Perhaps the most ubiquitous motion in the universe: "oscillatory motion". Results when any system is moved away from stable equilibrium.

Last chapter, we noted any $U(x)$ can be Taylor expanded

$$U(x) \approx U(x_0) + U'(x_0)(x-x_0) + \frac{U''(x_0)(x-x_0)^2}{2!} + \ldots$$

irrelevant constant = 0 if $x_0 = \text{equilibrium}$

So $U \sim c x^2$ and $F = -kx$

It's "mass on a spring", Hooke's law. A good model for many situations!

We'll start here, then add physics (e.g. damping or time-dependent driving) to get ever richer models

Examples: Molecules, quarks in nuclei, $RLC$ "oscillator" circuits, crystals, ...
Osc -2

When $F = -kx$, $U(x) = -\int_{x_0}^{x} \vec{F} \cdot d\vec{x} = -\int (-kx')dx' = \frac{1}{2} kx^2$

This is SHM, simple harmonic motion

"A" is the amplitude (turn around pt)

At endpoints, it's clear $E = \frac{1}{2} kA^2$.

Newton: $F = ma$ so $m\ddot{x} = -kx$, or

$\ddot{x} = -\omega^2 x$ with $\omega = \sqrt{\frac{k}{m}}$

We've seen this ODE many times (Ch.1 "pendulum", e.g.)

It's a 2nd order linear, homogenous ODE.
So, expect 2 linearly independent sol'n's.
We know the sol'n, it's:

$x(t) = B_1 \cos \omega t + B_2 \sin \omega t$

Check (plug it in + see that it works).

$\sin$ & $\cos$ are independent funs, $B_1$ & $B_2$ are any constants (determined by initial conditions)
Osc - 3

There are other ways to write / think about this sol'n, like e.g. \[ X(t) = C_1 e^{i \omega t} + C_2 e^{-i \omega t} \]

So, to see this, let's take a brief digression for **complex numbers** (see Boas 2.6, Taylor 2.6.)

These arise from \( \sqrt{-1} \). We define \( \mathbb{C} = \sqrt{-1} \)

A general complex number \( \mathbb{C} = x + i y \)

\[ \mathbb{C} = \text{Re}(\mathbb{C}) + i \text{Im}(\mathbb{C}) \]

You can "draw" any complex number in the "complex plane"

\[ \begin{array}{c}
\text{Im} \\
\downarrow \\
\text{Re} \\
\downarrow \\
x
\end{array} \]

This reminds me of polar coordinates, and we define

\[ |\mathbb{C}| = \sqrt{x^2 + y^2} = \sqrt{\text{Re}(\mathbb{C})^2 + \text{Im}(\mathbb{C})^2} \]

Modulus of \( \mathbb{C} \), or magnitude.

Note that \( x^2 \) & \( y^2 \) are both positive (the "i" is not part of \( y \), it's been pulled out.)
Definition: If \( z = x + iy \), then \( \overline{z} = x - iy \) = "complex conjugate".

\[
\overline{z} \cdot z = x^2 + y^2 = |z|^2,
\]

\[|z|^2 = \sqrt{z \cdot \overline{z}}\]

Note: The trick!

Can also divide, e.g.,

\[
\frac{2 + i}{3 - i} = \frac{2 + i}{3 - i} \cdot \frac{3 + i}{3 + i}
\]

\[
= \frac{6 + 5i + i^2}{9 + 1} = \frac{5 + 5i}{10} = \frac{1 + i}{2}
\]

Recall our Taylor series:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{for any} \ x
\]

\[
\sin x = x - \frac{x^3}{3!} + \ldots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \ldots
\]

Now check it out: \( e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots \)

\[
= 1 - \frac{\theta^2}{2!} + \ldots + i\left(\theta - \frac{\theta^3}{3!} + \ldots\right)
\]

\[
e^{i\theta} = \cos \theta + i \sin \theta\]

Euler's Formula.

Very useful!!
Using Euler: \[
\begin{array}{c}
\sqrt{x+iy} = r (\cos \theta + i \sin \theta) \\
= 121 e^{i \theta}
\end{array}
\]

Multiplying and dividing is very easy with this notation:
If \( z_1 = r_1 e^{i \theta_1} \) & \( z_2 = r_2 e^{i \theta_2} \)

\[ z_1 z_2 = (r_1 r_2) e^{i (\theta_1 + \theta_2)} \]
\[ z_1 / z_2 = (r_1 / r_2) e^{i (\theta_1 - \theta_2)} \]

Power \( (z_1)^\alpha = r_1^\alpha e^{i \alpha \theta_1} \) \( \Rightarrow \) Convince yourself! \( \sqrt{i} = e^{i \pi/2} \)

So e.g. \( \sqrt{i} = (i)^{1/2} = (e^{i \pi/2})^{1/2} = e^{i \pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \)

Actually there's another root, because \( i = e^{i 5\pi/2} \) also!

