Fourier - 1

We now have an exact, complete sol'n to the damped, driven oscillator if the driving force is $f(t) = f_0 \cos\omega t$ (or, $f_0 e^{i\omega t}$)

So if the source (driver) is sinusoidal, we know the system's response.

Sinusoidal $f(t)$ is common in both mechanical & electrical settings, but the real importance of this (special) sol'n is the following:

- If we have a "driver" ($f(t)$) that's periodic (with any shape or functional dependence at all), we can "build it up" out of a sum of sinusoids (with different ω's).

Thus we can solve the general case of any periodic "driver"

This is the "method of Fourier", or Fourier Series

Notation: $f(t)$ is periodic, with period τ, ("τ-periodic") if

$f(t+\tau) = f(t)$ for any/all times t.

Fourier's claim: Any τ-periodic function $f(t)$ can be uniquely written as

$$f(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin\omega t + b_2 \sin 2\omega t + \ldots$$

$$= \sum_{n=0}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \quad (\text{with } \omega = 2\pi/\tau)$$

(Any periodic fn looks like a superposition of pure sin's + cos's)

Fourier - 2.

Claim: If $y_1(t)$ solves $\ddot{y}_1 + 2\beta\dot{y}_1 + \omega_0^2 y_1 = \cos(\omega_1 t)$

and $y_2(t)$ solves $\ddot{y}_2 + 2\beta\dot{y}_2 + \omega_0^2 y_2 = \cos(\omega_2 t)$

then $y = C_1 y_1 + C_2 y_2$ solves $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = C_1\cos\omega_1 t + C_2\cos\omega_2 t$

(Just plug it in, the linearity ensures it, it's a 1-step proof!)

So since Fourier says any periodic $f(t) = \sum_n a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$

then apparently we know, by inspection, how to solve the ODE

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f(t);$$

it will simply be

$$y(t) = \sum_{n=0}^{\infty} a_n \, y_{cn}(t) + \sum_{n=1}^{\infty} b_n \, y_{sn}(t)$$

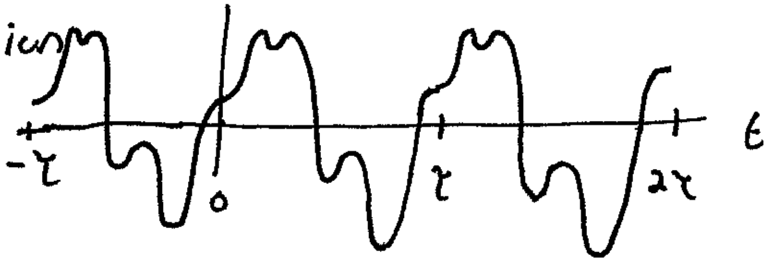
[where y_{cn} is the sol'n of our ODE driven by $\cos n\omega t$, and y_{sn} is the sol'n of our ODE driven by $\sin n\omega t$]

we just need to know the (constants!) a_n's + b_n's up here.

(And we need to remember, from the last section, what the sol'ns to the ODE are.)

Some examples of "τ-periodic" functions, anything that repeats!

- Hitting a nail once every τ seconds
- Singing a "pitch" of nominal frequency f (with overtones)
- Noisy electronic signals built on a base frequency like 60 Hz
- The function



← Air pressure from An oboe??

Fourier -3

Given $f(t)$, we need a method to find those a_n's + b_n's.
(Once we know them, we immediately know the response of a damped oscillator to this driver $f(t)$, i.e. $y(t)$.)
This method is straightforward, it's "Fourier's Trick".

Let me 1st give the result
then motivate it,
+ lastly "derive" it.

Result: If $f(t)$ is "τ-periodic", then define $\omega = 2\pi/\tau$,

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \cos n\omega t,$$ with coefficients:

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} f(t) \cos n\omega t \, dt$$

$$\left(a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, dt\right)$$ is a special case

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} f(t) \sin n\omega t \, dt$$

(Note: See notes p.11a if you want to know the math behind these "magic" formulas!)

That's it. We have formulas to compute a_n's + b_n's. These are definite integrals, a_n's + b_n's are all numbers, constants.

$f(t)$ is a "superposition" of pure sinusoids, (always!)
+ then the system's response, $y(t)$ is the same superposition (same coefficients) of "pure responses" (to pure frequency drivers)

Fourier – 4

Motivation: Any <u>vector</u> $\vec{V}$ can be uniquely expanded as

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

a constant, the "component" — a unit vector

[for some "basis set" of orthogonal unit vectors, the $\hat{e}_i$'s.]

If you know $\vec{V}$, you can find the components, by

$V_i = \vec{V} \cdot \hat{e}_i$ (so e.g $V_x = \vec{V} \cdot \hat{i}$ in Cartesian)

• Our unit vectors are orthogonal. In general, <u>any</u> 2 vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a} \cdot \vec{b} = 0$. In components

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i \, b_i = 0 \iff \text{orthogonality}$$

Now, imagine a vector in N-dimensional space. Can you see that

$\vec{a} \cdot \vec{b} = \sum_{i=1}^{N} a_i b_i = 0 \iff$ orthogonality. Now, let $N \to \infty$, + replace this <u>sum</u> with an integral! (This is a leap, we're making an analogy, not an equality) But the "dot product of two functions" (really called the <u>inner</u> product) a(t) and b(t) will be

$$\frac{\tau}{2} \int_{-\tau/2}^{\tau/2} a(t) \, b(t) \, dt \, .$$

(If this integral is 0, we say the fn's a and b(t) are orthogonal!)

(This factor is just for later convenience.)

Fourier - 5

So just as vectors can be "expanded" in a basis set of unit vectors
So too can functions be "expanded" in a basis set of functions!

Here, $\hat{e}_1, \hat{e}_2, \hat{e}_3$ was our 3-D basis set of vectors
and $\cos(\omega t), \cos(2\omega t), \cos(3\omega t), \ldots$ etc forms our "basis set" of functions!

— Just as the coefficient $V_n = \vec{V} \cdot \hat{e}_n$, so too, ← a dot product
the coefficient $a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos n\omega t \, dt$ ← our inner product.

when I see a Fourier series, $f(t) = \sum_n a_n \cos n\omega t$,
I think of this as "expanding f in the basis fn's $\cos n\omega t$"
the coefficients a_n are numbers, just like the "components" of a vector.
Just as $\{V_1, V_2, V_3\}$ completely + uniquely defines a vector in 3D
so $\{a_0, a_1, a_2, a_3, \ldots\}$ " " " " a function

<u>Recap</u>: Any τ-periodic fn $f(t)$ (with period $\tau = 2\pi/\omega$) can be uniquely written as

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$\hookrightarrow n=0$ is irrelevant, since $\sin 0 = 0$

you need to find those constants a_n and b_n, easy!

$$a_0 = 1/\tau \int_{-\tau/2}^{\tau/2} f(t)\, dt$$

$$a_n = 2/\tau \int_{-\tau/2}^{\tau/2} f(t) \cos n\omega t\, dt \qquad n \geq 1$$

$$b_n = 2/\tau \int_{-\tau/2}^{\tau/2} f(t) \sin n\omega t\, dt \qquad n \geq 1.$$

---

when I write $\vec{V} = \sum_{n=1}^{3} V_n \hat{e}_n$, I think of V_n as telling me "how much of $\vec{V}$ is in the $\hat{e}_n$ direction"

when I write $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$, I think of a_n as telling me "how much of the function $f(t)$ is like a <u>pure</u> sinusoid, $\cos(n\omega t)$".

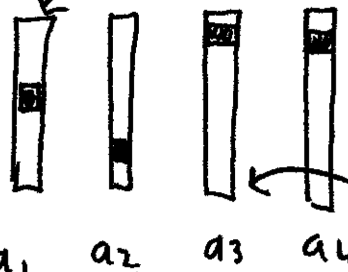
In music, different n's correspond to <u>harmonics</u>. E.g a singer singing concert A = 440 Hz = $\omega_0/2\pi$ produces a complex wave form $f(t)$

then a_1 tells me "How much is pure $\cos(\omega_0 t)$"

a_2 tells me "How much is the 1st overtone, $\cos(2\omega_0 t)$, at $f = 880$ Hz" (How strong)

etc.

On a stereo equalizer, each knob controls the strength of the a_n's.



I have some a_1 here, so some pure ω_0

I have a little a_2 here, not much $2\omega_0$

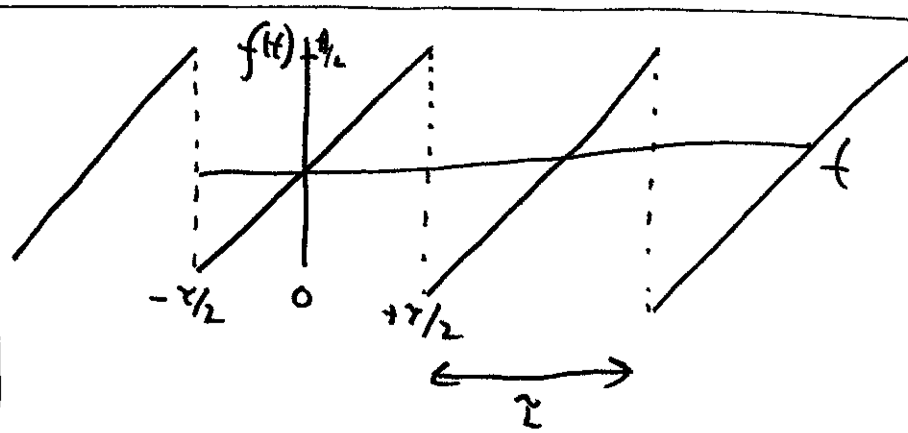
I have a lot of a_3, plenty of $3\omega_0$

Apparently I like "treble" here, + am adjusting the sound to emphasize the $n=3$ and 4 "high harmonics" of the base frequency

In this way, given an $\omega = 2\pi/\tau$, you can build up a complex periodic wave, with the same pitch (it's still periodic in τ seconds!) but with many overtones + a rich functional time dependence.

Example: Consider

$$\left[f(t) = A\frac{t}{\tau} \qquad -\frac{\tau}{2} < t < +\tau/2 \quad \text{repeating with period } \tau. \right]$$



- This is periodic, with period τ (not $\tau/2$, look at the graph!)
- This is certainly no sin wave! Not even close!

Fourier insists we can write it uniquely as

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t.$$

(Note: It's an odd fn, so I suspect the a_n's will all vanish! We'll see this shortly!)

Let's compute the b_n's, $b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin n\omega t\, dt$.

Here, we have $b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \frac{A}{\tau} t \sin n\omega t\, dt$.

Can integrate by parts! Or just let MMA do it, we get

$$b_n = \frac{2}{\tau}\cdot\frac{A}{\tau}\cdot\left[-\frac{t}{n\omega}\cos n\omega t + \frac{\sin n\omega t}{n^2\omega^2}\right]_{-\tau/2}^{+\tau/2}$$

But note $\omega = \frac{2\pi}{\tau}$, so we get $n\omega\frac{\tau}{2} \Rightarrow n\pi$

$$b_n = \frac{2A}{\tau^2}\left[-\frac{2\cdot\tau/2}{n\omega}\underbrace{\cos n\pi}_{\substack{=+1 \text{ for even } n \\ =-1 \text{ for odd } n}} + \frac{2}{n^2\omega^2}\underbrace{\sin n\pi}_{\text{always zero!}}\right]$$

$$b_n = \frac{2A}{2\pi}\cdot\frac{-1}{n}(-1)^n$$

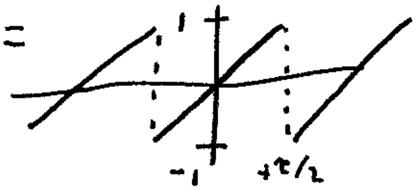
- you can do the a_0 and a_n integrals, in MMA, but no need to bother! Consider $a_n = \frac{2A}{\tau^2}\int_{-\tau/2}^{+\tau/2} \underbrace{t}_{\text{odd function}}\cdot\underbrace{\cos n\omega t}_{\text{even function}}\, dt$

But odd * even = odd,

and $\int_{-\tau/2}^{\tau/2} (\text{odd fn})\, dt = 0$!

