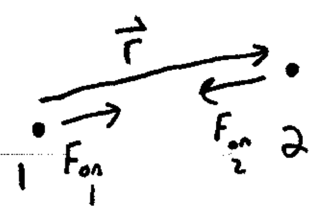
2210 – Gravity 1

We've talked about Force + Energy in general, with the key relation $\vec{F} = -\vec{\nabla} U$. Let's deviate briefly from Taylor's order, + zoom in on one particular force of enormous importance (+ familiarity!): Gravity

Newton started us off in 1666 with his Universal Law of Gravity (Principia published in 1687). It's still a hot topic of research in physics, astrophysics, + geophysics.

Reminder: for point-like particles



$$\vec{F}_{GRAV,\ on\ M_2} = -\frac{G M_1 M_2}{r^2}\,\hat{r}$$

with $\hat{r} = \vec{r}/r$ = unit vector from M_1 to M_2.

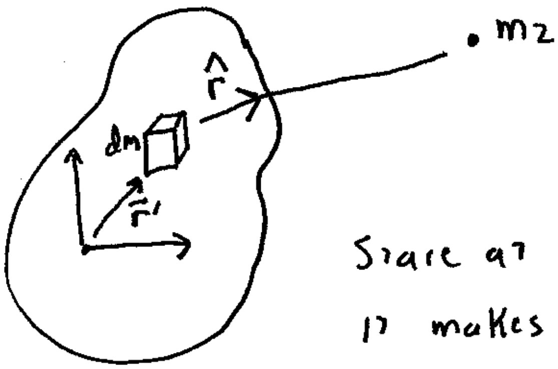
$G \approx 6.67 \cdot 10^{-11}\ \frac{Nm^2}{Kg^2}$. It's a fundamental physical constant. (Though interestingly, it's rather hard to measure, it's weak, + there's some controversy, maybe even in the 3rd or 4th decimal! It's not nearly as accurately measured as e.g c, $\hbar$, e, m_e, etc...)

2210 Grav 2

Gravity with real (extended) bodies:

It took Newton 20 years to convince himself he could accurately treat the earth-moon system as point masses @ their centers.

Consider e.g. an extended source (acting on point mass M_2):



$$\vec{F}_{\text{GRAV on } M_2} = -GM_2 \iiint \frac{dm}{r^2}\,\hat{r}$$

[Int. over volume where massive source is]

(with $dm = \rho(r')\,dV'$)

Stare at this + convince yourself it makes sense, just add up $\vec{F}_{\text{on } M_2}$ from all the little "chunks".

Setting up this integral is not much harder than our Center of Mass integrals, but you have to draw a careful picture to figure out that $\hat{r} = \dfrac{\vec{r}}{r}$, it's always the vector from our little chunk "dm" to our point of interest.

If Body #2 is extended, you must break it into chunks + integrate over it to to get $\vec{F}$. (Now you might see why Newton sweated for 20 years about this!)

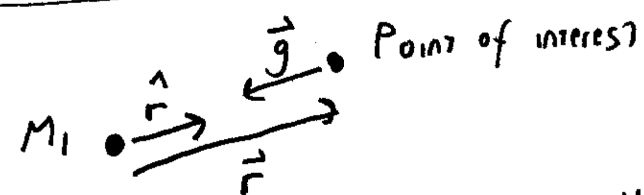
2210 - Grav 3

<u>The $\vec{g}$ - field:</u> In analogy to $\vec{E}$ fields, which are "force per unit charge", we can define

The $\vec{g}$-field $= \dfrac{\vec{F}_{\text{on a point, test-mass } m}}{m}$ = "Force per unit mass"

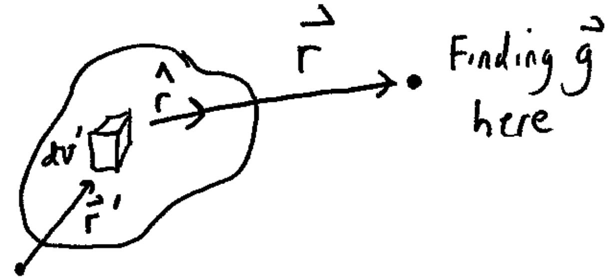
so

$$\vec{g}_{\text{from a point } M_1} = -\frac{G M_1}{r^2}\hat{r}$$



(Convince yourself that the ⊖ sign is correct!!)