This is \( e^{i (2\pi + \pi/2)} \)

and \( (e^{i 5\pi/2})^{1/2} = e^{i 5\pi/4} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \)

(There are no different answers, though.)

\[ \sqrt{i} \Rightarrow e^{i \pi/4} \]
\[ \text{Osc} = \xi. \]

Note: \[ e^{i\theta} = \cos \theta + i \sin \theta \]
\[ e^{-i\theta} = \cos (-\theta) - i \sin (-\theta) \]
\[ \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \]
\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \]

These will prove handy to us soon when we have integrals like

\[ \int_{-n}^{n} \cos 3x \cos 2x \, dx = \int_{-n}^{n} \frac{e^{i3x} + e^{-i3x}}{2} \cdot \frac{e^{i2x} + e^{-i2x}}{2} \, dx \]
\[ = \int_{-n}^{n} \frac{e^{i5x} + e^{-i5x} + e^{i5x} + e^{-i5x}}{4} \, dx = \int_{-n}^{n} \cos 5x + \cos x \, dx \]
\[ = \frac{1}{10} \sin 5x \bigg|_{-n}^{n} + \frac{1}{2} \sin x \bigg|_{-n}^{n} = 0 + 0 = 0 \]

(we'll return to this soon)

\[ \text{Bottom line} \]

Picture complex #\#s

as points in complex plane
Osc - 7

What's this got to do with our 2nd order ODE?

SHM is very closely connected to simple rotations.

Consider a point rotating (ccw) in the complex plane

\[ z = ae^{i\theta} \] 

with steady rate, \( \theta = \omega t \)

i.e. \( z(t) = a e^{i\theta(t)} = a \cos \omega t + i \sin \omega t \)

So "\( a \cos \omega t \) & "\( a \sin \omega t \)" are Re & Im parts of this z(t)

Return to \( \ddot{x} = -\omega^2 x(t) \)

we know \( x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \)

\( \Rightarrow \) plug it in,

(2 constants, 2 independent forms, as these should be).

But we can rewrite these to our familiar \( \sin \)'s & \( \cos \)'s:

\[ x(t) = C_1 (\cos \omega t + i \sin \omega t) + C_2 (\cos \omega t - i \sin \omega t) \]

\[ = (C_1 + C_2) \cos \omega t + i (C_1 - C_2) \sin \omega t \]

\[ = B_1 \cos \omega t + B_2 \sin \omega t \]

So these 2 different "forms" of \( x(t) \) are mathematically equivalent.

\[ B_1 = C_1 + C_2, \quad B_2 = i (C_1 - C_2) \]

\[ C_1 = \frac{B_1 - i B_2}{2}, \quad C_2 = (B_1 + i B_2)/2 \]
Consider a specific initial condition, \( x(0) = A, \dot{x}(0) = 0 \).

\[
x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}
\]

\[
\begin{align*}
x(0) &= A = C_1 + C_2 \quad &\text{or} &\quad A = B_1 = 0 \\
\dot{x}(0) &= 0 = C_1 i\omega e^{i\omega t} - C_2 i\omega e^{-i\omega t} \quad &\text{or} &\quad 0 = B_2 \omega
\end{align*}
\]

so \( \dot{x}(0) = 0 = C_1 (i\omega) - C_2 i\omega \) or \( 0 = B_2 \omega \)

Conclusion \( \quad C_1 = C_2 \quad B_2 = 0 \)

and \( \quad C_1 = C_2 = A/2 \quad \text{and} \quad B_1 = A \)

so

\[
x(t) = \frac{A}{2} (e^{i\omega t} + e^{-i\omega t}) \quad \text{or} \quad x = A \cos \omega t
\]

\[
= A \Re e^{i\omega t}
\]

of course, these are the same so\( ^n \)!

It is powerful, in general, to think of S/NM as \( \Re (\text{complex soln}) \)

\( e^{i\omega t} \) is called a "phasor". Think of it as a complex representation of a sin wave. (And, convince yourself that if \( x(0) = 0, \dot{x}(0) = +V_0 \), then \( x(t) = \frac{V_0}{\omega} \sin \omega t \).)
Recall, we're solving \( \ddot{x} = -\omega^2 x \), and found general sol'n
\( x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \)
\( \text{or} \quad B_1 \cos \omega t + B_2 \sin \omega t \)
these are equivalent, \( B_1 = C_1 + C_2 \), \( B_2 = i(C_1 - C_2) \)
In either case, these are periodic functions, with
Period \( T = \frac{2\pi}{\omega} \) \( (= \frac{2\pi \sqrt{m/k}}{k} \text{ for mass on spring}) \)
Proof for \( e^{i\omega t} \): \( e^{i\omega(t+T)} = e^{i\omega t} e^{i\omega T} = e^{i\omega t} e^{i\omega 2\pi n/w} \\
\text{definition of a period}! \\
\text{Period is independent of } B's \text{ or } C's! \\
\text{Note that if } \ddot{z} = -\omega^2 z \text{, with } z = a + ib \)
\( \text{then } \Re e^{i\omega t} = -\omega^2 \Re z \text{; so when you find a} \)
\text{complex sol'n, you can always "take the real part"} 
\( \text{and you still have a sol'n! This is one way to understand why we use complex sol'ns to "real" physics problems, the} \)
\( \text{mark of } e^{i\omega t} \text{ is easier than sines + cos's!} \)
Osc -10

There's yet a 3rd way to write/think of our general sol'n (It's particularly useful when \(x(0)\) is nothing special, i.e. neither "A" nor "O") I claim

\[ x(t) = A \cos(\omega t - \delta) \]

is again equivalent, a general sol'n to \(\ddot{x} = -\omega^2 x\)

with 2 independent constants, as needed!

