Energy is a rather subtle concept! It's a scalar quantity associated with all physical systems which is conserved (in total) for any/all physical processes.

I typically (if a bit naively!) think of it casually as a quantity that tells you about how much work a system can do.

In physics 2210 we focus on mechanical energy:

1) Kinetic Energy (associated with motion in a reference frame)
2) Potential Energy ("" particularly conservative forces")

The history of "energy" begins (mostly) well after Newton, Thompson ("Lord Kelvin") established the 1st law of thermodynamics, essentially "conservation of energy".

KE is called $T = \frac{1}{2}mv^2$ for a point mass.

The relation to work is key, that's our 1st order of business.
If a force \( \vec{F} \) acts on a moving point-like object, for small movements \( d\vec{r} \), the work done is \( dW = \vec{F} \cdot d\vec{r} \),

\[ dw = F \, dr \cos \theta. \]

- **Work is a signed scalar.** If \( \theta > 90^\circ \), \( dW \) is negative!

- \( dW = 0 \) if \( \vec{F} \perp d\vec{r} \). (So e.g., the ice rink does no work on the frictionless sliding puck, the earth does no work on a satellite in circular orbit!)

Other forces may be acting too. This formula gives \( dW \) by this \( \vec{F} \).

For larger movements, just **ADD UP** the little \( dW \)'s:

\[
W \text{ by } \vec{F} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} \\
\text{along path followed}
\]

This integral is a **sum**. (\( \int \) is not the "area under this curve"!!)

For point-like masses:

\[ \vec{F}_{\text{net}} \cdot d\vec{r} = (m \, \frac{d\vec{v}}{dt}) \cdot d\vec{r} = (m \, \frac{d\vec{v}}{dt}) \cdot \left( \frac{d\vec{r}}{dt} \right) = m \, \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt} = m \, \frac{d\vec{v}}{dt} \cdot \vec{v} \cdot dt \]

Trick: Convince yourself:

\[
= \frac{m}{2} \frac{d}{dt} \left( \vec{v} \cdot \vec{v} \right) dt = \frac{d}{dt} \left( \frac{1}{2} m \vec{v}^2 \right) dt
\]
You might prefer "running that proof backwards."

\[ \frac{d}{dt} \left( \frac{1}{2} m \dot{v}^2 \right) = m \dot{\vec{v}} \cdot \frac{d}{dt} \vec{v} = \vec{F}_{\text{net}} \cdot \vec{v} \] (cancelling \( dt \) everywhere in previous page's proof)

Bottom line, with \( T \equiv \frac{1}{2} m \dot{v}^2 \equiv \text{Kinetic energy}, \) for point masses.

\[ dT = \vec{F}_{\text{net}} \cdot d\vec{r} = dW_{\text{net}} \] where \( dW_{\text{net}} \) means "work done by the net force."

or, for finite movements

\[ \Delta T = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{\text{net}} \cdot d\vec{r} = W_{\text{net}}, \text{ going from } \vec{r}_1 \text{ to } \vec{r}_2 \]

This is the **Work- Energy theorem**. For point masses,

**Net work done = change in KE.**

- If \( W_{\text{net}} \) is negative, object slows down!
- Theorem is only true for point masses +
- Theorem is only true for net work of all forces.

\[
\begin{bmatrix}
\text{Consider e.g. a system of 2 masses} & \vec{F}_{\text{net},1} \rightarrow \text{m}_1 & \text{m}_1 \rightarrow \vec{F}_{\text{net},2} \\
\text{Here } \vec{F}_{\text{net,ext}} = 0, \text{ but } \Delta T > 0 \text{ as both masses speed up.} \\
\text{Also, } \leq W_{\text{by } \vec{F}_{i}} \neq W_{\text{by } \vec{F}_{\text{net}}} = 0, \text{ here!} \\
\text{Each is positive!}
\end{bmatrix}
\]
Example: Lift a mass at constant speed, so $\Delta T = 0$.

$W_{by\ you} = \vec{F}_{\text{you}} \cdot \Delta \vec{r} = (+mg) \hat{y} \cdot (h \hat{y}) = +mgh$

$W_{by\ gravity} = \vec{F}_{\text{grav}} \cdot \Delta \vec{r} = (-mg) \hat{y} \cdot (h \hat{y}) = -mgh$

$F_{net} = 0, \; W_{net} = 0, \; \Delta T = 0 \quad \text{Good!}$

Example: Slide a box across frictionfull floor at constant speed.