$$\vec{g}_{\text{from an extended body}} = -G\iiint_{\text{Volume of body}} \frac{\rho(r')\,\hat{r}\,dV'}{r^2}$$



Near earth, $\vec{g} = 9.8\frac{m}{s^2}(-\hat{z})$ if you call $+\hat{z}$ "local up"

really, it's $-\hat{r}$, where $\vec{r}$ is the vector from the center of the earth.

<u>Coming up Next</u>: ① Let's calculate $\vec{g}$ from some example distribution of mass. Then, we should discuss

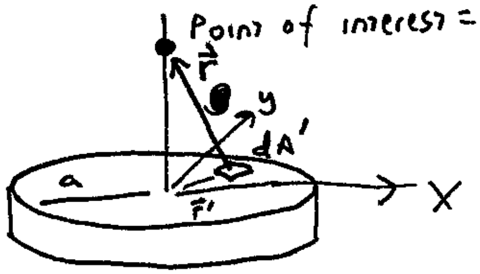
② Gravitational Potential Energy, i.e. find U so that

$$\vec{F}_{grav} = -\vec{\nabla} U.$$

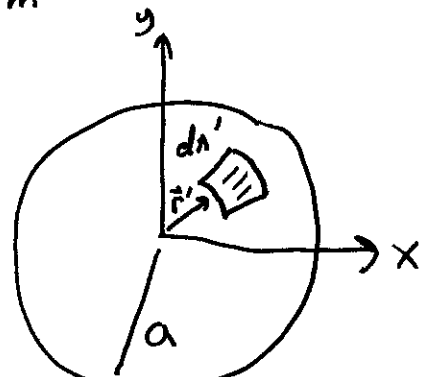
we start with ①, though...

2210 Grav 4

<u>Ex</u>: Let's find $\vec{g}$ above the center of a thin, massive "pancake" with given, constant mass density σ $\frac{\text{Kg}}{\text{m}^2}$ (mass per unit <u>area</u>)



or as seen from above



$$\vec{g}\,(\text{at } 0,0,z) = -G \iint_{\substack{\text{area of}\\ \text{disk}}} \frac{\sigma(r')}{r^2}\,\hat{r}\,dA'$$

<u>Watch out</u>! $\vec{r}\,'$ points <u>from</u> origin <u>to</u> our little "patch of source mass"

But $\vec{r}$ points <u>from</u> this patch <u>to</u> our point of interest.

The distinction is key, think about it, convince yourself!

Claim #1: $F_x = F_y = 0$ because of "matching patches" on opposite sides of origin. So, "by symmetry", $g_x = g_y = 0$ for this (special!) point of interest on the z-axis

(If we wanted $\vec{g}$ <u>off</u> the z-axis, this argument would not work! Do you see why not?)

So we <u>only</u> need to compute g_z. That helps!

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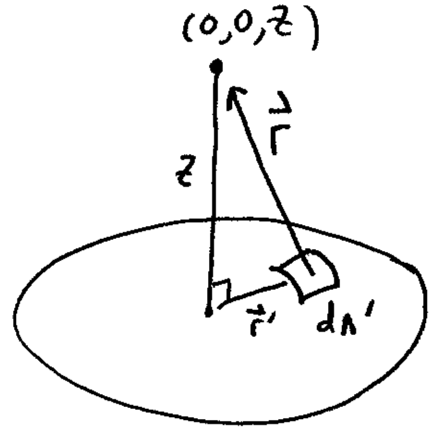

$$g_z = -G \underset{\text{over disc}}{\iint} \frac{\sigma}{|r|^2} \left(\hat{r}\right)_z \cdot dA'$$

think about this. $\hat{r} = \dfrac{\vec{r}}{|r|}$, so $\left(\hat{r}\right)_z = \overbrace{\dfrac{r_z}{|r|} = \dfrac{z}{|r|}}^{\text{Do you see this?!}}$