Fourier -9-

Let's recap + see what we've got. $b_n = \frac{A}{\pi}\frac{(-1)^{n+1}}{n}$

Let $A = 2$, so $f(t) =$ 

$b_1 = \frac{2}{\pi}$, $b_2 = -\frac{2}{2\pi}$, $b_3 = +\frac{2}{3\pi}$, $b_4 = \frac{-2}{4\pi}$, etc...

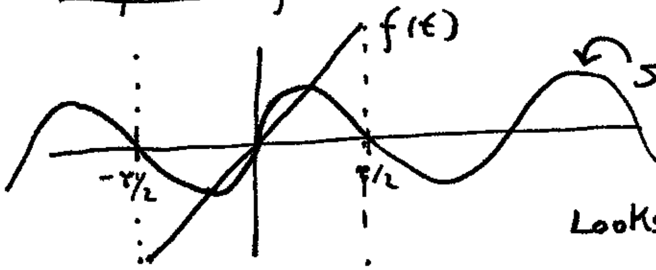
- As n grows, b_n shrinks. This is common - perhaps we only need a few terms to get a good approximation to $f(t)$!

- In general, $a_n \propto \int_{-\tau/2}^{+\tau/2} f(t) \underbrace{\cos n\omega t}_{\text{even}}$

  so if $f(t)$ is an odd fn, only get b_n's (sin functions)

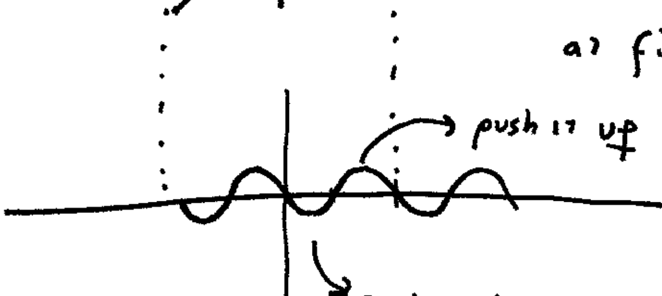
  if $f(t)$ is an even fn, only get a_n's (cos functions)

- We are sculpting $f(t)$ here out of pure $\sin(n\omega t)$ functions.

$n=1$ only: 

$\sin\omega t = \sin\frac{2\pi t}{\tau}$

Looks like this ↑ slightly overshoots at first, then undershoots. So, fix it up!

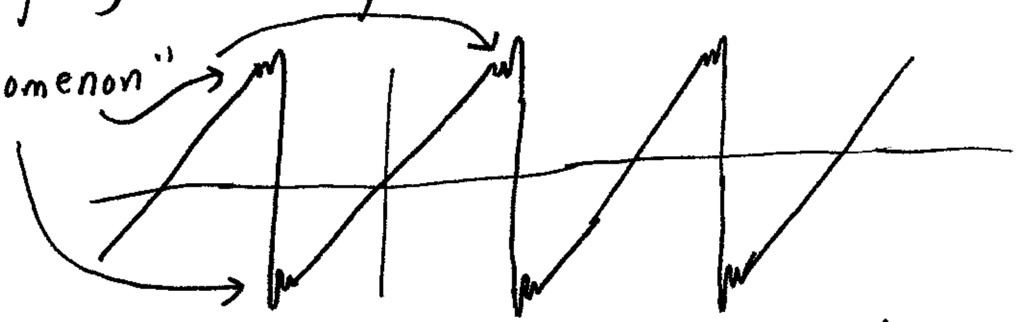
add in $n=2$ 

push it up a little here!

Pull it down a little, hence the negative b_2

Check out the PhET sim + try sculpting a bit yourself, you'll quickly see what those a_n's are doing!

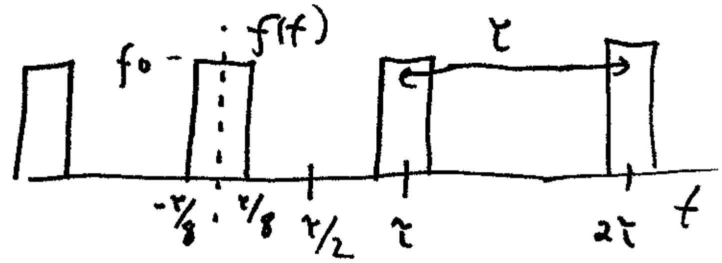
- Functions with discontinuities (like this example) turn out to generate some funny busines right <u>at</u> the discontinuities, the "Gibbs phenomenon"



If you truncate the series, there's a little "ringing" at the discontinuities. Even with $n \to \infty$, you can overshoot by ~9% " " " ! But the effect is localized to <u>just</u> the discontinuities, and of course in real life, nothing has true discontinuities!

Fourier -11-

Taylor works an example :



- It's an even function, he only gets a_n's. (No b_n's)
- It doesn't average to 0 (like $\cos(n\omega t)$ always does), so he must get an a_0 term!
- His a_n's also get smaller as n grows, slowly but surely.
- Even 4 or 5 terms is pretty good! 10 terms is a lovely fit!
- Printing 1.0 of Taylor's book has errors in the coefficients. Work it out, check for yourself! I get $\omega = 2\pi/\tau$ and

$$f(t) \approx f_0 \left(\frac{1}{4} + .45 \cos \omega t + .32 \cos 2\omega t + .15 \cos 3\omega t + 0 \cos 4\omega t + -.09 \cos 5\omega t + \cdots \right)$$

If this were a sound (pressure) wave, a_0 tells you there's some steady high pressure offset (not usually present in music!!)

This wave has a lot ($a_1 = .45$) of "fundamental", pure ω, and also quite a bit ($a_2 = .32$) of "1st overtone, pure 2ω, but progressively less & less higher harmonics

- The PhET sim can produce this & then play it for you (except $a_0 = 0$) so you can here & see the waveform!

Math digression, <u>Fourier's TRICK</u>. Where did that mystery formula for the a_n's + b_n's come from? Let's just focus on the a_n's: <u>Math Fact</u>

For integers n, m:

$$\frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} \cos n\omega t \cos m\omega t \, dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

you can easily prove/derive this yourself! Write $\cos n\omega t = \frac{e^{in\omega t} - e^{-in\omega t}}{2}$ and just do the integral!

I write δ_{nm} = "Kronecker delta" $\equiv \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$

Now, <u>assume</u> (a la Fourier) $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$

Fourier's <u>trick</u>: 1st multiply both sides by $\cos m\omega t$

$$f(t) \cos m\omega t = \cos m\omega t \sum_{n=0}^{\infty} a_n \cos n\omega t$$

$$= \sum_{n=0}^{\infty} a_n \cos n\omega t \cos m\omega t$$

$\cos m\omega t$ can be pulled into the sum, it's the <u>same</u> in each term of the sum

Next, integrate both sides: (with <u>same</u> limits + coefficient out front)

$$\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t) \cos m\omega t \, dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \sum_{n=0}^{\infty} a_n \cos n\omega t \cos m\omega t \, dt$$

Integral of sum = sum of integrals

$$= \sum_{n=0}^{\infty} \frac{2a_n}{\tau}\int_{-\tau/2}^{\tau/2} \cos n\omega t \cos m\omega t \, dt$$

My "math fact" from above

$$= \sum_{n=0}^{\infty} a_n \cdot \delta_{nm}$$

<u>every</u> term vanishes (!!) Except <u>one</u>.

$$= a_m$$

Fourier -11- 6

What I just got was

$$\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos m\omega t = a_m.$$

this is the magic formula we've been using (the dummy index is m here, but it's just a dummy!)

---

what just happened? The idea is, $\cos n\omega t$ and $\cos m\omega t$ are orthogonal functions, meaning the inner product $\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \cos n\omega t \cos m\omega t = 0$

So if we expand $f(t)$ in this basis set of orthogonal functions then the coefficient of $f(t)$ with one basis fn is the coefficient!

Just like, if $\vec{V} = \sum_{n=1}^{3} V_n \hat{e}_n$, then $V_n = \vec{V}\cdot\hat{e}_n$

similarly, if $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$, then a_n = inner product of $f(t)$ and $\cos n\omega t$.

Fact: $\cos n\omega t$ is orthogonal to $\sin m\omega t$ $\quad \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \underbrace{\cos n\omega t}_{\text{even}}\ \underbrace{\sin m\omega t}_{\text{odd}!} = 0$

Fact: $\sin n\omega t$ " " " $\sin m\omega t$ if $n \neq m$ $\quad \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \sin n\omega t \sin m\omega t = \delta_{nm}$

this gives the b_n formula!

Fact: $\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \cos n\omega t \cdot 1\, dt = 2\delta_{n,0}$, (this is why the funny factor of $\frac{1}{2}$ differs for the a_0 formula)

Let's use Fourier series, then, to solve the general driven oscillator.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$ (with $f(t)$ a τ-periodic fn. (assumed even for simplicity!)

- Fourier says $f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t$ (and we know a formula for each a_n here!)

- But we already solved

$$\ddot{x}_n + 2\beta\dot{x}_n + \omega_0^2 x_n = a_n \cos n\omega t.$$

Remember? It has a "homogeneous" part that dies off like $e^{-\beta t}$, and leaves behind the "particular" response (we've just got a driving freq $n\omega$ instead of ω)

$$x_n(t) \underset{\substack{\text{large times,}\\ \text{after transients}\\ \text{die away}}}{=} A_n \cos(n\omega t - \delta_n)$$

where
$$\begin{cases} A_n = a_n \Big/ \sqrt{(\omega_0^2 - (n\omega)^2)^2 + 4\beta^2 (n\omega)^2} \\ \delta_n = \tan^{-1} 2\beta n\omega / (\omega_0^2 - (n\omega)^2) \end{cases}$$

By the linearity of our ODE, $x(t)$ is given very simply

$$x(t) \underset{\text{large } t}{=} \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$$

($n=0$ works just fine - check it for yourself!)

Summary:

- Given a driver $f(t)$, write it as a Fourier series (this means compute all a_n's & b_n's. We have definite integrals to do, but they can always be numerically computed!)
- For each term, calculate A_n and δ_n. (these are a little ugly, but well-defined, formula on previous page) (If you have b_n's, you'll need to go back + take $\mathrm{Im}(z)$ when we first solved the driven SHO.)
- Add the sol'ns back up, to get the sum.

Most realistic (periodic) functions $f(t)$ will need only a few terms. This does seem like a job for a computer, but it gives excellent approximations, often needing only a couple terms

Comments:

(1) Energy of an oscillator (freq ω_n) is given (after transients die off!) by a steady $\frac{1}{2} K A_n^2$. Taylor then shows that if you drive an oscillator with a periodic $f(t)$

$$\frac{1}{2} K \underbrace{\langle x^2 \rangle}_{\text{time average}} = \frac{1}{2} K \left[A_0^2 + \sum_{n=1}^{\infty} \frac{1}{2} A_n^2 \right]$$

So knowing fourier coefficients $\Rightarrow$ A_n's $\Rightarrow$ energy of resulting response

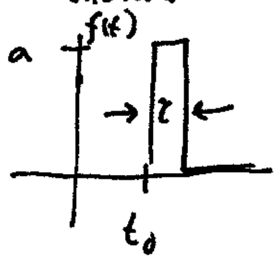
(2) If driver, $f(t) = \sum_n a_n \cos n\omega t$ needs many terms, and your system has a resonant frequency ω_0, then the "response" will be dominated by that one term in $f(t)$ where $n\omega \approx \omega_0$.

So e.g. if your driver has a low freq, $\omega \ll \omega_0$, you might not expect much response. But because of harmonics, one of them is likely to be close, $N\omega \simeq \omega_0$ for some N, so you may get a response after all! (you will respond at $N\omega$, not at ω. Response is $\approx$ resonant freq!)