$F_{\text{you}} \rightarrow \square \leftarrow F_{\text{friction}} = -F_{\text{you}}$

$W_{by\ you} = \vec{F} \cdot \Delta \vec{r} = (+F_{\text{you}} \hat{x}) (dx \hat{x}) = +F dx$

$W_{by\ friction} = (-F_{\text{you}} \hat{x}) (dx \hat{x}) = -F dx$

$F_{net} = 0, \; W_{net} = 0, \; \Delta T = 0 \quad \text{Good!}$

Example: Apple falls h, from rest.

$W_{by\ gravity} = (-mg) \hat{y} \cdot (-h \hat{y}) = +mgh$

$\Delta T = \frac{1}{2}m(v_f^2 - 0) = \frac{1}{2}m (2g \Delta y) = mgh$

Example: Sliding box slows due to friction, from $V_0 \rightarrow 0$.

$W_{by\ friction} = (-MK \cdot N) \hat{x} \cdot (\Delta x \hat{x}) = -MK mg \Delta x$

So here $\Delta T = W_{net} < 0$. Makes sense, slows down, (this formula relates $\Delta x$ to $V_0^2$ and $MK$.)

Note: No exceptions! Friction, gravity, human force, combos: $\Delta T = W_{net}$, always!
Computing \( W = \int_C \vec{F} \cdot d\vec{r} \) is a line integral (or "path") along a given path in general, to evaluate!

Many tricks! (See Boas 6.8) Best is often to "parametrize" path.

See Taylor's Ex 4.1!

My Example

1. Suppose \( \vec{F} = y \hat{z} \)
   - Can you picture this?

2. Particle moves from \((1,0,0)\) to \((0,1,-1)\)

The Procedure (it's pretty universal!)

1. Pick your coordinate system! Here, Cartesian. \( \vec{F} = x \hat{x} + y \hat{y} \)

2. In your coord system, generic \( d\vec{r} \) is always \( d\vec{r} = dx \hat{x} + dy \hat{y} \)

3. Compute \( \vec{F} \cdot d\vec{r} = (y \hat{z}) \cdot (dx \hat{x} + dy \hat{y}) = y \, dx + 0 \, dy \)

4. Write out \( \int_C \vec{F} \cdot d\vec{r} = \int_y y \, dx + \int_0^1 0 \, dy \)

5. Parametrize this path! Here, look at picture, \( y = x - 1 \)
   - Intercepts?
   - Then substitute in: \( \int_0^1 y \, dx = \int_0^1 (x-1) \, dx \)

6. Integrate: \( \int_0^1 (x-1) \, dx = \frac{x^2}{2} \bigg|_0^1 - (x \bigg|_0^1) = \frac{1}{2} + 1 = \frac{3}{2} \)
Parametrization step is not unique!

E.g., \( x = 1 - u \) \( y = -u \) \( u = 0 \) to \( u = 1 \) also generates that line!
\( \int_0^1 (-u)(-du) = \int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2} \).

So then, \( \int_{x=1}^{u=0} y dx = \int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2} \).

Another Example

- Suppose \( \vec{F} = \hat{i} \)

Can you picture this?

- Suppose particle moves along sideways parabola

\[(0,1) \quad x = 1 - y^2 \quad \text{Work done by } \vec{F} ?
\[(1,0)

Pick coord system. Cartesian is fine, \( \vec{r} = x \hat{i} + y \hat{j} \)

1) So (always, in Cartesian) \( d\vec{r} = dx \hat{i} + dy \hat{j} \)

2) \( \vec{F} \cdot d\vec{r} = (x \hat{i} + y \hat{j}) \cdot (dx \hat{i} + dy \hat{j}) = x dx + y dy \)

3) \[ \int \vec{F} \cdot d\vec{r} = \int_{x=1}^{x=0} x dx + \int_{y=0}^{y=1} y dy \]

- end

4) On path, \( x = 1 - y^2 \), so \( dx = -2y dy \).

\[ W = \int_0^1 (1-y^2)(-2y dy) + \int_0^1 y dy = -y^2 + \frac{2y^2}{2} + \frac{y^2}{2} \Big|_0^1 = 0 \]
Third Example
Suppose $\mathbf{F} = c\hat{\varphi}$ (in polar coords)

Can you picture

and you move along a quarter circle

from $(1,0)$ to $(0,1)$. Work done?