Now look carefully at the picture: $|r| = \sqrt{z^2 + r'^2}$

In plane-polar coordinates, $dA' = r'dr'd\varphi'$,

and $$g_z = -G \int_{r'=0}^{a} r'dr' \int_{\varphi'=0}^{2\pi} d\varphi' \cdot \underbrace{\frac{\sigma}{z^2+r'^2}}_{\text{this is } 1/|r|^2} \underbrace{\frac{z}{\sqrt{z^2+r'^2}}}_{\text{this is } \hat{r}_z}$$

$$= -G \cdot \underbrace{2\pi}_{d\varphi' \text{ integral}} \sigma \; z \int_{r'=0}^{a} \frac{r'\,dr'}{(z^2+r'^2)^{3/2}}$$

z → this is a constant as far as dr' is concerned!!

Not a terrible integral, if you "u-substitute", $u \equiv r'^2 + z^2$
$du = 2r'dr'$

So $g_z = -G \cdot 2\pi\sigma z \cdot \displaystyle\int \frac{1}{2}\frac{du}{u^{3/2}} = -G\pi\sigma z\left(-2u^{-1/2}\right)\Big|$

$$= +2G\pi\sigma z \cdot (r'^2 + z^2)^{-1/2}\Big|_{r'=0}^{r'=a} = +2\pi G\sigma z\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{|z|}\right]$$

2210 Gravity 6

So g_z (at point $0,0,z$) $= + G\sigma z \cdot 2\pi \left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{z}\right]$ (if $z > 0$)

Check: overall sign is OK: the thing in brackets is $\frac{1}{\text{big}} - \frac{1}{\text{small}} < 0$,

so $g_z < 0$, if $z > 0$, it points down as it should.

Check if $z \gg a$, that disc should "look like a point"!

watch carefully, this is an important trick in physics:

if $z \gg a$, then $\frac{a}{z} \ll 1$. Call this "ε" for a sec.

$$g_z \text{ (for big } z) = G\sigma z \cdot 2\pi \left[\frac{1}{z\sqrt{1 + a^2/z^2}} - \frac{1}{z}\right]$$

← exact! Factor out a z now...

$$= G\sigma \cdot \frac{z \cdot 2\pi}{z} \left[\frac{1}{\sqrt{1+\varepsilon^2}} - 1\right]$$

Taylor series says $(1+\varepsilon^2)^{-1/2} \simeq 1 - \frac{1}{2}\varepsilon^2 + \dots$, so

$(1+\varepsilon^2)^{-1/2} - 1 \simeq -\frac{1}{2}\varepsilon^2$, remember $\varepsilon = \frac{a}{z}$

$$\text{so } g_z \approx G\sigma \cdot 2\pi \left[-\frac{1}{2} \cdot a^2/z^2\right] = -G \cdot \frac{\pi a^2 \sigma}{z^2}$$

But $\pi a^2 \sigma$ = area * density = total Mass, this is

$g_z \simeq -G\frac{M_{total}}{z^2}$, yup, $\vec{g}$ from a point mass at the origin!

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One more "limiting case". What if $z \ll a$, so we're really close. Now, the pancake looks infinite, it's like a point near a giant plane of mass, (like humans near the earth!)

If $z \ll a$, our "small parameter" should be $\varepsilon \equiv z/a$

And now, we should write $\dfrac{1}{\sqrt{z^2+a^2}} = \dfrac{1}{a} \cdot \dfrac{1}{\sqrt{1+z^2/a^2}} = \dfrac{1}{a}\dfrac{1}{\sqrt{1+\varepsilon^2}}$

$$\text{so } g_z = G\sigma z \cdot 2\pi \left[\frac{1}{a\sqrt{1+\varepsilon^2}} - \frac{1}{z}\right]$$

For tiny ε, this is constant, this blows up, it dominates!

$$\text{so } g_z \simeq G\sigma z \cdot 2\pi \left[-\frac{1}{z}\right] = -G\sigma \cdot 2\pi.$$

oh look, g_z goes to a negative constant value, (like our neg. constant g_z near the giant earth!)