Ex: If you push a kid on a swing at low f, say once every 3 swings, she can still get going ... at the natural resonant frequency. It's not as good (she'll complain) because she's only picking up your "overtones" ... but the system will still "resonate".

Finally, what if $f(t)$ is not periodic? It turns out we can solve for the response from any driver! We won't work out the details this term, but here's the basic logic:

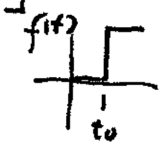
- Consider 1st how an underdamped system responds to this driver:



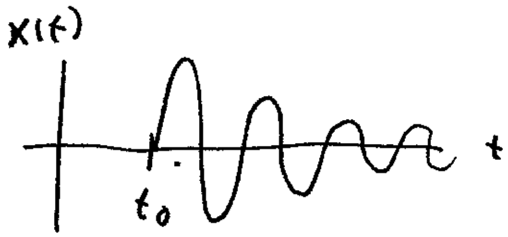
You can solve this. If $x = \dot{x} = 0$ before this "impulse", + if τ is short, then (with $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ as usual)

$$\underset{(t>t_0)}{x(t)} \approx \frac{a\tau}{\omega_1} e^{-\beta(t-t_0)} \sin\omega_1 (t-t_0)$$

Proof is a few short steps. Think about solving for a "step up" force first

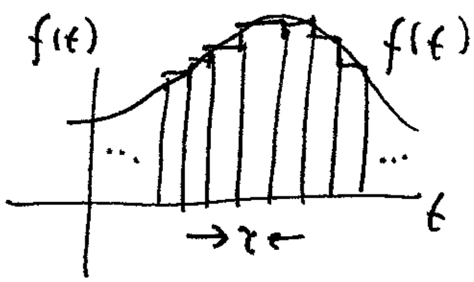


, and superpose with sol'n to "step down" force



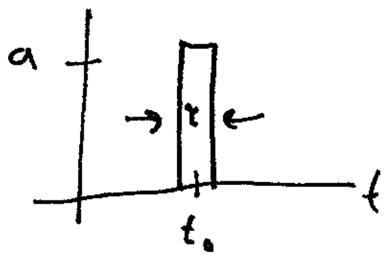
↓ Whack an oscillator, it rings + then dies away, nothing unusual here.

But any function $f(t)$ is just a superposition of such "little impulses"



So you can find the response to non-periodic drivers too. This is the method of "Green's functions", and involves Fourier Transforms rather than Fourier series. (our sum of $a_n \cos n\omega t$ turns into a continuous integration over all ω, i.e. $\int_{-\infty}^{\infty} A(\omega) \cos\omega t \, d\omega$)

- We won't pursue this (wait a semester!), but I do want to follow up on this "short impulse" function idea!

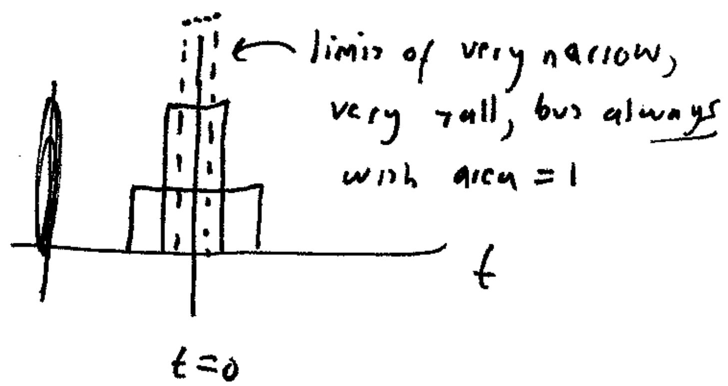


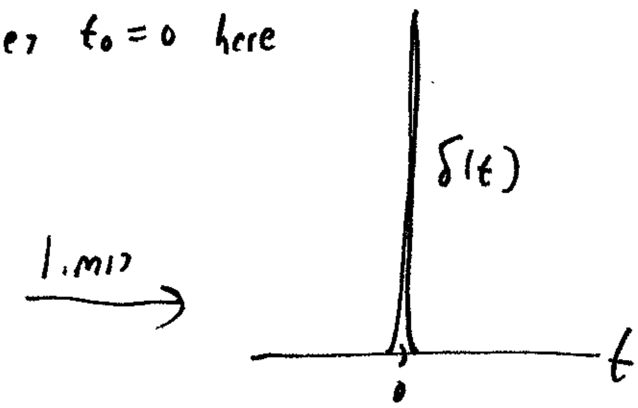
Consider a little quick impulse $f(t)$ as shown, in the limit that $\tau \to$ small (very quick!)

$a = 1/\tau$ (very strong!)

Note that the impulse $\equiv \int F(t)\,dt = a \cdot \tau = \frac{1}{\tau}\cdot\tau = 1$ is finite.

Now take the limit $\tau \to 0$. Let's let $t_0 = 0$ here





The Dirac Delta function, $\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0, \end{cases}$

such that $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$\delta(t)$ is not a legitimate mathematical fn, but is very useful, + integrals involving it are not problematic!

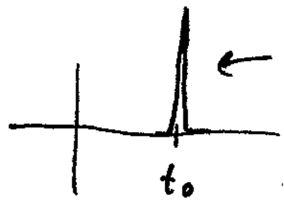
This function was introduced on previous page, but it has many applications. Let's investigate it just a bit more...

I claim $\int_{-a}^{+a} \delta(t)\,dt = 1$ ← For any a!

Limits of integration are irrelevant, since $\delta(t)$ is so narrow! It has area 1, as long as you integrate over the "spike"

Similarly, $\int_{a}^{3a} \delta(t)\,dt = 0$ because these limits don't "catch" the spike.

I argue $\delta(t-t_0)$ looks like this



← area is still 1! It's just shifted.

Now consider $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt$. The integrand is zero for all t, except the blip at $t=a$. So $f(t)$ is irrelevant except at $t=a$!

$$\text{so} \quad \int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = \int_{-\infty}^{\infty} f(a)\,\delta(t-a)\,dt = f(a)\int_{-\infty}^{\infty} \delta(t-a)\,dt = f(a)$$

Integrating $f(t) * \delta(t-a)$ "catches" the value of $f(a)$!

Physics: In 1-D, the charge density λ of an electron at $x=1$ would be $\lambda(x) = -e\,\delta(x-1)$.

Why? Physically, $\lambda = 0$ everywhere except $x=1$, + is infinite there (electrons are points!). Total charge, however, is

$$Q = \underbrace{\int_{-\infty}^{\infty} \lambda(x)\,dx}_{\text{as usual!}} = -e \underbrace{\int_{-\infty}^{\infty} \delta(x-1)\,dx}_{\text{one!}} = -e, \text{ as it should be!}$$

What is $\delta(Kt)$, where K is, say, a positive constant?

Think about $\int_{-\infty}^{\infty} f(t)\,\delta(Kt)\,dt$ for <u>any</u> function $f(t)$ at all!

Do a "u-sub", $u = Kt$, so $du = K\,dt$, and this integral is just

$$\int_{-\infty}^{\infty} f\left(\frac{u}{K}\right)\delta(u)\,\frac{du}{K} = \frac{1}{K} f\left(\frac{0}{K}\right)\int_{-\infty}^{\infty}\delta(u)\,du = \frac{f(0)}{K}$$

This is <u>identical</u> to $\int_{-\infty}^{\infty} f(t)\cdot\frac{1}{K}\delta(t)\,dt = \frac{f(0)}{K}$.

Since these integrals are equal for <u>any</u>/<u>all</u> functions, we equate the integrand!

So $\delta(Kt) = \frac{1}{K}\delta(t)$ if $K > 0$.

If $K < 0$, the u-sub changes the <u>limits</u> to $\int_{+\infty}^{-\infty}$, this is a sign flip!

so $\delta(Kt) = -\frac{1}{K}\delta(t)$ if $K < 0$.

or, $\boxed{\delta(Kt) = \frac{1}{|K|}\delta(t)}$

Units of $\delta(t)$? Well, $1 = \int_{-\infty}^{\infty} \underbrace{\delta(t)}_{\text{units?}}\,\underbrace{dt}_{\text{time}}$. Clearly $[\delta(t)] = \frac{1}{\text{time}}$!

We'll explore $\delta(t)$ many times in future classes!

So far, we've focused on physics problems involving ODE's: Equations for $y(x)$, functions of 1 variable. But many (most!) physics problems are richer than this. The unknown function you're after may depend on many variables.

E.g. Electric field $\vec{E}(x, y, z)$ or even (x, y, z, t)
(or Voltage, or temperature, or force, or velocity, or ...)

The eq'ns describing this function will thus involve partial derivatives, $\dfrac{\partial f(x,y,z,t)}{\partial t}$ for instance. (A partial differential eq!)

Examples of these include:

$$\frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0.$$

we abbreviate $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \equiv \nabla^2$, and write this

$$\nabla^2 V_{(x,y,z)} = 0$$ this is called "Laplace's equation".

It holds for Voltage in charge-free regions
or Temperature in steady-state with no sources
+ many other physical systems.

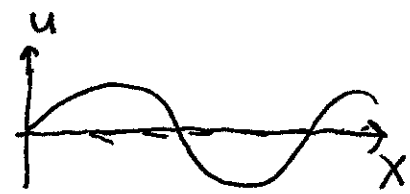
PDE - 2.

In E+M, you'll derive that eq'n (from Gauss' Law!), as well as

$$\nabla^2 V(x,y,z) = -\frac{1}{\varepsilon_0}\,\rho(x,y,z)$$ "Poisson's Equation"

$\rho \hookrightarrow$ charge density

If $u(x,t)$ is the sideways displacement of a little string at position x, time t,



then

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

the wave equation

See Taylor Ch. 16 for a derivation

in 3-D, you can have waves, the eq'n is

$$\nabla^2 u(x,y,z,t) = \frac{1}{v^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2} \longleftrightarrow \text{3D wave eq'n}$$

In Quantum, the wave function $\Psi(x)$ satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(x,y,z,t) + U(x,y,z)\,\Psi(x,y,z,t) = i\hbar\frac{\partial\Psi(x,y,z,t)}{\partial t}$$

Schrödinger's Eq'n.

Each has a story, you'll solve these (+ more) many times over in upcoming classes. The sol'n is not as easy as ODE's, the "boundary condition" can strongly impact how (+ whether!) you can solve it.

We're going to pick an example to see a common, <u>general</u> approach that works for many of the above PDE's.

The <u>Heat</u> Equation (or "Diffusion eq'n") is

$$\nabla^2 T(x,y,z,t) = \frac{1}{\alpha^2}\frac{\partial T(x,y,z,t)}{\partial t}$$

T = temperature.

Derivation sketched on next page.

It's a PDE, those are <u>all</u> partial derivatives. written out,...

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t} \longleftrightarrow \text{ in 3-D}$$

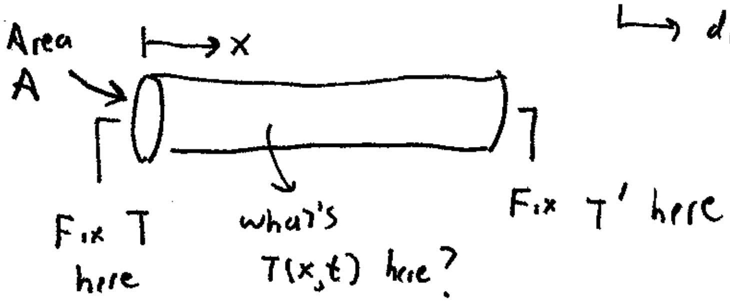
$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T(x,t)}{\partial t} \longleftrightarrow \text{ in 1-D}$$

We follow Boas ch.13 here, + use a common sol'n approach called <u>SEPARATION</u> of <u>VARIABLES</u>. *(we used that same name for a method of solving 1st order ODE's. This is totally different!)*

The heat equation describes how temperature varies over <u>time</u> and also over different <u>positions</u> for an ordinary solid object.