Just use \(\cos(a + b) = \cos a \cos b - \sin a \sin b\), so

\[ x(t) = (A \cos \delta) \cos \omega t + (A \sin \delta) \sin \omega t \]

\[ = B_1 \cos \omega t + B_2 \sin \omega t, \text{ ah ha, it is our exact same general sol'n! Picture this} \]

\[ z = A e^{i(\omega t - \delta)} \]

and \( \text{Re } z = A \cos(\omega t - \delta) \)

This is just simple rotation in complex plane. This sol'n is again equivalent.

(It's useful when \(x(0) = A \cos \delta\) is not at the "extreme" spot)

(See Taylor p.167)

For more!
In this form, \( x(t) = A \cos(\omega t - \delta) \), it's also particularly easy to look at energy.

Note: \( \dot{x}(t) = -A \omega \sin(\omega t - \delta) \)

So \( KE = \frac{1}{2} m \dot{x}^2(t) = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t - \delta) \)

\( PE = \frac{1}{2} K x^2(t) = \frac{1}{2} K A^2 \cos^2(\omega t - \delta) \)

But \( \omega^2 = k/m \), so \( mw^2 = k \), and thus

\( T + U = \frac{1}{2} k A^2 \left( \sin^2(...) + \cos^2(...) \right) = \frac{1}{2} k A^2 \)

Energy is conserved, with the value we noted back on p.1.

In general, for SHM "Simple Harmonic Motion"

- Periodic (sinusoidal) motion, \( x(t) = A \cos(\omega t - \delta) \)
- \( F = -kx \) Hook's law, force opposes motion
- \( U \sim x^2 \) Quadratic potential energy
- Period is independent of Amplitude
- There are 2 independent "constants" determined by initial conditions (e.g., \( x(0) \) & \( \dot{x}(0) \))
The 2 "conditions" can also be $x(t)$ & $\dot{x}(t)$ at some time, (or $T$ and $U$ at some time) or $A$ and $\phi$ ("Amplitude and phase shift").

If you know $x(t_0)$ & $\dot{x}(t_0)$, you know $x$ at all times, you can visualize:

Or something new, a "phase space diagram":

This point tells you $x$ & $\dot{x}$ at one time, so then you can watch this point move around as time goes by:

So e.g. slowing down as we move rightwards:

speeding up (neg), moving leftwards
2-D Harmonic Motion: If you have restoring forces

\[ m \ddot{x} = -k_x x \]
\[ m \ddot{y} = -k_y y \]

If \( k_x = k_y = mw^2 \), this is isotropic.
If \( k_x \neq k_y \), this is anisotropic.

If isotropic: \( x(t) = A_x \cos(wt - \delta_x) \)
\( y(t) = A_y \cos(wt - \delta_y) \)

Here it's nice to plot \( y(t) \) vs \( x(t) \). As time goes by, we "map out" the trajectory.

If \( \delta_x = \delta_y \), both are "in phase", \( x = \frac{A_x}{A_y} y \)

If \( \delta_y = \delta_x + \pi/2 \), \( x(t) = A_x \cos(wt - \delta_x) \)
\( y(t) = A_y \sin(wt - \delta_x) \)

So \( \frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1 \) (eccentric orbit if \( \frac{m}{\omega^2} \) is positive).
OSC-14

If anisotropic: w's are different. Pictures get more complicated, "Lissajous Patterns".

If \( \frac{w_x}{w_y} = \frac{n}{m} \) ratio of integers, then it "repeats" (closes on itself).

If \( \frac{w_x}{w_y} \) is irrational, it does not ever repeat (close).

Ex: \( w_y = 2w_x \), so \( n \) oscillates twice in y direction for every one in x direction.

\[\begin{array}{c}
\end{array}\]
Back to 1-D, let's add **DAMPING**

Consider **linear drag**, \( m \ddot{x} = -k x - b \dot{x} \)

(Quadratic drag \( \Rightarrow \) **non-linear** 2nd order ODE, this gets much more complicated). Rewrite Newton as

\[
\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0 \]

\( \Rightarrow \) A 2nd order, **linear**, homogeneous ODE with constant coefficients

Many physical systems obey this ODE (mechanical, electrical \( \Leftarrow \) RLC circuits, e.g.)