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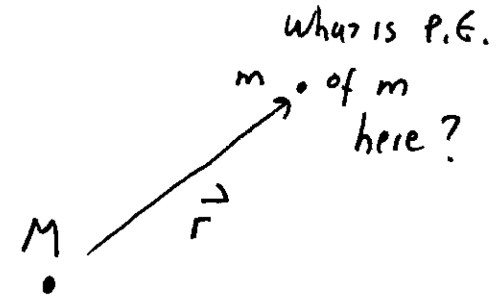
<u>Gravitational</u> <u>Potential</u> <u>Energy</u>



<u>Symmetry</u> says $|\vec{F}|$ depends only on $|r|$

Since $\vec{F} = -\vec{\nabla} U$, this implies U can only depend on $|\vec{r}|$ too.

(you can't have a different P.E. at different "polar angles" or "longitudes")
latitudes "azimuth"

In spherical coordinates, $\vec{\nabla} U(r) = \dfrac{dU(r)}{dr}\hat{r}$

and $\vec{F} = -\dfrac{GMm}{r^2}\hat{r} = -\vec{\nabla} U = -\dfrac{dU(r)}{dr}\hat{r}$

so "by inspection", if $\dfrac{dU}{dr} = \dfrac{GMm}{r^2}$, $\boxed{U(r) = -\dfrac{GMm}{r} + C}$

The constant C is our usual, arbitrary additive constant. We usually set it to 0 so that $U(\infty) = 0$, which seems reasonable, no P.E. far from our source.

If r gets <u>bigger</u>, $U(r) = -\dfrac{GMm}{r}$ gets <u>less</u> <u>negative</u>, that's an increase (right?). <u>Yes</u>, I like that: you must do + work on m to move it to bigger r, + thus it's P.E. <u>increases</u>.

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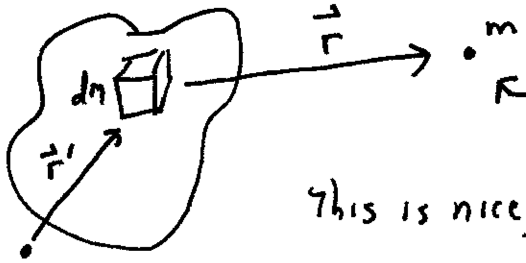
If M is extended, can you see that



$$U(\text{here}) = -Gm \iiint_{V'} \frac{\rho(r')\, dV'}{r}$$

This is nice, no vectors to integrate! And in the end, $\vec{F} = -\vec{\nabla} U$, so if you want the vectors, it's easier to take a gradient than to integrate vectors. P.E. is handy this way!

We can now define "Gravitational Potential" $\Phi \equiv \dfrac{U}{m}$

This is not Potential Energy, it's $\dfrac{\text{Potential energy}}{\text{unit mass}}$! (So, bad name!)

Convince yourself $\boxed{\vec{g} = -\vec{\nabla}\Phi}$ and $\boxed{\Phi_{\text{due to point } M \text{ at origin}} = -\dfrac{GM}{r}}$

Note: Φ (or U) are scalar functions of position.

The integrals (top of page, with m canceled out to get Φ) are much easier than those for $\vec{F}$ or $\vec{g}$, because there's no nasty $\hat{r}$ inside the integral.

And given $\Phi(\vec{r})$, $-\vec{\nabla}\Phi$ is always easy (right?)