→ The eq'n can represent <u>other</u> physics too, like neutrons <u>diffusing</u> through a material. (If you look carefully, it's mathematically <u>very</u> similar to the Schrodinger eq'n too!) Anything that <u>diffuses</u> will obey an ODE like this, (here it's <u>heat</u> that diffuses)

As a simple concrete example, consider a 1-D solid rod. If the ends are held at different temperatures, there will be a distribution $T(x, t)$ → depends on position; → depends on time

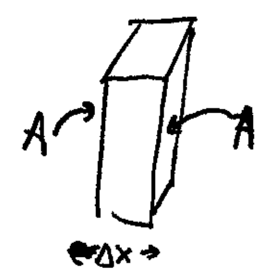


I won't derive the heat eq'n, but will motivate it, + then we'll solve it (in a couple of pages from here...)

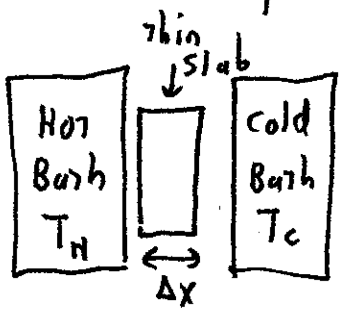
I want to start with a claim about heat flow in steady state

Define $H(x,t) \equiv \dfrac{\text{Amount of thermal energy passing by}}{\text{sec}}$, (in 1-D)

Then I claim $H(x,t) = -KA\, \partial T/\partial x$. → Some "thermal conductivity constant"



This is an experimental eq'n. A is the cross sectional area of our slab



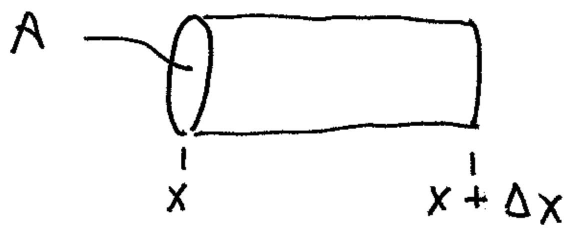
Here, $H(x,t) = \dfrac{\text{Joules passing}}{\text{sec}} = -KA \dfrac{\Delta T}{\Delta x}$

Makes sense! Big Area or $\Delta T \Rightarrow$ big heat flow.

Small/thin slab, $\Delta x \to 0 \Rightarrow$ big heat flow

Minus sign just says heat flows opposite ΔT, i.e. hot towards cold!

Now, consider our 1-D rod <u>not</u> yet in steady state:



Thermal energy flows <u>in</u> the left, that's $H(x)$ each sec

Thermal energy flows <u>out</u> the right, that's $H(x+\Delta x)$. each sec

Since $H = \dfrac{\text{thermal energy}}{\text{sec}}$, then $H\,\Delta t$ = energy passing through.

I claim $\underbrace{H(x)\,\Delta t}_{\text{energy in at left}} - \underbrace{H(x+\Delta x)\,\Delta t}_{\text{Energy out at right}}$ = energy <u>buildup</u> inside, in time Δt.

How can this be? If this is not zero, there's a <u>net</u> inflow of energy, + the rod <u>must</u> [redacted] get hotter! Remember <u>heat</u> capacity?

Heat Capacity $C = \dfrac{\text{Joules input}}{\text{mass} * (\text{change in temperature})} = \dfrac{J}{Kg \cdot {}^\circ C}$

or $\Delta T = \dfrac{1}{C} * \dfrac{\text{"Joules in"}}{\text{mass}}$. For our rod, above,

mass $= \rho * \text{Volume} = \rho * A * \Delta x$ (see figure)

and "Joules in" $= [H(x) - H(x+\Delta x)]\,\Delta t$ (see above)

(↳ I'm assuming no <u>other</u> heat sources!)

so $\Delta T = \frac{1}{c \rho A \Delta x} \cdot \Delta t \cdot \Big(H(x) - H(x + \Delta x) \Big)$

$$\Delta T = -\frac{\Delta t}{c \rho A} \cdot \left(\frac{H(x+\Delta x) - H(x)}{\Delta x} \right)$$ (This is just $\partial H / \partial x$)

Diving out Δt, + using $H = -KA \, \partial T / \partial x$ from 2 pages ago

so $\partial H / \partial x = -KA \, \partial^2 T / \partial x^2$

$$\frac{\partial T}{\partial t} = -\frac{1}{c \rho A} \cdot -KA \frac{\partial^2 T}{\partial x^2}$$

This is what we wanted, an eq'n for T, (eliminating the function H!)

$$\boxed{\frac{\partial T}{\partial t} = \frac{K}{\rho c} \frac{\partial^2 T}{\partial x^2}} \equiv \alpha^2 \frac{\partial^2 T}{\partial x^2} \qquad \text{with } \alpha = \sqrt{\frac{K}{c\rho}}$$

In 3-D, you can guess the generalization, $\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$

If you let things settle down + reach steady state, $\frac{\partial T}{\partial t} = 0$

+ you get

$$\nabla^2 T = 0$$ ← Laplace's eq'n, again.

(steady state)

The same eq'n that appears in electrostatics, (the math is identical!)

using $H = -KA \frac{\partial T}{\partial x}$

(For our 1-D rod in steady state $\Rightarrow H(x) = H(x+\Delta x) \Rightarrow \frac{\partial H}{\partial x} = 0$ so $\frac{\partial^2 T}{\partial x^2} = 0$)

So the PDE we must solve is $\nabla^2 T(x, y, z) = 0$ in steady state.

Looks simple. But, it's very rich + complicated! In fact, there is no generic one-size-fits all sol'n to this PDE! $T(x,y,z)$ depends not just on the eq'n $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$, but also on boundary conditions! (This is very different from ODE's!)

Let's look in lower dimensions first to learn something.

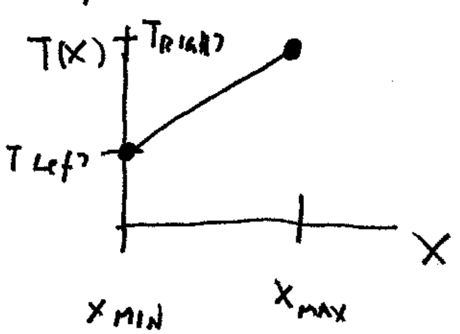
In 1-D, we have $\frac{d^2 T(x)}{d x^2} = 0$.

- Because we're in steady state, time t is gone.
- Because we're in 1-D, only x is variable

This is actually kind of trivial, because now there is only 1 variable, + we're back to an ODE!

Here, there is a generic sol'n, $T(x) = \underbrace{a + b}_{\text{2 undetermined constants}} x$

For ODE's, the form of sol'n is known, and Boundary conditions simple tell you what constants are. Here,



Rod temperature varies smoothly (linearly) from cold side to hot side.

[Voltage between 2 capacitor plates is same story! $\frac{d^2 V}{dx^2} = 0$]

What about 2-D? Imagine e.g a metal plate, with the edges set at fixed temperature distributions. In steady state, what is $T(x,y)$ everywhere else in the plate?

well, $\nabla^2 T(x,y) = 0$, i.e $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$

There is no generic form to solve this! you could imagine all sorts of complicated functions to try out, but if the boundary conditions are nice, we may find that

$$T(x,y) = (\text{some fn of } x) * (\text{some other fn of } y)$$

Like, say, $e^x * \cos y$ or something. (This works, it's just not very general!)

If that fails, maybe a simple linear combination of such functions like, say, $e^x \cos y + .337\, e^{2x} \cos 2y - 1.7\, e^{3x} \cos 3y + \dots$?

↳ This way, we could "build up" a lot of very complex functions!

So this will be our procedure, The Method of Separation of Variables, where we guess (hope!) that perhaps

$$T(x,y) = \mathcal{X}(x) * \mathcal{Y}(y) \quad \text{or, some sum of such fns}$$

a fn of x only * a fn of y only.

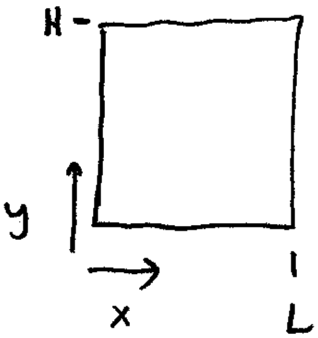
Remember, our goal is 2-fold:

a) Find $T(x,y)$ such that $\nabla^2 T(x,y) = 0$

b) $T(x,y)$ must satisfy our particular, given boundary condition.

This method (postulating $T = X(x)\,Y(y)$) won't always work, but it often will, + is quite general + powerful. This approach can also be used for all the ODE's I listed on pp. 1-2)

Let's pick a concrete example where this works.



Consider our 2-D metal plate, $\nabla^2 T(x,y) = 0$

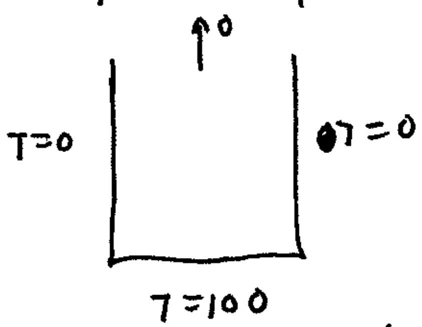
we want $T(x,y)$ (steady state).

As a first example, let us refrigerate ~~three~~ two sides (left + right, ~~[illegible]~~), fix $T = 0^\circ$ on those edges. Let's also fix $T = f(x)$ along the bottom. This could be anything, it's our choice. A simple case might be to put boiling water along this edge, so $T(x, y=0) = 100^\circ$. (But we can solve more complicated cases too.)

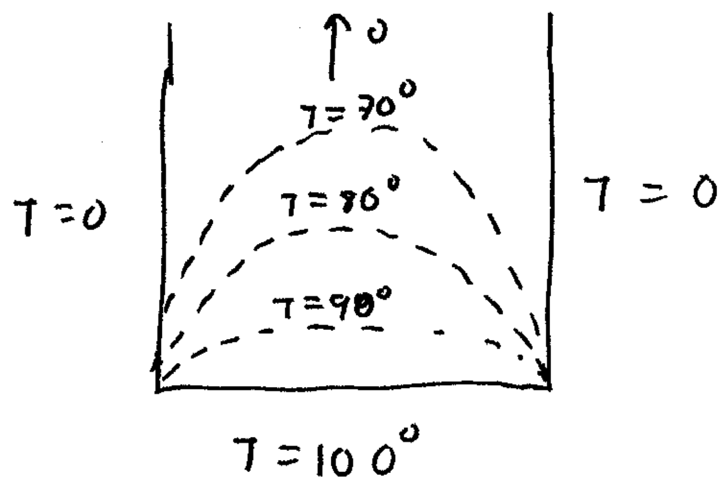
For the top, let's begin with a tall plate, so Height $H \to \infty$. (I'll assume the top is like the 2 sides, at 0°, just very far away)

PDE - 9.5

Before we proceed, what do we expect, physically?



For large y, I expect $T \approx 0$ everywhere. It's refrigerated all around! But, near the bottom, heat will flow from the hot base towards the cool walls. In steady state, I expect strong temp variation down there. Just guessing I predict some "equi-temperature" lines that might look like this:

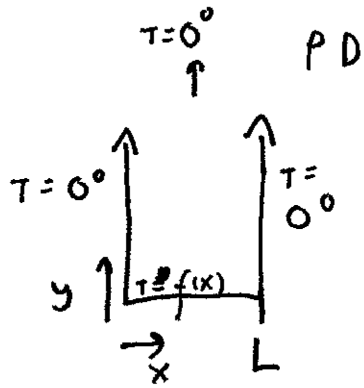


- I expect left/right symmetry
- I expect slow cooling as y increases up the middle
- Rapid variation near corners

the 2 bottom corners are odd, some sort of discontinuity there. (This is an artifact of my choice of $T = 100°$ at base + $T=0$ on sides, it is discontinuous at the corner! In real life, we wouldn't have this, and I expect to see some artifact of the discontinuity in our sol'n)

So, let's now do the math to get a formula for $T(x,y)$

such that $\begin{cases} \nabla^2 T = 0 & \text{everywhere} \\ T \text{ (at Boundaries)} = \text{what we specified here.} \end{cases}$



$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

what is $T(x,y)$? Let's try separation of v'bles

so assume (hope!) $T(x,y) = \mathcal{X}(x)\,\mathcal{Y}(y)$ + see what happens!