Let's take a brief digression here to look at 2nd order ODE's (Boas 8.5) The most general **linear** 2nd order ODE

\[ y'' + P(t)y' + Q(t)y = R(t) \]

\( \Rightarrow \) Homogeneous if \( R = 0 \)

- Should find 2 linearly independent solutions
- We'll deal with \( R(t) \neq 0 \) later. (Soon!)
What does "linear" imply? If also homogeneous,

1) If $y_1(t)$ solves $17$, so does $Cy_1(t)$
   (Convince yourself! $C$ comes thru $d/dt$ ...)

2) If $y_2(t)$ also solves $17$, so does $y_1(t) + y_2(t)$
   (Again, just plug it in to check)

- $y_1$ and $y_2$ are "linearly independent" if you cannot
  find any constants that make $C_1y_1(t) + C_2y_2(t) \equiv 0$
  for all times
  (except the trivial $C_1 = C_2 = 0$)

(Think of 2 vectors being independent similarly,
$C_1\vec{v}_1 + C_2\vec{v}_2$ can't be zero)

There is a nifty quick tool to test if functions
are linearly independent, you form the

\[ W = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} \]
If \( W = 0 \), they are dependent. If \( W \neq 0 \), independent?

\[
\text{Ex: } \begin{vmatrix} \cos wt & \sin wt \\ -(\sin wt)w & (\cos wt)w \end{vmatrix} = w \cos^2 wt + w \sin^2 wt = w \neq 0
\]

So \( \sin wt, \cos wt \) are independent (unless \( w = 0 \!\) !)

\[
\text{Ex: } \begin{vmatrix} e^{iwt} & -iwt \\ iwt & -e^{-iwt} \end{vmatrix} = -2iw \neq 0, \text{ again independent} (\text{unless } w = 0!)
\]

Ex: Can generalize to more \( \text{fn's} \),

\[
\begin{align*}
\text{fn's } \rightarrow & \begin{vmatrix} \cos wt & \sin wt & e^{iwt} \\ -w \sin wt & \cos wt & iwe \\ -w^2 \cos wt & -w^2 \sin wt & -we \\
\end{vmatrix}
\end{align*}
\]

\[
= w^3 e^{iwt} \begin{vmatrix} -\cos^2 w + i \sin \cos w + \sin w - i \sin \cos w - \sin^2 w + \cos^2 w \\
\end{vmatrix}
\]

\[
= 0! \text{ So they are not all 3 linearly independent,}
\]

(That's why we have \( A \cos + B \sin \) or \( Ae^{iwt} + Be^{-iwt} \)

but not some combo of 3 or more of these! \)
Osc - 18

Back to our ODE: \( \ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0 \)

Consider a simple, special case, just as an example

E.g. \( \ddot{y} + 3 \dot{y} + 2y = 0 \)

Here's the trick: Consider the operator \( D = \frac{d}{dt} \)

It's linear: \( D(3y) = 3Dy \) and \( D(y_1 + y_2) = Dy_1 + Dy_2 \)

So \( D^2y + 3Dy + 2y = 0 \) or \( (D^2 + 3D + 2)y = 0 \)

Now treat this like it was algebra \( (D+1)(D+2)y = 0 \)

This works if \( (D+1)y = 0 \)

\[ \implies (D+2)y = 0. \]

These are 2nd order ODE's, + I know their sol'n -

\( (D+1)y = 0 \) \( \implies \dot{y} = -y \) \( \implies y = C_1 e^{-t} \)

\( (D+2)y = 0 \) \( \implies \dot{y} = -2y \) \( \implies y = C_2 e^{-2t} \)

Check Wronskian

\[ \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = e^{-3t} (-2+1) \neq 0, \]

So \( y = C_1 e^{-t} + C_2 e^{-2t} \) is the general sol'n here.
In general, \((D-r_1)(D-r_2)y = 0\) is solved by
\[ y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad \text{(as long as } r_1 \neq r_2, \text{ see bottom of page 4)} \]
Indeed, if \(r_1\) and \(r_2\) are roots of the
"auxiliary algebraic eq'n"
(i.e. From \(a y(t) + b y(t) + cy = 0\),
(auxiliary eq'n is \(a D^2 + b D + c = 0\)), then roots of this
\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
generate our sol'n,
\[ y = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \]

* If roots are equal, I claim
\[ y(t) = (C_1 + C_2 t) e^{-rt} \quad \text{will solve the ODE.} \]
Can check by plugging \(t\) in, plus Wronskian to convince yourself \(e^{-rt}\) and \(te^{-rt}\) are independent.
Derivation of the previous claim:

\[(D-r)(D-r)y = 0.\]

Clearly \(y = c_1 e^{rt}\) is one sol'n.

Let \(U(t) = (D-r)y\). So \((D-r)U = 0 \Rightarrow U = c_2 e^{rt}\)

Thus, \((D-r)y = U = c_2 e^{rt}\). Recall, Boas has a trick to solve this, remember?

\[y + Py = Q\] (here, \(P = -r\), and \(Q = c_2 e^{rt}\))

Then \(I = \int P \, dt = -rt\)

And \(y = e^{-rt} \int Q \, e^{rt} \, dt + C e^{-rt} = e^{-rt} \int c_2 e^{rt} \, e^{-rt} \, dt + C e^{-rt} = e^{-rt} \cdot c_2 e^{-rt} + C e^{-rt} = e^{-rt} \cdot c_2 t + C e^{-rt} = (C + c_2 t) e^{-rt}.\)
Example: $\ddot{y} + \omega_0^2 y = 0$. Sol’n?

