <u>sort</u> of a "derivative" (it needs to <u>act</u> on a function!)

you can write $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ like you write $\vec{r} = (x, y, z)$

you can treat $\vec{\nabla}$ like a vector in many ways, e.g

$$\boxed{\text{Curl}} \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial z}\right)$$

I claim (+ it's easy to prove, just $\longrightarrow$
write this out!)

$\vec{\nabla} \times \vec{\nabla} f = 0$ for any function $f(\vec{r})$.

so $\vec{\nabla} \times \vec{\nabla} U = 0$ " " Potential $U(\vec{r})$

so $\boxed{\vec{\nabla} \times \vec{F} = 0 \quad \text{for } \underline{\text{any}} \text{ conservative force!}}$

The "curl of a conservative force vanishes"

"Conservative forces have no curl".

So, what's curl? What's it mean? I will invoke (w/o proof)

Stoke's theorem: $\underset{\text{around a closed loop}}{\oint \vec{F} \cdot d\vec{r}} = \underset{\text{over any surface } \underline{\text{bounded}} \text{ by the loop}}{\iint (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}}$

you'll study this much more in future classes! For now, what you should take away is

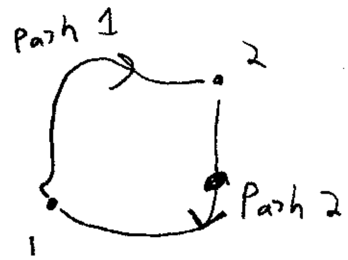
If $\vec{\nabla} \times \vec{F} = 0$ then by Stokes theorem, $\oint \vec{F} \cdot dr = 0$ <u>any</u> loop at all.

So Conservative force
means $\vec{\nabla} \times \vec{F} = 0$
means $\oint_{\text{any loop}} \vec{F} \cdot d\vec{r} = 0$

} All equivalent! If one is true, so are the others!

There's more! Consider integrating $1 \to 2$ and back on 2nd path

My loop:



If $\vec{F}$ is conservative, $\oint \vec{F} \cdot d\vec{r} = 0$, but this is for <u>any</u> loop

$$\Rightarrow \underbrace{\int_{1,\,\text{Path}_1}^{2} \vec{F} \cdot d\vec{r} + \int_{2,\,\text{Path 2}}^{1} \vec{F} \cdot d\vec{r}}_{\text{Together, Path 1 + Path 2 is a closed loop}} = 0$$

Now I claim $\int_{2,\,\text{Path 2}}^{1} \vec{F} \cdot d\vec{r} = -\int_{1,\,\text{Path 2}}^{2} \vec{F} \cdot d\vec{r}$, because reversing a path reverses the <u>sign</u> of $\vec{F} \cdot d\vec{r}$ everywhere

So $\int_{1,\,\text{Path 1}}^{2} \vec{F} \cdot d\vec{r} - \int_{1,\,\text{Path 2}}^{2} \vec{F} \cdot d\vec{r} = 0.$

True for <u>any</u> loop, thus <u>any</u> path. So we recover something we knew:

$\int_1^2 \vec{F} \cdot d\vec{r}$ is path independent for conservative forces.

All of the following are completely equivalent (any one implies all)

- $\vec{F}$ is a conservative force
- $\vec{\nabla} \times \vec{F} = 0$
- $\oint_{\text{any loop}} \vec{F} \cdot d\vec{r} = 0$
- $\int_1^2 \vec{F} \cdot d\vec{r}$ is independent of path
- $\vec{F} = -\vec{\nabla} U(r)$ for a well-defined potential function

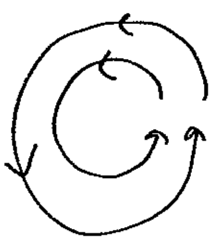
All different ways to think about meaning + consequences of conservative forces.

(The field does no work if you end up back where you start)

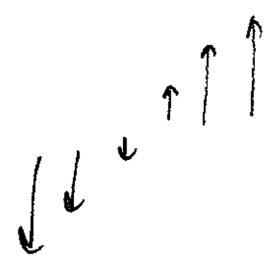
In particular, this helps me see what "curl free" means, a bit better - There is never a "circulation of $\vec{F}$", $\oint \vec{F} \cdot d\vec{r}$ is always 0 for any loop, small, or large!

$$\left(\vec{\nabla} \times \vec{F}\right)_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

carries information about "rotation" of the field around the z axis.



this field has curl! so does this:



If a tiny paddle wheel drops into the field, + it is made to rotate, There's a curl, + the field is not conservative.