1) $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{d^2 \mathcal{X}(x)}{dx^2}\,\mathcal{Y}(y)$ → $\mathcal{Y}$ is a constant as far as $\dfrac{\partial}{\partial x}$ is concerned

↳ this is a regular deriv, since $\mathcal{X}$ depends only on x

2) $\dfrac{\partial^2 T}{\partial y^2} = \mathcal{X}(y)\,\mathcal{Y}''(y)$ ← simpler notation, same as $\dfrac{d^2 \mathcal{Y}(y)}{dy^2}$.

So my PDE is $\mathcal{X}''(x)\,\mathcal{Y}(y) + \mathcal{Y}''(y)\,\mathcal{X}(x) = 0$.

Key trick in sep. of v'bles: Divide both sides by $\mathcal{X}(x)\,\mathcal{Y}(y)$!

Leaving $$\underbrace{\frac{\mathcal{X}''(x)}{\mathcal{X}(x)}}_{\text{a fn only of } x} + \underbrace{\frac{\mathcal{Y}''(y)}{\mathcal{Y}(y)}}_{\text{a fn only of } y} = 0$$

a fn only of x + a fn only of y $= 0$. For all x and y!

Huh? This looks nuts! x and y are independent! I can pick an x and vary y, and this eq'n says I always get 0! How can that be? It cannot, unless these "functions" don't depend on x or y!

so $\frac{\mathcal{X}''(x)}{\mathcal{X}(x)} + \frac{\mathcal{Y}''(y)}{\mathcal{Y}(y)} = 0$

requires

this is some constant $+c$

(It can't depend on x!)

this is some constant, must be $-c$.

If both are true... we have a sol'n!

"c" is called the separation constant.

so $\mathcal{X}''(x) = c\,\mathcal{X}(x)$. Well, I recognize this, it's an ODE.

The sol'n is familiar, $\mathcal{X}(x) = a_1 e^{\sqrt{c}\,x} + a_2 e^{-\sqrt{c}\,x}$

If $c > 0$, pure exponentials

If $c < 0$, pure sin's + cos's.

To have $\nabla^2 T = 0$, need Same constant, opposite sign!

At same time, $\mathcal{Y}''(y) = -c\,\mathcal{Y}(y)$

I Know this ODE too,

$\mathcal{Y}(y) = a_3 e^{\sqrt{-c}\,y} + a_4 e^{-\sqrt{-c}\,y}$

Here, If $c > 0$, pure sin's + cos's

If $c < 0$ pure exponentials.

Now, remember our particular problem. Boundary conditions are needed to proceed!

For our specific problem, we said $T(x,y) \to 0$ as $y \to +\infty$

sin's + cos's (y) wiggle, they don't settle down to 0.

$e^{-(\text{something})y}$ does what we want, it goes to 0 as $y \to +\infty$.

So for this specific problem with these particular boundary conditions, looks like we need $C<0$ to give us the exponential fn in y.

So let's rename our constant $C = -K^2$

↳ so it's obviously neg!

(This works out nicely, because it gives sin's + cos's in $X(x)$, which is needed to get T to vanish at two sides!)

So $X(x) = a_1 \sin Kx + a_2 \cos Kx$

$Y(y) = a_3 e^{+Ky} + a_4 e^{-Ky}$

↳ our B.C. as $y \to \infty$ also tells us $a_3 = 0$! otherwise $Y(y)$ would blow up.

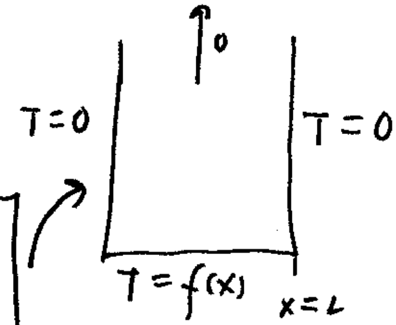
Now remember, $T(x,y) = X(y)\,Y(y)$, so we have

$T(x,y) = (a_1 \sin Kx + a_2 \cos Kx)\, e^{-Ky}$. This is our TRIAL sol'n

No need to include a_4 any more, just absorb it into a_1 and a_2.

This satisfies $\nabla^2 T = 0$, and our Bound. Condition as $y \to +\infty$.

PDE -13-



We have more BC's !

[we must have $T(x,y)=0$ for all/any values of y, whenever $x=0$]

the left edge B.C.

Plug in $x=0$, to get

$$T(x=0,y) = (a_1 \cdot \sin(0) + a_2 \cos(0))\, e^{-Ky} = 0$$

How can that be true for all y? $a_2 e^{-Ky} = 0$? for any y?!

only if $a_2 = 0$!

So if $a_2=0$, $T(x,y) = a_1 \sin Kx\, e^{-Ky}$ is our trial sol'n.

What about the right wall, where $x=L$? We need $T(x=L, \text{any } y)=0$

So $T(L,y) = a_1 \sin KL\, e^{-Ky} = 0$ for all values of y.

Could try $a_1 = 0$, but that's a FAIL, because then

$T(x,y) = 0$. Doesn't work at the bottom edge! And, is awfully trivial!

Are we stuck, did we fail? We're just trying to find something that works. Remember, K was a separation constant, we don't yet know what it is. Let's choose it, so that $\sin(KL)=0$.

i.e, pick K so that $KL = n\pi$. Any n (integer!) will work.

Let's label it $K_n \equiv \dfrac{n\pi}{L}$. Many different K's all work!

So check it out: Any trial function of the form

$$T_n(x,y) = a \sin K_n x \, e^{-K_n y}, \quad \text{with} \quad K_n \equiv n\pi/L$$

solves $\nabla^2 T = 0$ (by construction)

+ satisfies 3 of our 4 BC's. (Left, right, + "top" are all good)

Any integer n gives a different, yet valid sol'n.

we still need to satisfy our B.C. at bottom, $T(x, y=0) = f(x)$ ← given.

Useful observation:

If $T_1(x,y) = X_1(x)\, Y_1(x)$ solves $\nabla^2 T_1 = 0$

then so does $c_1 X_1 Y_1$ (for any c_1, ← this is linear!)

If $T_2(x,y)$ solves $\nabla^2 T_2 = 0$, then

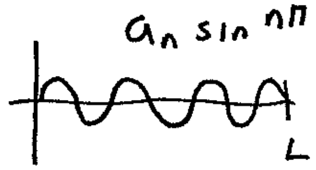
$$\nabla^2 (C_1 T_1 + C_2 T_2) = \nabla^2 C_1 T_1 + \nabla^2 C_2 T_2 = 0 \text{ too!}$$

So, if we have multiple valid sol'ns T_n, we can always

form $\sum_{n=1}^{\infty} C_n T_n(x,y)$, + this too will satisfy $\nabla^2 T = 0$!

The $T_n(x,y)$ at the top of the page gives, all by itself

$T_n(x,0) =$ $a_n \sin n\pi x/L$



If our given $f(x) = T(x,0)$ were a pure sin fn, like $f(x) = \sin \frac{17\pi x}{L}$ we'd be done. (Just pick $n = 17$)

But if not, we're ok, build what we want!

$$T(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-\frac{n\pi y}{L}}$$

we can choose these at will!

By linearity, this combo still satisfies $\nabla^2 T = 0$. And, convince yourself, it also still satisfies ALL 3 other boundary c's.

we will pick them to satisfy B.C.#4,

$$T(x,0) = \underbrace{f(x)}_{\text{given}} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

since $y=0$, $e^{-\frac{n\pi y}{L}} = 1$ in every time!

This is a Fourier sum. I know how to find these constants.

Remember? $a_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

Before, we called these the b_n's. Also, before we integrated from $-T/2$ to $T/2$, but this is fine, it's just a shift of origin.

Check it out: We found our sol'n! $\nabla^2 T(x,y) = 0$, + we satisfy all B.C.'s. And, here's some joy, there is a uniqueness theorem that says if we solve Laplace's eq'n + our B.C's, there is no other sol'n. We're done!!

Suppose e.g. $f(x) = 100°$ along the base. So, we need

$$T(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = 100$$

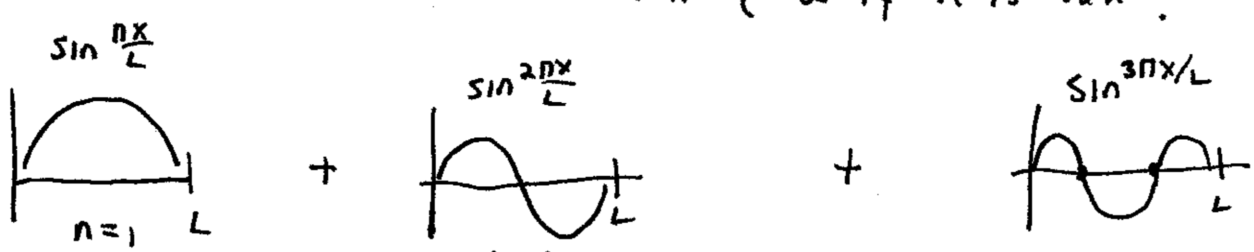
thus

$$a_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\cdot 100\,dx = \frac{200}{L}\cdot\frac{L}{n\pi}\left.-\cos\frac{n\pi x}{L}\right|_0^L$$

$$= \frac{200}{n\pi}\left(1-\cos n\pi\right)$$

$$= \frac{200}{n\pi}\begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd}\end{cases}.$$

Add up:

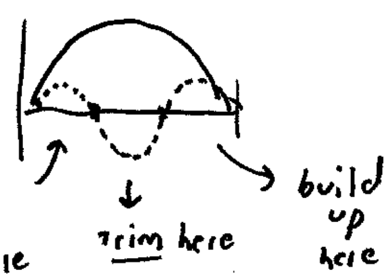


Lots of this, $a_1 = 400/\pi$

None of this! Bad symmetry, why would left + right half be different? $a_2 = 0$!

Some of this, but less 'cause of $\frac{1}{n}$ factor

when add



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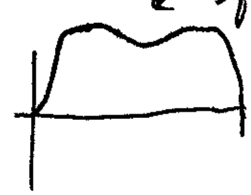
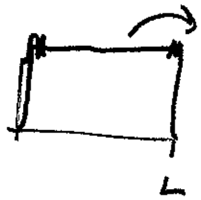


squared off the top a bit!

After many terms, we get



constant, $T = 100$.

Gibbs "rings" at edges, 'cause of our ugly discontinuous corners.

Don't forget, that was all just to find the a_n's by looking at the boundary, $y=0$. The full sol'n is

$$T(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\, e^{-n\pi y/L}$$

$$\boxed{T(x,y) = \sum_{\substack{n=1 \\ (\text{odd } n \text{ only!})}}^{\infty} \frac{400}{n\pi} \sin\frac{n\pi x}{L}\, e^{-\frac{n\pi y}{L}}}$$

Full sol'n $\forall\, x,y$.

Mustn't forget this!