2210 - Grav 10

Some analogies to Physics 1120 (E+M):

	GRAVITY	E+M
Force:	$\vec{F}_G = -\frac{GMm}{r^2}\hat{r}$	$\vec{F}_E = +\frac{KQq}{r^2}\hat{r}$
	$\vec{g} = \frac{\vec{F}}{m} = \frac{-GM}{r^2}\hat{r}$	$\vec{E} = \frac{\vec{F}}{q} = +\frac{KQ}{r^2}\hat{r}$
Potential Energy:	$U(\vec{r}) = -\frac{GMm}{r}$	$U(r) = +\frac{KQq}{r}$
Potential:	$\Phi(\vec{r}) = \frac{U(r)}{m} = -\frac{GM}{r}$	$V(\vec{r}) = \frac{U(r)}{q} = \frac{KQ}{r}$
Gauss' Law:	$\oiint \vec{g}\cdot d\vec{A} = -4\pi G\, M_{enc}$	$\oiint \vec{E}\cdot d\vec{A} = +4\pi K\, Q_{enc}$

Stare at each row, think about what it means, how gravity + electricity examples are similar. (The signs are different, can you think why?)

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Gauss' Law: It's just like for electricity:

$$\oiint \vec{g} \cdot d\vec{A} = -4\pi G \underbrace{M_{inside}}$$

closed area, surrounding a volume

This is $\iiint_{volume} \rho(r')\,dV'$

It's always true (!). It's only useful to find/compute $\vec{g}$ if $\vec{g}$ can be "pulled out" of the integral. So, you need symmetry to argue

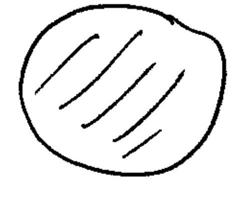
1) Direction of $\vec{g}$ is $\perp$ to $d\vec{A}$ (or parallel) so that dot product simplifies

2) $|g|$ is constant all around the integral

($d\vec{A}$ is "d (Area)", + points outwards.)

Example: Conside a sphere (radius a), w. Uniform ρ inside. (like a planet, or a galaxy filled uniformly w. stars) can we find $\vec{g}$ both inside + outside?

$\cdot\vec{g}$?



Method ① $\vec{g} = -G \iiint \frac{\rho(r')\hat{r}}{r^2} dV'$ Ugly!

But, double

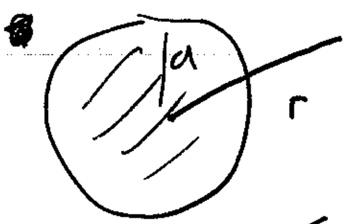
② $\Phi = -G \iiint \frac{\rho(r')}{r} dV$ Easier, but still a painful integral

(+ then $\vec{g} = -\nabla\Phi$)

or...

Method (3) Gauss' Law! $\oiint \vec{g} \cdot d\vec{A} = -4\pi G M_{enclosed}$.

Case #1: $r > a$, let's find gravity <u>OUTSIDE</u> the galaxy.

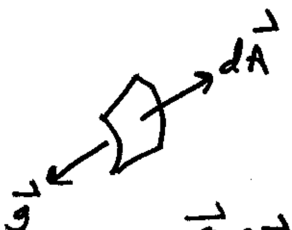


• $\vec{g}$ here? <u>Symmetry</u> says $\vec{g}$ points towards origin (Nowhere else makes sense. Can you convince yourself?)

So this means $\vec{g} = -g(\vec{r})\,\hat{r}$

<u>Symmetry</u> <u>also</u> says $g(\vec{r}) = g(|\vec{r}|)$, i.e. the magnitude of g depends on distance, but not angle. This is <u>different</u> than the above, do you see why? So, $\vec{g}(\vec{r}) = -g(r)\,\hat{r}$ by symmetry

Now,



On a <u>large</u> imaginary sphere (radius r)

$$\vec{g}(\vec{r}) \cdot d\vec{A} = -g(r)\, dA$$

↳ Directions are opposite!

So on this large sphere, $\oiint \vec{g} \cdot d\vec{A} = -\oiint g\, dA = -g(r) \oiint dA$

g is <u>same</u> everywhere on this sphere

so $\oiint \vec{g} \cdot d\vec{A} = -g(r) \cdot \underbrace{4\pi r^2}$

that's $\oiint dA$, do you see why?