0) Pick coord system: plane polar! $\mathbf{F} = r \hat{r}$

1) So (Always, in polar coords!) $d\mathbf{r} = dr \hat{r} + r d\varphi \hat{\varphi}$

Look back at our early derivation of $\mathbf{V} = \frac{d\mathbf{r}}{dt}$, just cancel out the $dt$!

2) $\mathbf{F} \cdot d\mathbf{r} = c \hat{\varphi} \cdot d\mathbf{r} = cr d\varphi$

3) $\int_{\varphi = 0}^{\frac{\pi}{2}} r \cdot c \cdot d\varphi = c \cdot \frac{\pi}{2}$ along our path

4) Parametrize our path. Here, $r = 1$, so it's simple, nothing to do

4) Parameterize our path. Here, $r = 1$, so it's simple, nothing to do

$\int_{\varphi = 0}^{\frac{\pi}{2}} (1)(c) d\varphi = c \cdot \frac{\pi}{2}$ (makes sense, $\mathbf{F} \cdot d\mathbf{r}$ is the same) all along the path

Suppose our path had been up a straight line.

Given the polar form of $\mathbf{F}$, I would stick with polar coords, all the way to step 3)

$\int_{\varphi = 0}^{\frac{\pi}{2}} r \cdot c \cdot d\varphi$ look @ the path, that is the final $\varphi$ value

4) Along this path, $x = r \cos\varphi = 1$, so $r = 1 / \cos\varphi$ "parametrizes" the path!

so we need $\int_{\varphi = 0}^{\pi/4} c \cos\varphi d\varphi = .88 c$, from Mathematica
Calculating line integrals is a bit of an art, but it's just a sum. The main physics story here is, for point particles:

**Work-Energy Theorem:**

\[ \Delta T = W_{net}(1 \rightarrow 2), \text{ or } T_2 - T_1 = \int_1^2 \vec{F}_{net} \cdot d\vec{r} \]

along your particle's path

Where \( \vec{F}_{net} = \sum_i \vec{F}_i \), and \( W_{net} = \sum_i W_{bi} \)

In many physics cases (Gravity, electrostatics, springs, ...)

1) \( \vec{F} = \vec{F}(\vec{r}) \), not explicitly \( \vec{F}(\theta) \) or \( \vec{F}(\vec{v}) \) or \( \vec{F} \) (anything else)

and

2) \( W(1 \rightarrow 2) \) is independent of the path taken.

when both are true, we say \( \vec{F} \) is conservative, + lovely consequences follow!

Note: friction (\( \mu mg \)) violates condition 2, \( W \) depends on path. Drag (\( bv \) or \( cv^2 \)) \( \mu \) 1 and 2! \( Neither \) is conservative.

Normal force might satisfy condition 2 since \( W(1 \rightarrow 2) = 0 \) (normal force is \( \vec{F} \) to \( d\vec{r} \) along the path), but it is not just a fn of \( \vec{r} \), e.g., \( \vec{N} \) increases if you go faster around a curve, it depends on \( v \! \). So, Normal force is not conservative.
If all forces on an object are conservative, we can define PE or \( U(\vec{r}) \), the "potential energy", as a function of position. We construct it so that \( E_{\text{mech}} = T + U \) is conserved, \( \Delta E = 0 \).

I claim that \( U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}' \) does the trick!

- That - sign is key, see below.
- \( \vec{r}_0 \) is arbitrary, it's your choice. A different \( \vec{r}_0 \) gives you a slightly different function \( U(\vec{r}) \) (it differs by addition of a constant).

So PE is arbitrary up to one undetermined constant. But, the functional form \( U(\vec{r}) \) is what is determined by \( \vec{F} \).

Since we're posulating \( W(\vec{r}_0 \to \vec{r}) \) is path independent, this \( U(\vec{r}) \) is perfectly unique & well-defined (given \( \vec{r}_0 \)).

**Example:** \( \vec{F}_{\text{grav}} = -mg \hat{y} \), so \( U_{\text{grav}}(y) = -\int_0^y (-mg \hat{y}) \cdot d\hat{y} = mg y \)

The - sign was key, makes no sense for PE to become smaller as \( y \) increases; it must get bigger to conserve energy.
Let's show that this definition of \( U(r) \) leads to Energy conservation.