This does produce the temp pattern we expected back on p. 9.5

- It's left-right symmetric because of $\sin\frac{(\text{odd } n)\pi x}{L}$
- It dies off (exponentially) as you climb in y.
- A few terms is probably enough to get a good approximation

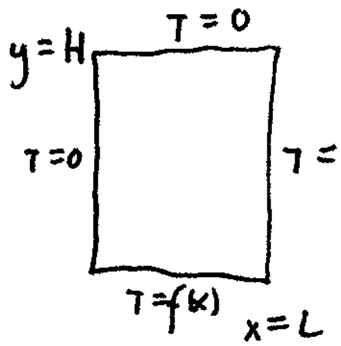
This solved $\nabla^2 T = 0$ for very specific, (artificial!) boundary c's.

Let's consider some more cases.

- If $f(x)$ is something besides $100°$ along the base, no problem.

  Just recompute the $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$, that's all that changes.

• what if it had a finite height, $y = H$, with $T = 0$ at the top?



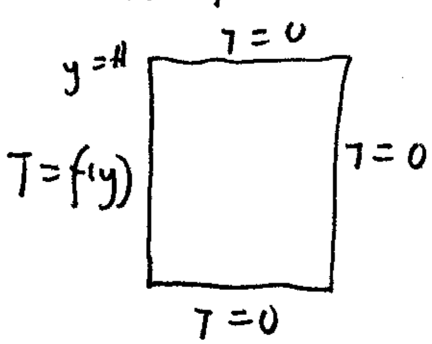
Since $T(x=0) = T(x=L) = 0$, we must have $X(x)$ = sinusoidal. ($ae^{Kx} + be^{-Kx}$ never vanishes twice along x, for any a, b.)

So, we still want our separation constant $= -K^2$

so $X''(x) = -K^2 X(x)$ gives $a_1 \sin Kx + a_2 \cos Kx$

But now, $Y(y) = a_3 e^{+Ky} + a_4 e^{-Ky}$, we need both terms to ensure $Y(H) = 0$. Otherwise, the process is the same, + we can solve this problem pretty much the same as before.

• What if we had a B.C. on the left wall that was the non-zero one?



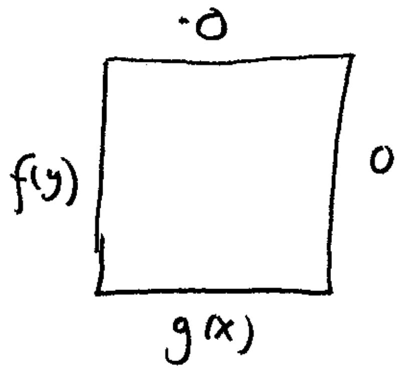
Now I want T to vanish for two y-values so I need the separation constant to have the other sign, giving

$X''(x) = +K^2 X(x) \rightarrow$ these give e^{Kx} and e^{-Kx}

$Y''(y) = -K^2 Y(y) \longrightarrow \sin Ky$ and $\cos Ky$

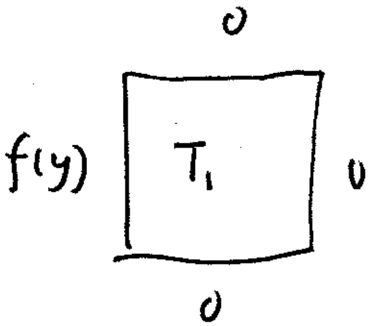
It's like what we did, but swapping $x + y$!

• What if

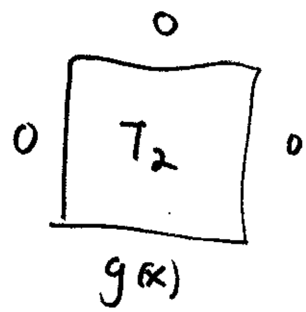


? Use superposition!!

the idea would be to solve

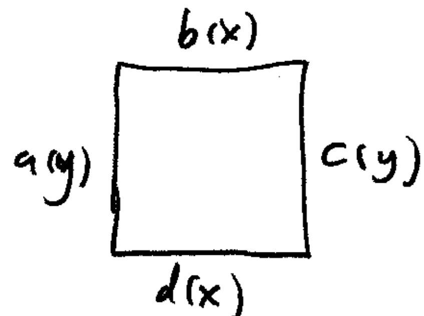


and <u>add</u>



, $T_1 + T_2$ will work!

So we can in fact solve



By summing 4 sol'ns!

Pretty <u>general</u> problem ↗ !

So our method is really more robust/general than it may have 1st appeared.

<u>Recap</u> To solve $\nabla^2 T(x,y) = 0$ (or any other PDE!)

1) Hope that $T(x,y) = X(x)\,Y(y)$ (just try!)

2) Plug this into the ODE, (partials all become "total derivatives")

Divide through by T, + discover you have <u>separate</u> <u>ODE's</u> with some new (as yet undetermined) separation constants

$X''(x) = +k^2 X(y)$ $\qquad\qquad$ $Y''(y) = -k^2 Y(y)$

a constant to be determined

the <u>same</u> constant, but opposite in sign, to make $\nabla^2 T = 0$.

Pick the <u>sign</u> of the constant depending on whether you B.C.'s <u>need</u> $X(x)$ to be sinusoidal, $Y(y)$ exponential, or vice versa

3) Solve the <u>separate</u> ODE's, + make sure $T(x,y)$ satisfies your B.C.'s (one by one)

This will fix your separation constant (there may be <u>many</u> options) and most other ODE constants

4) you can <u>sum up</u> valid sol'ns (superposing) if that helps!

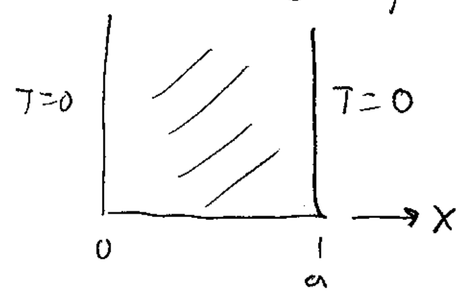
Fourier TRANSFORMS

We first encountered Fourier Series when we had a periodic

$f(t) = \sum_n b_n \sin n\omega_0 t \qquad (\omega_0 = 2\pi/T)$

We encountered them again, just now when solving Laplace's Eq'n with a rectangular plate:

The periodic bounary condition of our example problem



led to solutions of the form $X(x) = \sin\frac{n\pi x}{a}$, and thus to a Fourier series.

In Fourier series, there is a "base" frequency ω_0, and an infinite set of other higher frequencies $n\omega_0$, but not all frequencies are present.

- What if that plate was not finite in x, but extended forever?
- Or similarly, what if our oscillator is driven by an $f(t)$ that has "infinite period" (i.e, it is not periodic)

How do we deal with such situations? Since $\omega_0 = 2\pi/T$, then if $T \to \infty$, it hints that perhaps there is no "base ω_0", + we will need to start from $\omega = 0$, and include all ω's.

This leads to an integral over ω, not a sum, Fourier Transforms

(See Boas 7.12)

COMPARISONS:

Fourier Series	Fourier Transform
$f(x) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 x + b_n \sin n\omega_0 x$	→ think of letting $\omega_0 \to 0$, + this sum over discrete $n\omega_0$'s becomes an integral over continuous α's:
or, using $e^{ix} = \cos x + i \sin x$	
$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 x}$ ⟷	$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} \, d\alpha$
• Fourier's trick tells us	Analogy to Fourier's trick gives
$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{\overset{\text{Note this sign}}{-} in\omega_0 x} \, dx$ ⟷	$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{\overset{\text{Note this sign}}{-} i\alpha x} \, dx$
• Given $f(x)$, you can (thus) find $\{C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots\}$	• Given $f(x)$, you can (thus) find $g(\alpha)$
(This ∞ set of #'s "represents" f, and vice versa)	(This ∞ set of #'s, i.e. this function, "represents" f, and vice versa)
• n is a dummy index. (could use any name for it)	• α is a dummy variable. (could use any name for it)
• The C_n's are called the "Fourier Coefficients for $f(x)$"	• $f(\alpha)$ and $g(\alpha)$ are called "Fourier Transforms" of each other (or, $g(\alpha)$ is the Four. Transform of f and $f(x)$ is the inverse " " of g.)

In Fourier TRANSFORMS:

So $g(\alpha)$ corresponds to C_n ← continuous function rather than discrete set

α corresponds to n (or maybe $n\omega_0$) ← continuous frequencies now, rather than just integers

$\int_{-\infty}^{\infty}$ corresponds to $\sum_{-\infty}^{\infty}$ ← continuous integration, rather than discrete sums.

The factor $\frac{1}{2\pi}$ in front of the Fourier Transform is subtle, I didn't derive it for you. Some texts have different conventions! E.g., sometimes you see $\frac{1}{\sqrt{2\pi}}$ in both formulas for $f(x)$ + $g(\alpha)$.

Note: If $f(x)$ is an odd function, we get only $b_n \sin n\omega_0 x$ terms.

With TRANSFORMS: $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$ Use Euler's theorem

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{odd}} \underbrace{\cos\alpha x}_{\text{even}}}_{\text{Vanishes!}} - \frac{i}{2\pi}\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{odd}} \underbrace{\sin\alpha x}_{\text{odd}}\, dx$$

so replace with $2\int_0^{\infty}\ldots$

Pure sin integral.

Also, this $g(\alpha) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx$ is itself an odd function (convince yourself, let $\alpha \to -\alpha$!)

$$\text{so } f(x) = \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha = \int_{-\infty}^{\infty} \underbrace{\underset{\text{odd}}{g(\alpha)}\, [\underset{\text{even}}{\cos\alpha x}}_{\text{vanishes}} + i\underset{\text{odd}}{\sin\alpha x}]\, d\alpha = 2i\int_0^{\infty} g(\alpha)\sin\alpha x\, d\alpha$$

Summarizing, for ODD functions, we get Fourier sine Transform

$$f(x) = 2i \int_0^\infty g(\alpha)\sin(\alpha x)\,d\alpha$$

and the Fourier sine transform of $f(x)$ is

$$g(\alpha) = -\frac{i}{2\pi}\int_0^\infty f(x)\sin\alpha x\,dx$$

As before, there are various conventions regarding the constants.

For instance, if I define $g_s(\alpha) \equiv i\sqrt{2\pi}\,g(\alpha)$ ← Just a simple rescaling

then (convince yourself!) $f_{odd}(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g_s(\alpha)\sin\alpha x\,d\alpha$

and $g_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_{odd}(x)\sin\alpha x\,dx$

Looks nicer, more symmetric, and all real!

Similarly, if you have an even function $f(x)$, you'll get

$$f_{even}(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g_c(\alpha)\cos\alpha x\,d\alpha$$

and $g_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_{even}(x)\cos\alpha x\,dx$

Lastly, if f is a function of time $f(t)$, the story is basically the same (just swap $t \leftrightarrow x$). In this case, it's quite common to rename α back to "ω", which seems very natural!

Summarizing that last point

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$ ← Looks a lot like Fourier series!

where $$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$ ← Looks a lot like Fourier coefficients

here, $g(\omega)$ is the Fourier transform of $f(t)$
and $f(t)$ is the inverse fourier transform of $g(\omega)$.

As mentioned, different texts have different conventions which amount to multiplying through by a convenient constant.
For instance, we can make the eq'ns look more <u>symmetric</u> if we redefine $\tilde{g}(\omega) \equiv \sqrt{2\pi}\, g(\omega)$. (or, $g(\omega) = \frac{1}{\sqrt{2\pi}} \tilde{g}(\omega)$)

Plug this in at the top of the page + convince yourself that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{i\omega t} d\omega$$

where $$\tilde{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

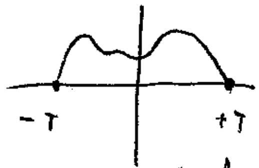
Nice + symmetric!

PDE - 24.5 - Optional SIDE NOTE

Here's a crude sketch (not proof!) of the Fourier transform formulas

Recall

For Fourier series:



$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}, \quad \omega_0 = \frac{2\pi}{2T}$$

see figure!

and, as usual, $C_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-in\omega_0 t} \, dt$

As $T \to \infty$, let's define a variable $\alpha \equiv n\omega_0 = n\pi/T$

As $T \to \infty$, this α becomes continuous, + as n marches from $-\infty$ to $+\infty$, α continuously marches from $-\infty$ to ∞ too.