Auxiliary eq’n is $(D^2 + \omega_0^2) = 0$

roots are $\pm i\omega_0$, so general sol’n is $y = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$—precisely our sol’n we’ve used before!

What about $\ddot{y} + \frac{b}{m} \dot{y} + \omega_0^2 y = 0$, which is what we’ve been after? Let’s define a damping constant $\beta = \frac{b}{2m}$ units of $[\beta] = \text{units of C}W$

so we’re solving $\ddot{y} + 2\beta \dot{y} + \omega_0^2 y = 0$

auxiliary eq’n $D^2 + 2\beta D + \omega_0^2 = 0$

roots are $r_{1,2} = -\beta \pm \sqrt{\beta^2 - 4\omega_0^2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

Assuming $\beta \neq \omega_0$, these roots are distinct, and we have a sol’n $\hat{y}(t) = e^{-\beta t} (C_1 e^{\sqrt{\beta^2 - \omega_0^2} t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2} t})$

(if $\beta = \omega_0$, we use the trick from prev page, we’ll come back to this)
\[ \ddot{y} + 2\beta \dot{y} + \omega_0^2 y = 0 \quad \text{(Damped SHM)} \]

There are 3 possible cases:

1. \( \beta < \omega_0 \), "weak damping", \( \sqrt{\beta^2 - \omega_0^2} \) is imaginary!
2. \( \beta > \omega_0 \), "strong damping", \( \sqrt{\beta^2 - \omega_0^2} \) real.
3. \( \beta = \omega_0 \), "critical damping", use "double root" trick.

Case 1: \( \beta < \omega_0 \), also called UNDERDAMPED.

\[
\sqrt{\beta^2 - \omega_0^2} = \delta \sqrt{\omega_0^2 - \beta^2} = i \omega_1 \quad \text{(Defines} \omega_1, \text{For small} \beta, \omega_1 \approx \omega_0) \]

\[ y(t) = e^{-\beta t} (C_1 e^{i \omega_1 t} + C_2 e^{-i \omega_1 t}) \]

Or, looking back a few pages, can also rewrite as:

\[ y(t) = e^{-\beta t} A \cos(\omega_1 t - \delta) \]

"Amplitude" is decaying

\[ \beta = 0, \text{undamped} \]

Damped

Undamped \( \omega_0 \).

\[ e^{-\beta t} \] "Envelope"
Case 2, $\beta > \omega_0$, "over damped"

$$y(t) = C_1 e^{-\beta t + \sqrt{\beta^2 - \omega_0^2} t} + C_2 e^{-\beta t - \sqrt{\beta^2 - \omega_0^2} t}$$

Both terms are dying exponentials. (Convince yourself the $C_1$ term is decaying, the $+ past$ is smaller than the $- past$!)

No oscillations, too much damping! It might cross the axis (once) if you pick $C_1$ & $C_2$ cleverly, but no "ringing".

Note: $C_2$ term has a much more negative coefficient for $t$ so it $\to 0$ faster. At large $t$, $C_1$ term dominates.

Only 1 left is

\[Ce^{-(\beta - \sqrt{\beta^2 - \omega_0^2}) t}\]

if $C_1, C_2 > 0$

so $y(0) > 0$.

\[Ce^{-(\beta + \sqrt{\beta^2 - \omega_0^2}) t}\]

if eg $C_2 = -C_1$.

so $y(0) = 0$, $\dot{y}(0) > 0$

Note: As $\beta$ gets bigger, $\beta - \sqrt{\beta^2 - \omega_0^2} \to 0$, so somewhat surprisingly, large damping in fact slows the rate of decay!
Case 3: Critical damping, $\beta = \omega_0$

Double root of $-\beta$, so $y(t) = (C_1 + C_2 t) e^{-\beta t}$

The dying exponential always "wins" eventually, so this does die away, very much like $e^{-\beta t}$. This is a faster dropoff than the $e^{-(\beta^2 - \sqrt{\beta^2 - \omega_0^2}) t}$ we had in the over-damped case.

Think of the pneumatic tube on a screen door, or shocks on your car:

Too little damping $\Rightarrow$ oscillations (or banging)
Too much damping $\Rightarrow$ very slow to settle down
Critical $\Rightarrow$ fastest return to equilibrium w/o oscillations.
What if we drive/force our oscillator?
\[ y'' + 2\beta y' + \omega_0^2 y = R(t) \quad \text{— inhomogeneous} \]

From Newton, this is \( \frac{F(t)}{m} \)

(or, e.g. in an RLC circuit, it might be a voltage source)

The approach for inhomogeneous linear ODE's is:

1. Solve the homogeneous case to get \( y_c = C_1 y_1(t) + C_2 y_2(t) \) complementarily
2. Then find any particular solution \( y_p \).
3. \( y = y_c + y_p \) is your fully general solution w. 2 constants!