Consider this sequence of parts:

\[
\begin{aligned}
\vec{r}_0 & \rightarrow \vec{r}_1 & \rightarrow & \vec{r}_2
\end{aligned}
\]

\[
\text{Work (0 → 2) = Work (0 → 1) + Work (1 → 2)}
\]

For conservative forces, all of these terms are path independent, so this is generically true. Subtracting to solve for Work (1 → 2),

\[
\begin{aligned}
W(\vec{r}_1 \rightarrow \vec{r}_2) &= W(\vec{r}_0 \rightarrow \vec{r}_2) - W(\vec{r}_0 \rightarrow \vec{r}_1) \\
&= -U(\vec{r}_2) + U(\vec{r}_1) \quad \text{By my def of } U! \\
&= -\Delta U(1 \rightarrow 2)
\end{aligned}
\]

That's what \( \Delta \) means.

If \( E = T + U \), then

\[
\begin{aligned}
\Delta E(1 \rightarrow 2) &= \Delta T(1 \rightarrow 2) + \Delta U(1 \rightarrow 2) \\
&= +W(1 \rightarrow 2) + (-W(1 \rightarrow 2)) \\
&\quad \text{By work-energy theorem!}
\end{aligned}
\]

From lines above,

\[
\Delta E(1 \rightarrow 2) = 0
\]

Ahh (with our definition \( U(r) \equiv -\oint W(\vec{r}_0 \rightarrow \vec{r}) \))

we proved \( \Delta E(1 \rightarrow 2) = 0 \). This \((-)\) sign was critical in our proof!

So **Conservative forces means Energy is conserved**.
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If there is only one force (think "projectile"), e.g.:

\[ E = T + U_F(r) \] is conserved, with \( U_F = \text{"P.E. of force F"} \)

If you have many conservative forces (think e.g. "projectile with E-field")

\[ E = T + U_1(r) + U_2(r) + \ldots \] is conserved: \( \Delta E (1 \rightarrow 2) = 0 \)

for any points on trajectory

Mechanical Energy.

Remember, this is all for point particles.

Example: \( q \) in uniform \( \vec{E} \) field: \( \vec{E} = E_0 \hat{x} \), \( \vec{F} = q \vec{E} \)

\[ U(r^2) = - \int_{r_0}^{r} (q \vec{E}) \cdot d\vec{r} \]

\[ = - q \int_{r_0}^{r} E_0 dx = - q E_0 (x - x_0). \]

\[ \Rightarrow U(r^2) = - q E_0 x \]

\( \text{Bigger } x \Rightarrow \text{lower } U. \text{ Makes sense!} \)

\[ \text{high } U \quad \text{for} \quad q \quad \text{for} \quad q \]

\[ \Rightarrow x \]

\( \text{high } U \text{ for mass} \)

\( \text{low } U \text{ for mass} \)

Reminds me of

\( \text{gravity} \)
What about non-conservative forces?

Suppose $\vec{F}_{\text{net}} = \vec{F}_{\text{cons}} + \vec{F}_{\text{Non-cons}}$

So $\Delta T(1 \rightarrow 2) = W_{\text{net}}(1 \rightarrow 2) = W_{\text{c}}(1 \rightarrow 2) + W_{\text{nc}}(1 \rightarrow 2)$

(work-energy theorem still holds!)

and so

$\Delta E(1 \rightarrow 2) = \Delta (T+U) = \Delta T - W_{\text{cons}}(1 \rightarrow 2)$

(see ref. of $U$, see p. 66)

$\uparrow$

Definition of $E = T + U$.

$= W_{\text{nc}}(1 \rightarrow 2)$ by using the line above.

So $E_{\text{mech}}$ is not conserved! And, $W_{\text{nc}}(1 \rightarrow 2)$ may depend on path.

The change of mech. energy goes to/comes from some other form (e.g., often, thermal energy). So total energy is still (always!) conserved, but mechanical " is not.

See Taylor Ex. 4.3, block sliding on incline:

$\Delta T + \Delta U = W_{\text{nc}}(1 \rightarrow 2) = 0 + W_{\text{friction}}$

from $F_{\text{normal}}$ which is $-\mu mg \cos \theta \cdot \text{distance}$

E.g. start at rest + slide down hill,

$\frac{1}{2}mv^2 - 0 + (0 - mgh) = -\mu mg \cos \theta \cdot d$. Let's you find $V_f$.

$\left[ \text{If } \theta \rightarrow 0, \frac{1}{2}mv^2 = mg(0) - \mu mgd. \text{ Sign is nonsense... but of course, } \theta c \right]

\text{ block would never slide if } \theta = 0!!$
Relating $U$ to $\mathbf{F}$ (and vice versa!)

We defined $U(r) = \int_{r_0}^{r} \mathbf{F}(r') \cdot dr'$. So if you know $\mathbf{F}$, you can find $U(r)$.