C_n should now be considered $C(n)$, or better yet, $C(\alpha)$, (a function.)

Finally, note that $\Delta\alpha \equiv \alpha_{n+1} - \alpha_n = (n+1)\omega_0 - n\omega_0 = \omega_0 = \pi/T$

so, $T \frac{\Delta\alpha}{\pi} = 1$. OK, we're set, go back to the usual expressions above

---

$$f(t) = \sum_{\alpha=-\infty}^{\infty} C(\alpha) e^{i\alpha t} = \sum_{\alpha=-\infty}^{\infty} C(\alpha) e^{i\alpha t} \underbrace{\frac{\Delta\alpha T}{\pi}}_{\text{this is just 1!}}$$

As $T \to \infty$, $\Delta\alpha \to d\alpha \to 0$, + this is basically

$$f(t) = \int_{-\infty}^{\infty} \frac{T}{\pi} C(\alpha) e^{i\alpha t} \, d\alpha$$

C_n From top of page

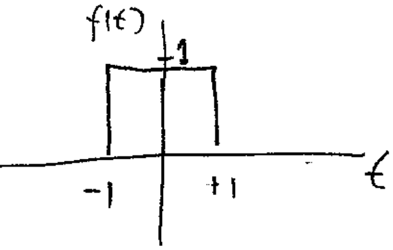
Let's define $g(\alpha) \equiv \frac{T}{\pi} C(\alpha) = \frac{T}{\pi} C_n = \frac{T}{\pi} \cdot \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\alpha t} \, dt$

Look at what we have: $g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} \, dt$

and, recapping, $f(t) = \int_{-\infty}^{\infty} g(\alpha) e^{+i\alpha t} \, d\alpha$

OUR expressions we claimed earlier, with the 2π's now at least somewhat justified!

Example: we initially suggested that Fourier transforms would be useful if we have a non-periodic (i.e. one-time-only) impulsive force. Let's work this out using our new Fourier transform formulas.



Let's suppose $f(t) = \begin{cases} 1 & \text{for } |t|<1 \\ 0 & \text{for } |t|>1 \end{cases}$

This is not periodic, it's a single blip.

It's even, so we can use our Fourier-cosine expressions, or we can just stick with the full complex form (it's all the same). Let's do the latter, & we'll stick with Boas's "2π" convention: → Because $f(t)$ vanishes everywhere else

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{2\pi}\int_{-1}^{1} e^{-i\omega t}\, dt$$

$$= \frac{1}{2\pi}\,\frac{1}{-i\omega}\, e^{-i\omega t}\Big|_{t=-1}^{t=+1} = \frac{1}{-2\pi i\omega}\left(e^{-i\omega} - e^{+i\omega}\right)$$

By Euler, this is $-2i\sin\omega$

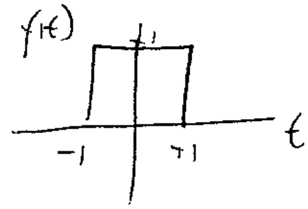
$$g(\omega) = +\frac{\sin\omega}{\pi\cdot\omega}$$

Remember, ω is a dummy index. This function is the continuous version of our old C_n's. Before, we had $f(t) = \sum_n C_n e^{in\omega_0 t}$

now we have $f(t) = \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha t}\, d\alpha$

Before, we had $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-in\omega_0 t}\, dt$

Now we have $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\alpha t}\, dt$



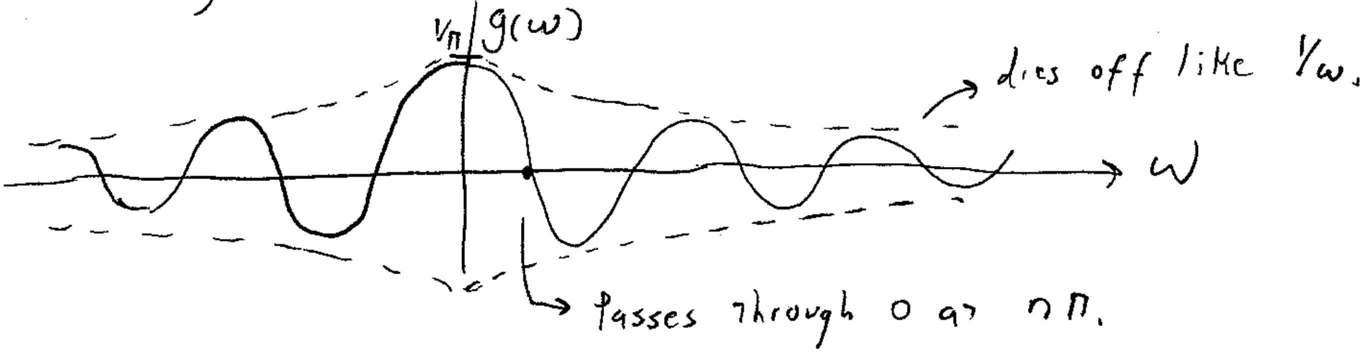
Continuing with this example:

we found $g(\omega) = \frac{\sin\omega}{\pi\omega}$. Let's sketch this.

As $\omega \to 0$, need L'Hopital's rule, $g(\omega)\Big|_{\omega\to 0} = \left.\frac{\frac{d}{d\omega}\sin\omega}{\frac{d}{d\omega}\pi\omega}\right|_{\omega\to 0} = \left.\frac{\cos\omega}{\pi}\right|_{\omega\to 0} = \frac{1}{\pi}$

(that's a surprise!)

As $\omega \to \infty$, the $1/\omega$ kills us off. So here it is



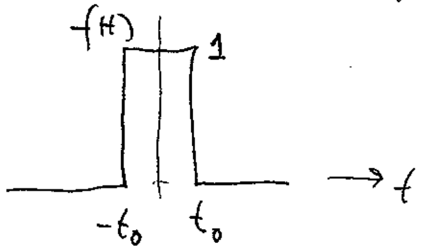
Our $f(t)$ was localized in time. But when we think of it as a sum of sinusoids $e^{i\omega t}$, with strength $g(\omega)$ i.e. $\int g(\omega) e^{i\omega t}$

we see that $g(\omega)$ is spread out, you need many ω's (all, in fact!)

Now, $g(\omega)$ is dominated (big amplitude) for ω up to, oh, the 1st or 2nd zero, i.e. out to $\omega \sim$ couple $* \pi$, before it fades away.

Let's pursue this by considering an $f(t)$ that's more limited in time.

So if



In this case, $g(\omega) = \frac{1}{2\pi}\int_{-t_0}^{t_0} e^{-i\omega t}\,dt$

$= \frac{1}{\pi\omega}\sin \omega t_0$

Nearly same as before, but $g(\omega)$ hits zero first when $\omega = \pi/t_0$

So if t_0 gets smaller, $g(\omega)$'s first zero gets bigger, i.e. $g(\omega)$ is getting wider.

Mathematically (or physically!) this says you need more ω's, more "strength at higher frequency" to build up a narrower $f(t)$.

It's a very general feature of Fourier TRANSFORMS that

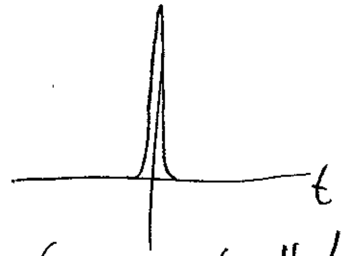
$$(\text{width of } f(t)) \propto \frac{1}{(\text{width of } g(\omega))}$$

This has many consequences!

To get a signal that's a quick blip, you need many frequencies (so e.g short laser pulses have many "colors" involved)

As we'll see, the Heisenberg uncertainty principle also arises from this!

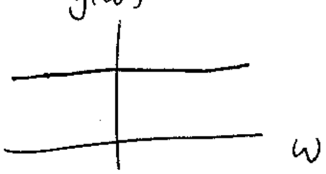
Example: Consider $f(t) = f_0\,\delta(t)$



This is the skinniest, most localized function of all!

Let's find $g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_0\, \delta(t)\, e^{-i\omega t}\, dt = \frac{f_0}{2\pi}$

Aha!



This is a constant!

Infinitely narrow $f(t) \Longleftrightarrow$ infinitely wide $g(\omega)$!

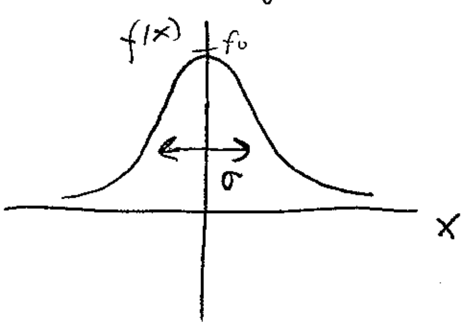
you need all frequencies (equally strong) to "build up" a δ function.

$\Rightarrow$ Short laser pulses have no definite color (!)

Short percussive claps have no definite pitch.

Example: Consider the Gaussian function $f(x) = f_0\, e^{-x^2/2\sigma^2}$

- Note I'm going back to $f(x)$, + thus will revert to $g(\alpha)$



σ is called the standard Deviation, + it directly tells you the width of this function.

you can compute $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

This would make a nice homework problem! I'll show you some tricks to help find the answer

$$g(\alpha) = \frac{\sigma f_0}{\sqrt{2\pi}}\, e^{-\alpha^2\sigma^2/2}$$

PDE - 28.5 - -Aside-

The trick to this integral is called "completing the square".

We have $g(\alpha) = \frac{f_0}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2} - i\alpha x}\, dx = \frac{f_0}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma^2} + i\alpha x\right)} dx$

The thing is, I happen to know $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

you can look this up, or ask me for a clever proof!

So the trick is to "complete the square" + do a u-sub. First

Write $\left(\frac{x^2}{2\sigma^2} + i\alpha x\right) = \left(\frac{x}{\sqrt{2}\sigma} + \text{something}\right)^2 - \text{same something}^2$

By inspection (check for yourself!) this is

$$\left(\frac{x}{\sqrt{2}\sigma} + \frac{i\alpha\sigma}{\sqrt{2}}\right)^2 + \frac{\alpha^2\sigma^2}{2}$$

So letting $u = \left(\frac{x}{\sqrt{2}\sigma} + i\frac{\alpha\sigma}{\sqrt{2}}\right)$, so $du = \frac{dx}{\sigma\sqrt{2}}$, you get $g(\alpha)$ as claimed

So $g(\alpha) = \frac{\sigma f_0}{\sqrt{2\pi}} e^{-\alpha^2\sigma^2/2}$

This is <u>also</u> a Gaussian. The standard deviation is $\frac{1}{\sigma}$ (!!)

So this is consistent with our pattern: <u>narrow</u> in $f(x) \Leftrightarrow$
<u>wide</u> in $g(\alpha)$ (+ vice versa)

In quantum mechanics, we will find that

$$\underbrace{\Psi(x)}_{\text{the wave fn}} \propto \int_{-\infty}^{\infty} \underbrace{\Phi(p)} e^{2\pi i p x/h}\, dp$$

This looks mathematically just like "g", the fourier transform (except for some <u>unit</u> sneakiness)

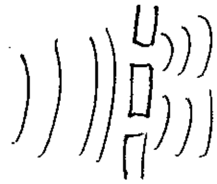
$\Phi(p)$ is the wave function too, in terms of <u>momentum</u>.

<u>Narrow</u> $\Psi(x)$ (well defined x) $\Leftrightarrow$ wide $\Phi(p)$
(poorly defined momentum)

+ vice-versa, the Heisenberg Uncertainty principle!

Physics is filled with applications of Fourier transforms!

When light scatters from an object



e.g. a slit, or pair of slits, or lattice ...

The transmitted light is a sum of waves (this is Huygen's principle)

The intensity of detected light $\propto \left| \text{Fourier transform of } f(x) \right|^2$

where $f(x)$ describes the spacial distribution of scatterers / holes ...