We know how to find \( y_c \), so we just need \( y_p \).

Sometimes: "by inspection" works! e.g. \( y' + 3y + 3y = 5 \)

I claim \( y_p = 5/3 \) is clearly a solution. (Do you see why? Try it!)

General solution: Roots of \( D^2 + 4D + 3 \) are \( 1 \) & \( 3 \), so
\[ y = Ae^t + Be^{3t} + 5/3 \] is the fully general solution

(So if \( R(t) = R_0 = \text{constant} \), use \( y_p = \frac{R}{\omega_0^2} \) ... )
Osc -25

What if \( R(t) = \int_0^t e^{ct} \) (where \( c \) can be imaginary!)

This is really much more general than it looks, because \( e^{iwt} \) as we've seen has \( \cos(wt) \) and \( \sin(wt) \) "built in", and we can build up any oscillatory \( R(t) \) by summing up \( \sin(wt) + \cos(wt) \) with different \( w \)'s. We'll come back to do that, but this is why this example is so fundamental and essential.

Note that if we find \( y_{p1} \) for \( R_1(t) \) on the right side and \( y_{p2} \) for \( R_2(t) \), then

\[
[ y_{p1} + y_{p2} \text{ will solve the ODE with } R = R_1(t) + R_2(t) ]
\]

We can derive \( y_p \) for \( R(t) = \int_0^t e^{ct} \), but the nice thing about \( y_p \) is you just need to find anything that works, so the method of "guess and check" is just fine!
\[
\begin{align*}
0 &= \text{osc} - 2c \\
\text{we're solving }\quad \ddot{y} + 2\beta \dot{y} + w_0^2 y &= f_0 e^{ct} \\
\text{looking for any } y_p. \text{ Let's try } y_p(t) = C_3 e^{ct} ! \\
\text{[This doesn't always work (e.g. if } c \text{ happens to be one of the roots of the auxiliary eq'n, we'll need to try } C_3 t e^{ct}, \text{ if it equals a double root, } C_3 t^2 e^{ct}, \text{ but those are really special cases.)}
\end{align*}
\]

Important: \( C_3 \) is not a new "unknown, arbitrary" constant! This is \( y_p \), we need to plug it in, \( + \) the ODE itself will require a very particular value for \( C_3 \), it's not set by "initial conditions"! Let's do it: \( \dot{y}_p = C_3 \dot{y}_p \)
\[
\begin{align*}
\dot{y}_p &= C_3 y_p \\
so \quad (c^2 + 2\beta c + w_0^2) C_3 e^{ct} &= f_0 e^{ct} \\
\text{so} \quad C_3 &= \frac{f_0}{c^2 + 2\beta c + w_0^2} \quad \text{is required, it's fixed!} \\
\text{(This is all good as long as } c \text{ isn't accidentally a root of the auxiliary eq'n, then the denom-}0) 
\end{align*}
\]
Important example: \( R(t) = \text{fo} \, e^{i\omega t} \), (oscillating driver) (here, \( C = i\omega \))

so \( y_p = \frac{\text{fo} \, e^{i\omega t}}{y_p = \csc^{i\omega t} \rightarrow (-\omega^2 + 2i\beta \omega + \omega_0^2)} = \frac{\text{fo} \, e^{i\omega t}}{(-\omega^2 - \omega_0^2) + 2i\beta \omega} \)

If the driving force is real, say \( \text{fo} \, \cos\omega t \), no problem, just take the real part of this soln!!

Our \( C_3 = \frac{\text{fo}}{(-\omega^2 - \omega_0^2) + 2i\beta \omega} \) is complex. Any complex # can always be written in the form \( C_3 = A e^{-i\delta} \) (Note that "taking the real part" is much easier in this form, \( \text{Re} (C_3) = A \cos \delta \)).

Let's pause to work out \( C_3 \), then:

To get \( A = |C_3| \), use \( \left| \frac{1}{a+bi} \right| = \left| \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \right| = \frac{1}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}} \)

(To get \( \delta \), rewrite \( C_3 \) in the form \( \text{Re}(e^{i\delta} = \text{Re} + i\text{Im}) \))
\[ C_3 = \frac{f_0}{(w_0^2 - w^2) + i \cdot 2\beta w} \]

so \[ |C_3| = \frac{f_0}{\sqrt{(w_0^2 - w^2)^2 + 4\beta^2 w^2}} \]

(use \[ \frac{1}{a+bi} = \frac{1}{\sqrt{a^2 + b^2}} \]

also, \[ C_3 = \frac{f_0}{(w_0^2 - w^2)^2 + 4\beta^2 w^2} \cdot \left[(w_0^2 - w^2) - 2\beta w i\right] \]

so \[ \delta = \tan^{-1} \frac{2\beta w}{w_0^2 - w^2} \]

From the picture!