But, can you invert this? If you know $U(r)$, can you deduce $\mathbf{F}$?

$$\Delta U (1 \rightarrow 2) = - \int_{r_1}^{r_2} \mathbf{F}(r') \cdot dr'$$

If your displacement is tiny, i.e., you go from $r$ to $r + dr$,

$$dU = - \mathbf{F}(r) \cdot dr = -(F_x \, dx + F_y \, dy + F_z \, dz)$$

do you see this?

But $dU = \frac{\partial U}{\partial x} \, dx + \frac{\partial U}{\partial y} \, dy + \frac{\partial U}{\partial z} \, dz$. (Chain rule).

Since $dx, dy, dz$ can be anything you choose (e.g., $dy = dz = 0$ is a choice)

I claim $\frac{\partial U}{\partial x} = -F_x$ and \quad $\frac{\partial U}{\partial y} = -F_y$ and \quad $\frac{\partial U}{\partial z} = -F_z$.

So

$$\mathbf{F} = -\left( \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k} \right)$$

(hint: conservative force is derivable from its potential energy)

We write this in “shorthand” as $\mathbf{F} = -\nabla U$ from its potential energy.

So

$$U(r) = -\int_{r_0}^{r} \mathbf{F}(r') \cdot dr' \iff \mathbf{F}(r) = -\nabla U(r)$$

$\nabla$ and line integrals are “inverses”.
Let's talk about the **Gradient** \( \nabla \):
\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\]

- \( \nabla \) is an operator. (A "vector differential operator".)

It acts on **fns** and returns a **vector function**.

To me, its basic meaning comes from:
\[
\nabla f = \nabla f \cdot \hat{r}
\]

True for any scalar function \( f(\vec{r}) \). We derived this on the previous page!

\[
\begin{align*}
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} &= (\nabla f)_x \, dx + (\nabla f)_y \, dy + \ldots \\
&= \frac{df}{dx} \, dx + \frac{df}{dy} \, dy + \ldots
\end{align*}
\]

\( \Theta \) \( df = \nabla f \cdot d\vec{r} \) tells me how to interpret \( \nabla f \).

It says how for \( f \) changes if you move "d\vec{r}" away!

\( \nabla f \) is a vector and points in the **direction** of **max** rate of change of \( f(\vec{r}) \).

In that direction, \( |\nabla f| = \frac{df}{dr} \) \( \Rightarrow \) it's the "directional derivative".

If \( \nabla f = 0 \), \( f \) is not changing in any direction, we're at a **local max** or **min**. (Or **inflection /Saddle**)

\( \text{In spherical coords, } \nabla f \neq \hat{r} \frac{df}{d\hat{r}} + \hat{\theta} \frac{df}{d\hat{\theta}} + \hat{\phi} \frac{df}{d\hat{\phi}} \) (Units ???)

Instead, use \( \nabla f \cdot d\hat{r} = df \) to figure out \( \nabla f \).
Back to $\vec{F}(\vec{r}) = -\nabla U(\vec{r})$

* The sign says Force points opposite the direction of increasing $U$. This makes sense, $\vec{F}$ points "downhill", not "uphill".

$\vec{F}$ is \( \perp \) to "equipotential lines".

Because $dU = \nabla U \cdot d\vec{r}$, Equipotential lines follow locations of constant $U$, meaning $dU = 0$, or $\nabla U$ is \( \perp \) to $d\vec{r}$. $d\vec{r}$ follows an equipotential.

So $\vec{F} \perp d\vec{r}$, if $d\vec{r}$ is following an equipotential line.

$|\vec{F}| = |\nabla U|$, so when you draw equipotential lines, if they are closely spaced, meaning $\frac{dU}{dr}$ is big there, then $|\vec{F}|$ is big.

$\frac{\Delta U}{\Delta r}$ is big here, it's a cliff! Large Force.
\[ \vec{\nabla} \text{ is a funny beast. It's sort of a vector (x component is } \frac{\partial}{\partial x}, \text{ or) and sort of a "derivative" (it needs to act on a function!)} \]

You can write \[ \vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \] like you write \[ \vec{r} = (x, y, z) \]

You can treat \( \vec{\nabla} \) like a scalar vector in many ways, e.g.

\[ \text{Curl} \quad \vec{\nabla} \times \vec{F} = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{array} \right| = \hat{\imath} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) - \hat{\jmath} \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) + \hat{k} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]

I claim (+ it's easy to prove, just write this out!) \[ \vec{\nabla} \times \vec{\nabla} f = 0 \] for any function \( f(\vec{r}) \).