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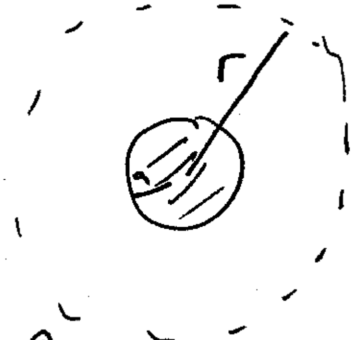


Remember, $\underbrace{\oiint \vec{g}\cdot d\vec{A}}_{\text{this is } -g(r)\cdot 4\pi r^2} = -4\pi G \underbrace{M_{enclosed}}_{\text{This is } \left(\frac{4}{3}\pi a^3\right)\rho}$

this is $-g(r)\cdot 4\pi r^2$

This is $\left(\frac{4}{3}\pi a^3\right)\rho$ ← This is the sphere we're integrating over

$= \text{Volume}\cdot\rho$

Notice we use "a", not "r" here, because mass stops at $r=a$!

$$\text{so } +g(r)\cdot 4\pi r^2 = +4\pi G\cdot \tfrac{4}{3}\pi a^3 \rho = 4\pi G\,(M_{total})$$

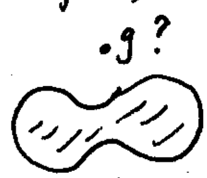
$$\text{or } g(r) = \frac{4\pi G\cdot M_{total}}{4\pi r^2} = \frac{G M_{tot}}{r^2} \qquad (\text{and } \vec{g} = -g(r)\hat{r})$$

Ahhh! Outside, the galaxy acts just like a point mass at the center.

(This result took Newton 20 yrs to prove - but he didn't know about Gauss' law, + so had to use our "Method 1" approach)

This result depended on spherical symmetry to argue that $g(r)$ was the same everywhere on the surface, and $\vec{g}\cdot d\vec{A} = -g\,dA$

So, the result is NOT true for e.g



Peanut shaped galaxy

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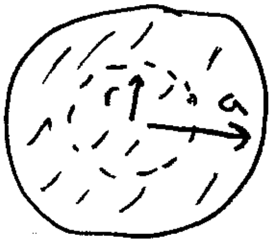
Example continued: Case #2, $r < a$. Inside the galaxy.



Once again, symmetry tells us

$\vec{g}(\vec{r}) = -g(r)\,\hat{r}$ ← convince yourself!

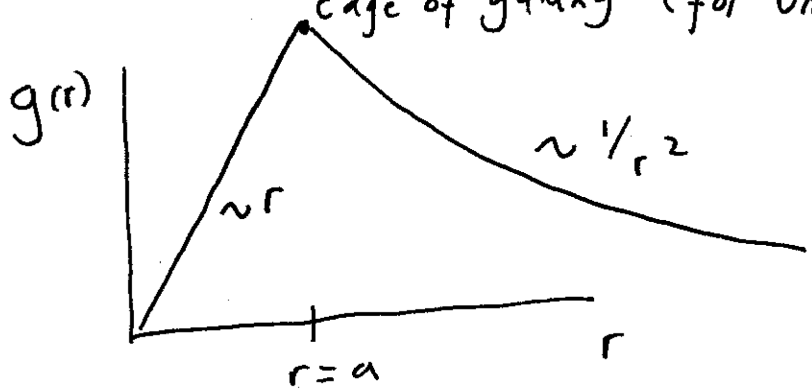
so again, $\oiint \vec{g} \cdot d\vec{A} = -g(r)\, 4\pi r^2$

dashed surface

But now, $M_{enclosed} = \underset{\text{out to } r}{\iiint} \rho(r')\,dV' = \underbrace{\frac{4}{3}\pi r^3}_{\text{volume}}\, \underbrace{\rho}_{\text{density}}$ ← I assume ρ is constant

so $g(r) = \dfrac{4\pi G}{4\pi r^2} \cdot \frac{4}{3}\pi r^3 \rho = G\left(\frac{4}{3}\pi\rho\right) r$. Gravity increases with radius!

And $\vec{g}(\vec{r}) = -\frac{4}{3}\pi\rho G\, r\,\hat{r} = -\frac{4}{3}\pi\rho G\, \vec{r}$ (for $r < a$, remember)



(Note: at $r = a$, $g = \frac{4}{3}\pi G \rho a$, which agrees with $\dfrac{G M_{tot}}{a^2}$. check!)