Example $\vec{F}_{coul} = \frac{kQq}{r^2}\hat{r} = \frac{kQq}{r^2}\frac{\vec{r}}{r} = \frac{kQq}{r^3}\vec{r}$

you can compute $\vec{\nabla}\times\vec{F}$. Taylor does it: $\frac{1}{r^3} = \frac{1}{(x^2+y^2+z^2)^{3/2}}$, $\vec{r} = (x,y,z)$

It's a little painful, but do the determinant formula, + get $\vec{\nabla}\times\vec{F} = 0$

OR use Taylor's back flyleaf in spherical coordinates.

this $\vec{F}$ has no $\hat{\theta}$ or $\hat{\varphi}$ component, and $F_r = \frac{kQq}{r^2}$

Take a look, every entry gives you 0. So $\vec{\nabla}\times\vec{F} = 0$

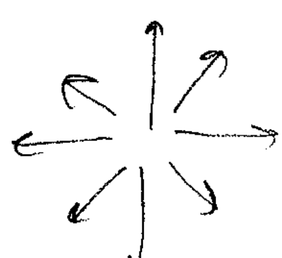
Thus: Coulomb force is conservative

- Work done by E-field is independent of path taken
- No work done if move q around any loop + return to start
- There is a well-defined potential energy. We can find it by

$U(r) = -\int_{r_0}^{r}\vec{F}\cdot d\vec{r}$. I'll show on next pages that

$U(r) = \frac{kQq}{r}$ works. (Back flyleaf of Taylor shows $\vec{\nabla}U$ in spherical coords, so you can quickly check that it gives $\vec{F}$ at top of page)

- Coulomb field is "curl free"



No "rotation" in this field.

Let's compute $U(\vec{r}) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$ with $\vec{F} = \frac{kQq}{r^2}\hat{r}$

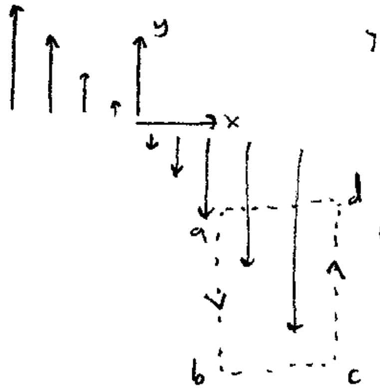
I'll pick $r_0 \to \infty$, this is my arbitrary choice.

1) In spherical coordinates, always, $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$

2) $$-\int_{r_0}^{r}\vec{F}(\vec{r}\,')\cdot d\vec{r}\,' = -\int_{\infty}^{r}\frac{kQq}{r'^2}\,dr' \underset{\text{easy!}}{=} +\left.\frac{kQq}{r'}\right|_{\infty}^{r} = +\frac{kQq}{r}$$

As claimed, $PE = \frac{kQq}{r}$, like in Phys 1120!

More curl intuition! Consider a non-zero curl function, like



the "pinwheel" turns, anywhere, this has non zero curl!

let's integrate $\oint \vec{F}(\vec{r})\cdot d\vec{r}$ this dashed loop

As you integrate $a \to b$, you get a small positive result, do you see why?

from $b \to c$, $\vec{F}\cdot d\vec{r} = 0$, no contribution

From $c \to d$, $\vec{F}\cdot d\vec{r}$ is negative ($\vec{F}$ is opposite $d\vec{r}$) and bigger.

From $d \to a$, $\vec{F}\cdot d\vec{r} = 0$

so $\oint \vec{F}\cdot d\vec{r} \neq 0$. Mathematically

$$\vec{F} = -x\,\hat{y}$$

(can you see this matches the picture?)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -x & 0 \end{vmatrix} = -\hat{K}$$

Not zero

Negative, ("circulates" around z axis!)

One last curl example. In E+M, you will learn

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

If B is "time independent", this is a "static" problem, $\vec{\nabla} \times \vec{E} = 0$,

all the 1120 stuff! $\Bigg\{$ and so $\vec{E}$ is conservative

Potential energy is defined

$\vec{E}$ does no work around closed loops

But if $\vec{B}$ depends on time, (Faraday's Law!)

$\vec{E}$ becomes non-conservative!

P.E. is <u>not</u> well defined

$\vec{E}$ <u>does</u> do work as you go around closed loops.

(This is what power generators do!)

Energy for 1-D systems:

"1-D" here means one variable defines our location.

Could be literally 1-D (e.g train on flat track)

or a pendulum (θ tells all!)

or a roller coaster (distance from start tells all!), etc.

If $\vec{F}$ acts on a 1-D particle (no vector needed in 1-D, call it $F_x(x)$ or simply $F(x)$)

$$W(1\to 2) = +\int_{x_1}^{x_2} F(x)\,dx$$

If F is conservative, recall
1) F depends only on x (not t, or V, ...)
2) Work (1→2) is independent of path

(Taylor p.124 shows that in 1-D, these 2 are coupled, either one ⇒ the other!)