So \[ \vec{\nabla} \times \vec{\nabla} U = 0 \] is the Potential \( U(\vec{r}) \)

So \[ \vec{\nabla} \times \vec{F} = 0 \] for any conservative force!

The "curl of a conservative force vanishes"

"Conservative forces have no curl"

So, what's curl? What's it mean? I will invoke (w/o proof)

Stoke's theorem: \[ \oint \vec{F} \cdot d\vec{r} = \iint (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \]

around a closed loop over any surface bounded by the loop
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You'll study this much more in future classes! For now, what you should take away is:

If \( \nabla \times \mathbf{F} = 0 \) then by Stokes' theorem, \( \oint \mathbf{F} \cdot d\mathbf{r} = 0 \) for any loop at all.

So, conservative force

means \( \nabla \times \mathbf{F} = 0 \)

means \( \oint \mathbf{F} \cdot d\mathbf{r} = 0 \) for any loop

All equivalent! If one is true, so are the others!

There's more! Consider integrating from 1 to 2 along path 1 + path 2:

\[
\oint \mathbf{F} \cdot d\mathbf{r} = \int_{\text{Path 1}} \mathbf{F} \cdot d\mathbf{r} + \int_{\text{Path 2}} \mathbf{F} \cdot d\mathbf{r} = 0
\]

Together, path 1 + path 2 is a closed loop.

Now I claim \( \int_{\text{Path 2}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\text{Path 1}} \mathbf{F} \cdot d\mathbf{r} \), because reversing a path reverses the sign of \( \mathbf{F} \cdot d\mathbf{r} \) everywhere.

So \( \int_{\text{Path 1}} \mathbf{F} \cdot d\mathbf{r} - \int_{\text{Path 2}} \mathbf{F} \cdot d\mathbf{r} = 0 \).

True for any loop, thus any path. So we recover something we knew:

\( \int_{1}^{2} \mathbf{F} \cdot d\mathbf{r} \) is path independent for conservative forces.
All of the following are completely equivalent (any one implies all)

- \( \vec{F} \) is a conservative force
- \( \nabla \times \vec{F} = 0 \)
- \( \oint \vec{F} \cdot d\vec{r} = 0 \) any loop
- \( \int F \cdot d\vec{r} \) is independent of path
- \( \vec{F} = -\nabla U(r) \) for a well-defined potential function

All different ways to think about meaning + consequences of conservative forces.
(The field does no work if you end up back where you start)

In particular, this helps me see what "curl free" means, a bit better

There is never a "circulation of \( \vec{F} \)". \( \oint \vec{F} \cdot d\vec{r} \) is always 0 for any loop, small or large!

\[
(\nabla \times \vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}
\]
carries information about "rotation" of the field around the \( z \) axis.

This field has curl! So does this:

If a tiny paddlewheel drops into the field, it is made to rotate.

There's a curl, and the field is not conservative.
Example: \[ \mathbf{F}_{\text{coul}} = \frac{kQq}{r^2} \mathbf{r} = \frac{kQq}{r^2} \mathbf{r} = \frac{kQq}{r^3} \mathbf{r} \]

You can compute \( \mathbf{\nabla} \times \mathbf{F} \). Taylor does it: \[ \frac{1}{r^3} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \]
\[ \mathbf{r} = (x, y, z) \]

It's a little painful, but do the determinant formally, and get \( \mathbf{\nabla} \times \mathbf{F} = 0 \)

\( \implies \) use Taylor's back flyleaf in spherical coordinates.

This \( \mathbf{F} \) has no \( \hat{\theta} \) or \( \hat{\phi} \) component, and \( \mathbf{F}_r = \frac{kQq}{r^2} \)

Take a look, every entry gives you 0. So \( \mathbf{\nabla} \times \mathbf{F} = 0 \)

Thus: Coulomb force is conservative

- Work done by \( \mathbf{E} \)-field is independent of path taken
- No work done if move \( q \) around any loop and return to start
- There is a well-defined potential energy, we can find it by

\[ U(r) = - \int_{r_0}^{r} \mathbf{F} \cdot d\mathbf{r} \]

\[ U(r) = \frac{kQq}{r} \] works.