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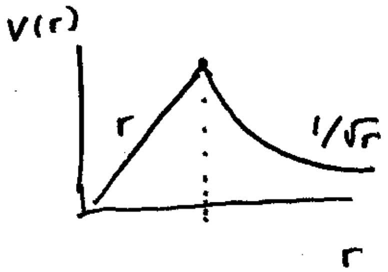
Application: Consider a star in circular orbit in/around a galaxy.

$F_{grav} = mv^2/r$ for circular orbits.

So inside galaxy, $\frac{v^2}{r} = \frac{4}{3}\pi G \rho r \Rightarrow V_{star} \propto r$

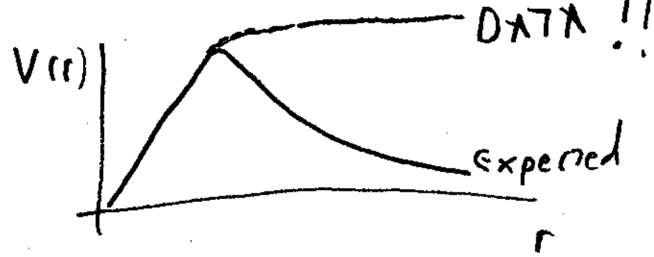
outside ", $\frac{v^2}{r} = \frac{G M_{tot}}{r^2} \Rightarrow V_{star} \propto \sqrt{\frac{1}{r}}$

So



This is our predicted plot of V(r).
This can be measured by doppler shifts!

If the galaxy "edge" isn't sharp, this might smooth out, but for many galaxies/star clusters the data is not what we expect.



Conclusion? Is Newton wrong? (I doubt it!)
Perhaps there is non-visible mass (not stars!) extending well beyond the visible edge of the galaxy!

$\boxed{\text{Dark Matter}}$ There's other evidence for this: Cosmic Background radiation, Galactic Clustering, lensing ...

Current best estimate: 80% of Universe's Mass is "Dark Matter"!!

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So what *is* the dark matter? (why don't we see it? why doesn't it form stars, "ignite", + glow?)

Is it "planets", brown stars (lumps of coal? Lost socks?)

"Machos": Massive Compact Halo Objects?

Or maybe it's novel, weakly interacting particles like massive neutrinos, or even more exotic new undiscovered particles?

"Wimps": Weakly interacting massive particles.

↳ Best guess right now, *very* hot topic of research in *both* laboratory particle physics + observational cosmology

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Geophysics:

The earth is *not* a sphere, so $g \neq -\frac{G M_{earth}}{r_{earth}^2}\hat{r}$

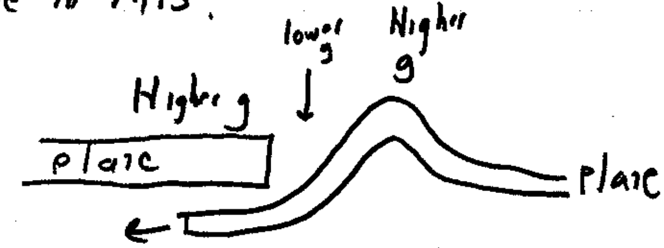
Close, but not exactly. Eg, g depends on

- *altitude* (Mexico City @ 10,000 ft, g = 9.779 m/s^2
  Taipei, same latitude @ sea level, g = 9.790 m/s^2)

- *latitude* (earth is bulging!)
  $g_{Helsinki} = 9.819$
  $g_{Rio\ de\ Janeiro} = 9.788$

- *local geology* ("gravitational anomalies" due to local rock composition - can help you search for metallic ores!)

Satellites can measure $\vec{g}$ (by effectively dropping test masses) + thus map out $\vec{g}$ (earth). It convolves all the above effects (+ more), + occupies a lot of current geophysicist's efforts. There's a CU group active in this!



Ex: Tonga/Kermadec subduction zone:
pacific plate goes under Australian plate)

- Let's you "map" underground features otherwise invisible/undetectable more directly.