So if F is conservative in 1-D,

$$U(x) = -\int_{x_0}^{x} F(x')\,dx'$$ is well-defined

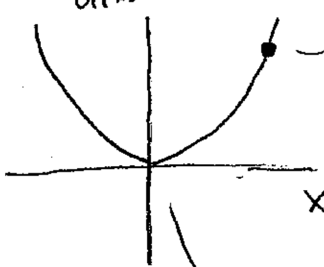
$$F(x) = -\frac{dU(x)}{dx}$$ (like $\vec{F} = -\vec{\nabla}U$, but in 1-D!)

Example: If $F = -kx$ (spring)

then $U = +\frac{1}{2}kx^2$ (setting $x_0 = 0$ as my choice)

<u>$U(x)$ carries all the information that $F(x)$ does!</u>

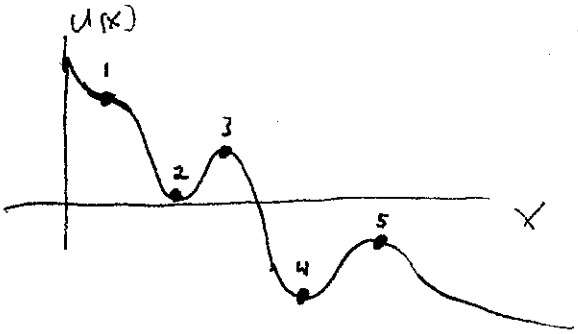
Ex:



→ F here is given by $-\frac{dU}{dx}$. In this example graph, $-\frac{dU}{dx}$ is negative, leftwards. Makes sense, particles "fall" towards lower PE!

→ At a minimum of $U(x)$, we're in equilibrium, $F=0$

Ex:



think of a roller coaster. $U(x) = mg\,y(x)$

(or, a molecule, show energy as a fn of separation of atoms)

$F = -\frac{dU}{dx} = 0$ at all the numbered dots

At 2 & 4 it's <u>stable</u>: Look at $U''(x)$ there, to decide! ($U''(x) > 0$)

at 3 & 5 it's unstable: $U''(x) \le 0$.

at 1 it's an inflection point - must investigate, but looks <u>unstable</u> to "runaway to the right" in this case!

Consider a Taylor series for $U(x)$ around any x_0 "equilibrium" pt:

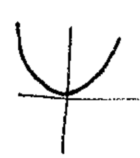
$$U(x) = \underbrace{U(x_0)} + U'(x_0)(x-x_0) + U''(x_0)\frac{(x-x_0)^2}{2!} + \dots$$

a constant added to $U(x)$ has no physically significance

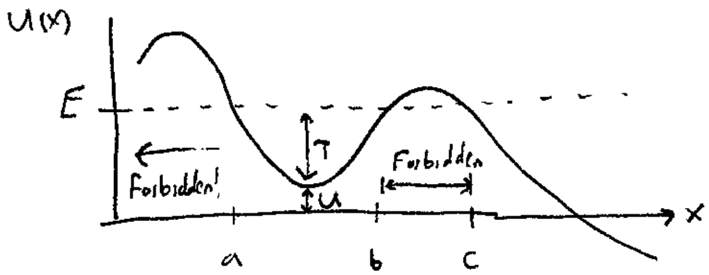
If x_0 is equilibrium, $U'(x_0) = 0$.

so $U(x) \approx \frac{U''(x_0)}{2!}(x-x_0)^2$.

For stability, need $U'' > 0$,



Consider a roller coaster with total Energy E.



If system is conservative

$E = T + U$ is always the same

If it's moving, $T \geq 0$. So, can't be anywhere that $U > E$.

If $E = U$, then $T = 0$, it's stopped. "Turning points"

In Fig above, we could be trapped, oscillating $a \leftrightarrow b$

or could be escaping with $x \geq c$ at all times.

Knowing about Energy can help us skirt solving N-II ODE's!

Ex: $T = E - U(x) \Rightarrow \frac{1}{2} m \dot{x}^2 = E - U(x)$

so speed $\dot{x} = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$ (gives $V(x)$, sometime useful... like, roller-coaster designers!)

Bummer. Energy doesn't tell us sign of V. Might need more physics to solve for $x(t)$.

In 3-D, alas, the problem is worse, since direction of $\vec{V}$ is not determined by KE. So, this trick helps mostly if you want $|V|(x)$.

If you do know the sign of v, this ODE gives $x(t)$, an alternative to N-II (that doesn't require forces, force diagrams, etc)

Need a trick: $\dot{x} = \frac{dx}{dt}$, so $\frac{dx}{\sqrt{E-U(x)}} = \pm\sqrt{\frac{2}{m}}\,dt$ separates!

integrating: $t(x) = \pm\sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E-U(x')}}$ + for "rightward travel", − " "leftward " "

Inverting gives $x(t)$.