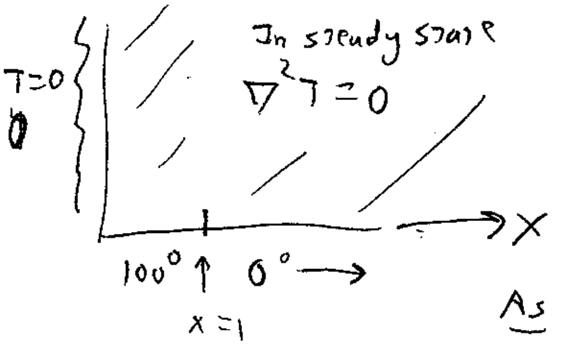
So you see the fourier transform of the crystal lattice or slits!

(For a crystal, this requires a 2-D generalization of what we've done)

But again, a smaller hole / slit / scatterer $\Rightarrow$ a wider pattern
(more diffraction!)

Example Back to our square plate. Let it be ∞ big now!

Let's set $T(x, y=0) = f(x) = \begin{cases} 100^\circ & 0<x<1 \\ 0^\circ & x>1 \text{, all the way} \ldots \end{cases}$



$T(x=0, y) = 0^\circ$.

As before, try $T(x) = X(x)\, Y(y)$

As before, $\nabla^2 T = 0 \Rightarrow X''(x) = -k^2 X(x) \Rightarrow A\sin kx + B\cos kx$

$Y''(y) = +k^2 Y(y) \Rightarrow Ce^{ky} + De^{-ky}$

As before, ambiguity of sign choice on k^2, but this will work out.

As before, $T(y \to \infty) = 0 \Rightarrow C$ vanishes, no growing exponential

As before, $T(0,y) = 0 \Rightarrow B = 0$, only $\sin kx$ vanishes at $x=0$.

But now, no second B.C. in x, no period, all possible k's are ok!

So, instead of summing over sol'ns, we integrate over all k

$$T(x,y) = \int_0^\infty B(k) \cdot \sin kx \, e^{-ky} \, dk$$

The B.C at $y=0$ will help, $T(x,0) = f(x) = \int_0^\infty B(k) \sin kx \, dk$

This is just our Fourier sin expression, + from early (p. 23) we can read off the sol'n for $B(k)$.

Letting "α" become "K", and using the "g_s" convention (p.23)

If $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(K) \sin Kx \, dK$

Then $g_s(K) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin Kx \, dx$

Here, letting $B(K) = \sqrt{\frac{2}{\pi}}\, g_s(K)$ we get $f(x) = \int_0^\infty B(K) \sin Kx \, dK$,

+ so

$$B(K) = \sqrt{\frac{2}{\pi}}\, g_s = \underline{\frac{2}{\pi} \int_0^\infty f(x) \sin Kx \, dx}$$

That's it! I can find $B(K)$ for any $f(x)$

For the given one, I get

$$B(K) = \frac{2}{\pi} \int_0^1 100^\circ \sin Kx \, dx = \frac{200}{\pi} \left. \frac{\cos Kx}{-K} \right|_{x=0}^{x=1} = \frac{200}{\pi K}(1-\cos K)$$

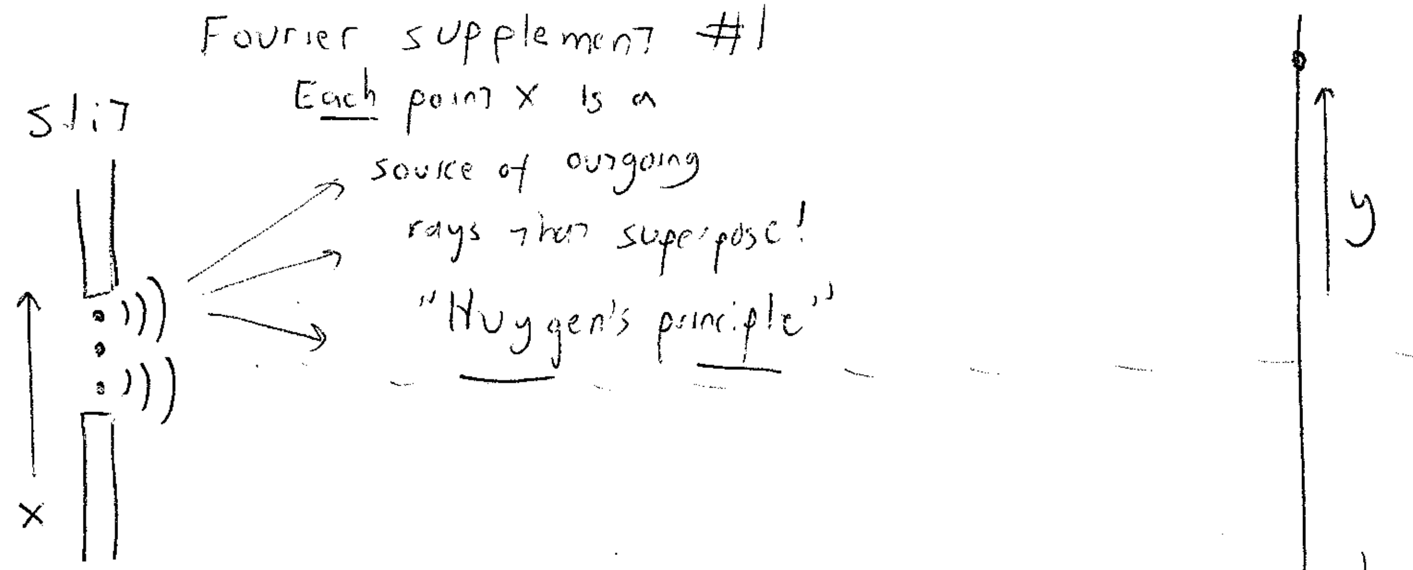
+ thus,

$$T(x,y) = \int_0^\infty \frac{200}{\pi} \left(\frac{1-\cos K}{K} \right) e^{-Ky} \sin Kx \, dK.$$

It's a 1-D integral, just compute it. If you can't do it analytically, it's a definite integral, so you can always compute it numerically for any desired x, y.

This ends our (first!) treatment of Fourier Transforms; their power is clear, Fourier series limit us to periodic functions, transforms do not!

Fourier supplement #1



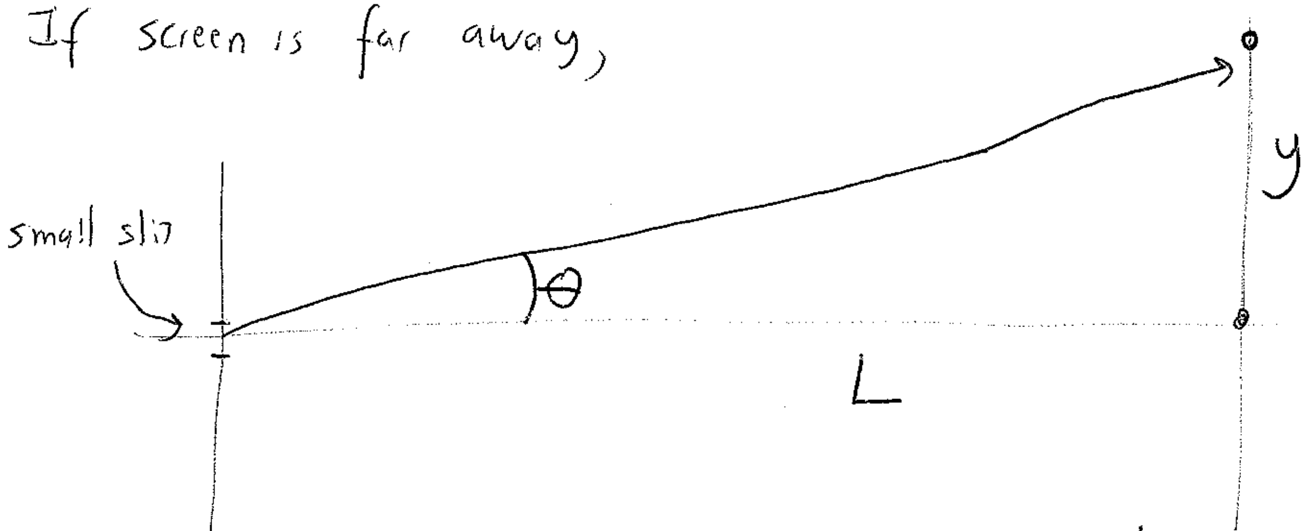
Each point x is a source of outgoing rays that superpose! "Huygen's principle"

some E field pattern given by $A(x)$, the "spatial distribution of slits"

screen, where you Detect $E(y)$ (or in fact, $|E(y)|^2$ = "brightness")

- At a given point y, where does $E(y)$ come from? It's the superposition, the sum, of all the E's arriving from all the different x's.

If screen is far away,

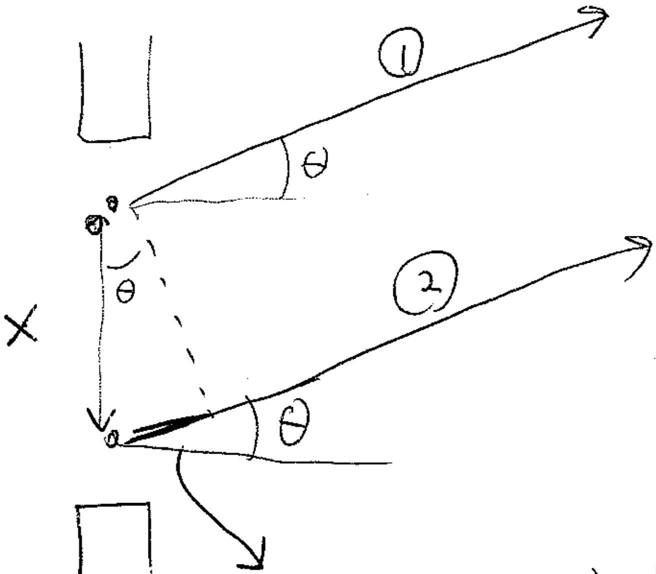


Big Distance $L \Rightarrow \theta \simeq y/L$

so y is just a measure of the ray direction θ

Fourier Supplement #2

Back to the slit, blown up.



If L is big
These two "parallel" rays both reach the <u>same</u> point y (!)
(parallel rays converge at ∞)

This extra path length is $x\sin\theta \approx x\theta$

It's the <u>extra</u> distance one ray goes compare to the other.

So the E fields of the two rays (which we add)

are $E_1 + E_2 = E_1(1 + e^{i\delta})$

↑ a phase shift caused by that <u>extra</u> path length

I claim $\delta = 2\pi$ for each λ extra that the path length represents

So $\delta = \frac{2\pi}{\lambda} * (\text{extra distance})$

$= \frac{2\pi}{\lambda} x\theta$

so $E = E_1(1 + e^{i\delta_{x_2}} + e^{i\delta_{x_3}} + e^{i\delta_{x_3}} + \ldots)$

Fourier supplement #3

Thus $E(y) \propto \underbrace{\int dx}_{\text{Sum over all sources}} \underbrace{A(x)}_{\text{the spatial distribution of slits}} \cdot e^{i\delta\theta}$

$$= \int dx\, A(x)\, e^{i\frac{2\pi x\theta}{\lambda}}$$

$$= \int dx\, A(x)\, e^{i\frac{2\pi x}{L}y}$$

$$= \int dx\, A(x)\, e^{ix\left(\frac{2\pi y}{L}\right)}$$

This is thus saying that $E(y)$ at the screen is the <u>fourier</u> Transform (with constants, $2\pi/L$) of $A(x)$ (and vice versa)

The <u>pattern</u> (image) on screen ($E(y)$) is the F.T. of the <u>slit</u> pattern $A(x)$!

So e.g <u>narrow</u> slits ⇒ <u>wide</u> patterns, + vice versa.