It's gotten a little ugly, let's recap:

we're solving \[ y' + 2\beta y + w_0^2 y = f_0 e^{i\omega t} \]

we know how to find \( Y_c(t) \) (solving "ancillary", homog eq's)

we just found \( Y_p(t) = C_3 e^{i\omega t} = A e^{-i\delta} e^{i\omega t} \)

(If we have a real driver, \( f_0 \cos(\omega t) \), we'll simply tak't the Real part of our sol'n \( \Theta \), giving \( \text{Re}(Y_p) = A \cos(\omega t - \delta) \))

Let's do an example!
Osc-27

Underdamped, driven oscillator. (Like, resonant circuit)

Recall, the homogeneous $y_c(t)$ sol'n has an overall $e^{-\beta t}$, so it dies away, it's transient. In general, that sol'n is $y_c(t) = A e^{-\beta t} \cos (\omega_1 t - \delta_{fr})$

These are our 2 arbitrary coefficients, found from initial conditions

so $y(t) = y_p + y_c$

$= A \cos(\omega_1 t - \delta) + A e^{-\beta t} \cos (\omega_1 t - \delta_{fr})$

where $A = \frac{\omega_1}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4 \beta^2 \omega_1^2}}$

and $\delta = \tan^{-1} \frac{2 \beta \omega_1}{\omega_0^2 - \omega_1^2}$

After some time $\gg 1/\beta$, only $y_p = A \cos(\omega_1 t - \delta)$ remains. (true also if overdamped, or critically damped, so this is quite general)

If you drive an oscillator, it settles down by oscillating at the driving frequency (but, phase shifted)
Bottom line: \[ \ddot{y} + 2\beta \dot{y} + \omega_0^2 y = f \cos \omega t \]

large \& small: \[ y = A \cos (\omega t - \delta) \]

A is fixed, \( \text{it's} \; \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \; \text{not initial conditions} \)

\( \delta \) is fixed, \( \text{it's} \; \tan^{-1} \frac{2\beta \omega}{\omega_0^2 - \omega^2} \; \text{again} \; \text{it's large response} \)

\( \omega \) is the driver's \( \omega \), not the natural \( \omega \). (Neither \( \omega_0 \) nor \( \omega \))

The Amplitude \( A \propto f_0 \), so "strong drivers" \( \Rightarrow \) large response

---

A tells you the "strength of response", the long-term amplitude of motion. Energy in an oscillator \( \propto \frac{1}{2} k A^2 \), so it also tells you how much energy the "final state" has.

(For this reason, we're often interested in \( 1A^2 \).)

\( A \) depends \( \propto \) (linearly) on \( f_0 \), the driving force, but also on \( \omega_0 \) (natural freq), \( \beta \) (damping), \( \omega \) (driving freq)

when \( \beta \) is small, interesting things happen, let's investigate.
In some situations, \( \omega \) is set somehow, but you can vary or control \( \omega_0 \). E.g., if PBS broadcasts a radio wave at 98.5 MHz, that drives your stereo. \( \omega = 2 \pi \times 98.5 \) MHz. But by twisting a knob (thus varying a capacitance or inductance), you can "tune" \( \omega_0 \) as will! We'll come back to the details of "RLC circuits" soon.

In other situations, \( \omega_0 \) is set somehow, but you can vary or control the driving frequency \( \omega \). (Consider a bridge or building with natural vibrational frequency, driven by a controllable or variable outside force.)

Situation 1 is slightly simpler mathematically, but for small drag (\( \beta < \omega, \omega_0 \)) both are qualitatively similar.

So, let's first consider case 1), \( \omega \) is fixed, and we are free to vary \( \omega_0 \) as will. How does Amplitude \( A \) respond? (Or, how does \( 1A^2 \) respond, since that's energy)
Osc -32. (Case 1: Fix \( w \), vary \( w_o \))

\[
1A^2 = \frac{s_0^2}{(w_o^2 - w^2)^2 + 4\beta^2w^2},
\]

Think of this as a peak when \( w_o = w \)!

\( \text{fn of } w_o: \frac{1}{c^2 + (w_o^2 - w^2)^2} \)

\( \frac{1A^2}{w_o} \)

Small but not 0.

The max occurs when denom is min, i.e. that's \( w_o = w \)

If \( \beta \) is small, \( 1A^2 \) grows very large when \( w_o = w \), in fact.

\[
1A_{\text{max}}^2 = \frac{s_0^2}{4\beta^2w^2} \quad (\to \infty \text{ if } \beta \to 0)
\]

This "Blowup" when \( w_o = w \) is Resonance.

The system responds strongly when you drive it at the natural frequency.

(If \( w_o \) is far from \( w \) in either direction, the response is much weaker.)
Osc -33  (Case 2: Fix \( \omega_0 \), vary \( \omega \))

\[ |A|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2} \]

Again, \( \text{Max} \) occurs when denominator is minimum, i.e.

\[ \frac{d}{d\omega} \left( \omega_0^4 - \omega^4 \omega_0^2 + \omega^4 + 4\beta^2 \omega^2 \right) = 0 \]

i.e. \(-2\omega_0^2 \omega + 4\omega^3 + 8\beta^2 \omega = 0\)

\[ \omega = 0 \text{ solves this, but that turns out to be a max, not a min.} \]

[of the denominator, at least if \( \beta \ll \omega_0 \).]