(Back flyleaf of Taylor shows \( \mathbf{\nabla} U \)

in spherical coords, so you can quickly check that it gives \( \mathbf{F} \) as top of page)

- Coulomb field is "curl free"

No "rotation" in this field.
Let's compute $U(\mathbf{r}) = -\int_0^\infty \vec{F}(\mathbf{r}') \cdot d\mathbf{r}'$ with $\vec{F} = \frac{kQq}{r^2} \hat{r}$

I'll pick $r_0 \to \infty$, this is my arbitrary choice.

1) In spherical coordinates, always, $d\mathbf{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$

2) $-\int_0^\infty \vec{F}(\mathbf{r}') \cdot d\mathbf{r}' = -\int_0^\infty \frac{kQq}{r^2} dr' = \left. \frac{kQq}{r} \right|_0^\infty = \frac{kQq}{r}$

As claimed, $PE = \frac{kQq}{r}$, like in Phys 1120!

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More curl intuition! Consider a non-zero curl function, like

The "pinwheel" turns, anywhere, this has non-zero curl!

As you integrate a→b, you get a small positive result, do you see why?

From b→c, $\vec{F} \cdot d\mathbf{r} = 0$, no contribution

From c→d, $\vec{F} \cdot d\mathbf{r}$ is negative (F is opposite dr) and bigger.

From d→a, $\vec{F} \cdot d\mathbf{r} = 0$

so $\int \vec{F} \cdot d\mathbf{r} \neq 0$. Mathematically

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & y \end{vmatrix} = -\hat{z}$$

(Can you see this matches the picture?) 

\text{Not zero, Negative,} 

\text{"Circulaires" around z axis!}
One last curl example. In E+M, you will learn

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

If \( \vec{B} \) is "time independent", this is a "static" problem, \( \nabla \times \vec{E} = 0 \), and so \( \vec{E} \) is conservative

Potential energy is defined

\( \vec{E} \) does no work around closed loops

all the 1120 stuff!

But if \( \vec{B} \) depends on time, (Faraday's Law!)

\( \vec{E} \) becomes non-conservative!

P.E. is not well defined

\( \vec{E} \) does do work as you go around closed loops.

(This is what power generators do!)
Energy for 1-D systems:

"1-D" here means one variable defines our location.

Could be literally 1-D (e.g. train on flat track)
or a pendulum (θ tells all!)
or a roller coaster (distance from start tells all!), etc.

If F acts on a 1-D particle (no vector needed in 1-D), call it

If \( F \) acts on a 1-D particle (no vector needed in 1-D), call it

\[
W(1 \to 2) = \int_{x_1}^{x_2} F(x') \, dx'
\]

If \( F \) is conservative, recall

1) \( F \) depends only on \( x \) (not \( t \) or \( v \))
2) Work (1→2) is independent of path

(Taylor p.124 shows that in 1-D, these 2 are coupled, either one \( \Rightarrow \) the other!)

So if \( F \) is conservative in 1-D,

\[
U(x) = -\int_{x_0}^{x} F(x') \, dx' \quad \text{is well-defined}
\]

\[
F(x) = -\frac{dU(x)}{dx}
\]

(like \( \vec{F} = -\nabla U \), but in 1-D!)

Example: If \( F = -kx \) (spring)

then \( U = \frac{1}{2} kx^2 \) (setting \( x_0 = 0 \) as my choice)
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U(x) carries all the information that F(x) does!

Ex: \( U(x) \)

- \( F \) here is given by \(- \frac{du}{dx} \). In this example graph, \(- \frac{du}{dx} \) is negative, forwards. Makes sense, particles "fall" towards lower PE!

- At a minimum of \( U(x) \), we're in equilibrium, \( F = 0 \)

Ex:

- Think of a roller coaster. \( U(x) = mg y(x) \)
  - (or, a molecule, show energy as a fn of separation of atoms)

- \( F = - \frac{dy}{dx} = 0 \) at all the numbered dots

At 2 & 4 it's stable: Look at \( U''(x) \) there to decide! (\( U''(x) > 0 \))

At 3 & 5 it's unstable: \( U''(x) \leq 0 \).

At 1 it's an inflection point - must investigate, but looks unstable to "runaway to the right" in this case!

Consider a Taylor series for \( U(x) \) around any \( x_0 \) of "equilibrium" pt:

\[
U(x) = U(x_0) + U'(x_0)(x-x_0) + U''(x_0)(x-x_0)^2 + \ldots
\]

- a constant added to \( U(x) \) has no physically significance

- If \( x_0 \) is equilibrium, \( U'(x_0) = 0 \)

So \( U(x) \approx U''(x_0)(x-x_0)^2 \)

For stability, need \( U'' > 0 \).
Consider a roller coaster with total energy $E$.