Familiar Example: Free fall! Pick $+y\downarrow$ $\bullet\ y=0$

we know $U_{grav} = -mgy$ (minus because I called down "+")

If $v(0) = y(0) = 0$, $E_0 = 0$ is conserved

Since it falls down after release, use "+" sol'n.

$$t = \sqrt{\frac{m}{2}} \int_0^y \frac{dy'}{\sqrt{+mgy'}} = \frac{1}{\sqrt{2g}}\, 2y^{1/2}\Big|_0^y = \sqrt{2y/g}$$

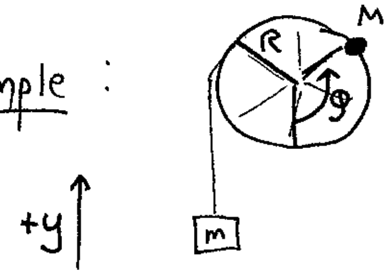
Inverting, $y = \frac{1}{2}gt^2$, ahh!

Stability.

Recall: Equilibrium if $\partial U/\partial x = 0$

stability if $U'' = 0$.

Example:



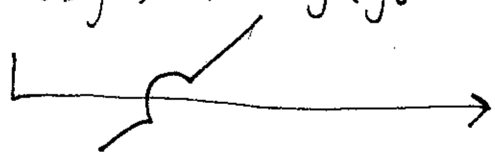
- Massless, frictionless pulley
- M glued to pulley rim
- m hangs from string (at $y = y_0$ when $\varphi = 0$)
- φ = angle of wheel ($\varphi = 0$ when M is "straight down")

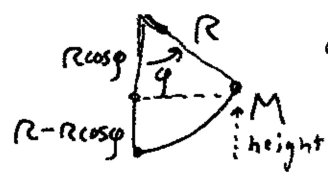
This is a 1-D system. φ alone determines all!

Are there equilibrium values for φ, where the system is stable?

$$U(\varphi) = U_{\text{of } M}(\varphi) + U_{\text{of } m}(\varphi) = +Mg\,(\text{height of } M) + mg\,(\text{height of } m)$$

$$= MgR(1-\cos\varphi) + mg(y_0 - R\varphi)$$





convince yourself

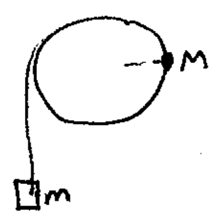
As $\varphi \uparrow$, m goes down, and $R\varphi$ is the amount of string let out!

Equilib if $U'(\varphi) = 0 \Rightarrow MgR\sin\varphi - mgR = 0$

or $\sin\varphi = m/M$.

If $m > M$ No equilibria. It just falls forever!

If $m = M$, one sol'n, $\varphi = \pi/2$.



If $m < M$, two sol'ns, one with $0 < \varphi < \pi/2$

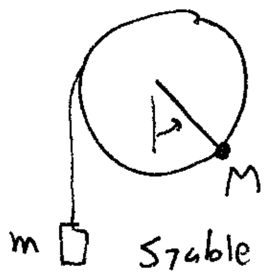
another $\pi/2 < \varphi < \pi$.

What about stability?

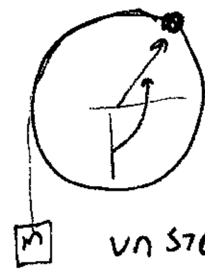
$U''(\varphi) = MgR\cos\varphi$.

If $m < M$, the $0 < \varphi < \pi/2$ sol'n has $\cos\varphi > 0 \Rightarrow$ stable

$\pi/2 < \varphi < \pi$ " " $\cos\varphi < 0 \Rightarrow$ unstable



m M Stable



m unstable!

If $m = M$, $U'' = 0$. Must look at next term

$U''' = -MgR\sin\varphi \big|_{\varphi=\pi/2}$ is negative

$$U(\varphi) = \underbrace{U(\pi/2)}_{\substack{\text{a constant,}\\ \text{unimportant}}} + \underbrace{U'}_{0} + \underbrace{U''}_{0} + \underbrace{U'''}_{\text{neg}} \frac{(\varphi - \pi/2)^3}{3!}$$

If $\varphi \to \pi/2 - \varepsilon$, U goes ~~down~~ up, that's stable, we return

If $\varphi \to \pi/2 + \varepsilon$, U goes down, that's a runaway situation.

So, this is not stable.

(By the way, no sol'ns for $\varphi > \pi$ makes physical sense if you think about torques, draw the picture!)

This concludes what we'll cover in ch. 4

4.8, central forces is a good review of polar coords for you!

4.9 + 4.10 is about energy of systems.

Read it if you're interested.