The other soln: \( \omega^2 + \omega_0^2 = -2\beta^2 \) (Divided out \( \omega \))

So \( \omega_{\text{peak}}^2 = \omega_0^2 - 2\beta^2 \) gives "resonant \( \omega \)"

\[ |A|^2 \sim \text{Peak just shy of } \omega_0 \]

\[ |A|_{\text{max}}^2 = \frac{f_0^2}{4\beta^4 + 4\beta^2 (\omega_0^2 - 2\beta^2)} \approx \frac{f_0^2}{4\beta^2 (\omega_0^2 - \beta^2)} \] (Blows up as \( \beta \to 0 \))

Again, resonance when \( \omega \approx \omega_0 \).
Summarizing: \[ w_0 = \sqrt{K/m} = \text{natural undamped freq} \]
\[ w = \text{driving freq} \]
\[ \omega_i = \sqrt{w_0^2 - \beta^2} = \text{nrz damped freq} \]
\[ \omega_{\text{peak}} = \sqrt{w_0^2 - 2\beta^2} = \text{resonance freq (for fixed } w_0, \text{)} \]

1A_{\text{max}} \approx f_0 / 2 \beta w_0 \quad \text{(if } \beta \ll w_0) 

Kids on swings know all this! \( w_0 \) is the "natural swing freq", you get the best ride if you pump at that freq. If there's drag, \( 1A \) is finite, but can be quite big.

\[ 1A^2 \]

\[ \rightarrow \text{smaller } \beta \Rightarrow \text{higher peak (nearly at } w_0) \]
\[ \rightarrow \text{bigger } \beta \Rightarrow \text{lesser peak (twice as "left" of } w_0) \]

Width of the resonance curve: Let's look for the \( \omega \) where \[ 1A^2(\omega) = \frac{1}{2} 1A^2(\text{peak}) \]. That's "half max".

It will be (next page) \( \frac{\omega}{\text{max}} = \frac{\omega_0 \pm \beta}{\text{max}} \).

So, smaller \( \beta \) \Rightarrow taller and narrower resonance.
\[ |A|^2 = \frac{\gamma_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2} \]

This is "half max" when the 1st term = 2nd term (if \( \beta \) is small)

so this is when \( \omega_0^2 - \omega^2 = \pm 2\beta \omega_0 \)

if \( \beta \) is small, \( \omega_0^2 - \omega^2 \) is thus small, so our \( \omega = \text{"Half max" is in fact still very near } \omega_0 \).

In that case \( \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0 (\omega_0 - \omega) \)

and \( \pm 2\beta \omega \approx \pm 2\beta \omega_0 \), so

\[ (\omega_0 - \omega) \approx \pm \frac{2\beta \omega_0}{2\omega_0} = \pm \beta , \text{ as claimed.} \]
Osc -36

It's common to define a "Quality Factor"\( Q = \frac{\omega_0}{2\beta} \)

\[Q \text{ unless!}\]

Narrow resonance \(\Leftrightarrow\) small \(\beta\) \(\Leftrightarrow\) big \(Q\).

\[A_{\text{peak}} \propto \frac{f_0^2}{4\beta^2\omega_0^2} = \frac{f_0^2}{4\beta^2\omega_0^4} = Q^2 \frac{f_0^2}{\omega_0^4} \propto Q\]

\[Q = \frac{\omega_0}{2\beta} = \frac{2\pi}{T_{\text{natural}}}\]

Recall that with no drive, \(\omega\) \(\sim\) \(e^{-\beta t}\) dies off in time \(t \approx 1/\beta\)

so \(Q = \frac{2\pi}{T_{\text{natural}}}\)

Big \(Q\) means "die-off time" is long compared to natural period.

So, big \(Q\) means you will "ring" many times without a drive and, with a drive, you'll get strong resonance.

you can also show \(Q\) tells you Energy stored in oscillator

\[\text{Energy lost to damping in one cycle} \]
I've referred to RLC circuits. They might look like this:

**Kirchhoff says** $AV_{\text{loop}} = 0$

$s + V_{o}\cos(\omega t)$

But $I = dQ/dt$, so

$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = V_{o}\cos(\omega t) \Rightarrow \text{our usual ODE!}$

$\ddot{Q} + \frac{R}{L} \dot{Q} + \frac{1}{LC} Q = \frac{V_{o}}{L} \cos(\omega t)$

$\omega = \omega_{0}$

$R$ causes damping, makes sense!

$\frac{1}{LC} = \omega_{0}^2$ is the "natural frequency"

For a 30 pF capacitor

For a 100 nH inductor

Cheap, simple, ordinary values

$\frac{\omega_{0}}{2\pi} \approx 90.9$ MHz