If system is conservative, $E = T + U$ is always the same.

If $E > 0$, So, can't be anywhere that $U > E$.

If $E = U$, then $T = 0$, it's stopped. "Turning points."

In Fig above, we could be trapped, oscillating a $\leftrightarrow b$

or could be escaping with $x \geq c$ at all times.

Knowing about energy can help us skirt solving N-D ODE's!

\[ T = E - U(x) \Rightarrow \frac{1}{2}m \dot{x}^2 = E - U(x) \]

so speed $\dot{x} = \pm \sqrt{\frac{2}{m} \sqrt{E - U(x)}}$ gives $V(x)$, sometime useful... like roller-coaster design!)

However, energy doesn't tell us sign of $V$, might need more physics to solve for $x(t)$.

In 3-D, alas, the problem is worse, since direction of $\nabla V$ is not determined by KE. So, this trick helps mostly if you want $|V(x)|$. 
If you do know the sign of $V$, this ODE gives $x(t)$, an alternative to N-II (that doesn't require forces, force diagrams, etc).

Need a trick: $\dot{x} = \frac{dx}{dt}$, so $\frac{dx}{\sqrt{E - U(x)}} = \pm \frac{a}{m} dt$ separates!

Integrating: $t(x) = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}} + \text{for "rightward travel"}$

Inverting gives $x(t)$.

Familiar Example: Free fall! Pick $y = 0$

\[ t = \sqrt{\frac{m}{2}} \int_{y_0}^{y} \frac{dy'}{\sqrt{E_{grav} + mg y'}} = \sqrt{\frac{2}{g}} \frac{y}{\sqrt{y}} \bigg|_{y_0}^{y} = \sqrt{\frac{2y}{g}} \]

Inverting, $y = \frac{1}{2} g t^2$, ahh!
Stability. Recall: **Equilibrium** if \( \frac{dU}{dx} = 0 \)

**Stability** if \( U'' = 0 \).

Example:

- Massless, frictionless pulley
- \( M \) glued to pulley rim
- \( m \) hangs from string (at \( y = y_0 \) when \( y = 0 \))
- \( \theta \) = angle of wheel (\( \theta = 0 \) when \( M \) is "straight down"

This is a 1-D system. \( \theta \) alone determines all!

Are these equilibrium values for \( \theta \), where the system is **stable**?

\[
U(\theta) = U_{of \, m}(\theta) + U_{of \, M}(\theta) = Mg \text{ (height of } M) + mg \text{ (height of } m)
\]

\[= MgR(1-\cos \theta) + mg(y_0 - R\theta)\]

As \( \theta \uparrow \), \( m \) goes down, and \( R\theta \)

is the amount of string let out!

Equilib \( \text{f} U'(\theta) = 0 \Rightarrow Mg \cos \theta - mg \theta = 0 \)

or \( \theta = \frac{m}{M} \).

If \( m > M \) **No** equilibria. It just **falls** forever!

If \( m = M \), one soln, \( \theta = \pi/2 \).

If \( m < M \), two solns, one with \( 0 < \theta < \pi/2 \)

another \( \pi/2 < \theta < \pi \).
What about stability?

\[ U''(\phi) = M g R \cos \phi. \]

If \( m < M \), the \( 0 < \phi < \pi/2 \) solution has \( \cos \phi > 0 \implies \text{stable} \)

\[ \pi/2 < \phi < \pi \] \( \cos \phi < 0 \implies \text{unstable} \)

If \( m = M \), \( U'' = 0 \). Must look at next term:

\[ U''' = -M g R \sin \phi \bigg|_{\phi = \pi/2} \quad \text{is negative.} \]

\[ U(\phi) = \frac{U(\pi/2)}{a \text{ constant}} + \frac{\phi'}{2} + \frac{\phi''}{4} + \frac{U'''}{6} (\phi - \pi/2)^3 \]

\( \text{a constant} \quad 0 \quad 0 \quad \text{neg} \quad 3! \)

If \( \phi \to \pi/2^- \), \( U \) goes up, that's stable, we remain.

If \( \phi \to \pi/2^+ \), \( U \) goes down, that's a runaway situation.

So, this is not stable.

(By the way, no solution for \( \phi > \pi \) makes physical sense if you think about torques, draw the picture!)
This concludes what we'll cover in ch. 4.

4.8 Central forces is a good review of polar coords for you.

4.9 + 4.10 is about energy of systems.

Read it if you're interested.