

Transformed E&M I homework

Biot-Savart Law (Griffiths Chapter 5)

Biot-Savart Law

Question 1. Magnetic field and power lines

Purcell, 6-10 pg. 246

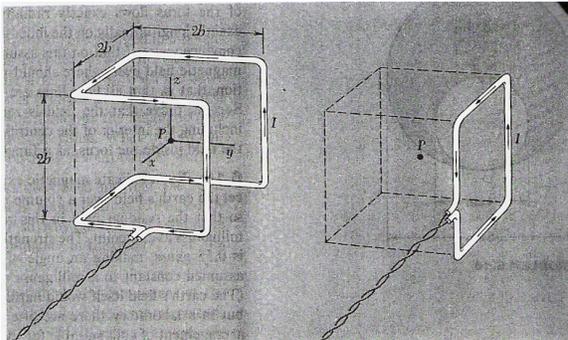
A 50-kilovolt direct-current power line consists of two conductors 2 meters apart. When this line is transmitting 10 megawatts, how strong is the magnetic field midway between the conductors?

Question 2. Magnetic field of square loop versus u-shaped loop

Purcell, 6-12 pg. 247

Current I flows around the wire frame in the figure.

- What is the direction of the magnetic field at P , the center of the cube?
- Show by using superposition that the field at P is the same as if the frame were replaced by the single square loop shown on the right.



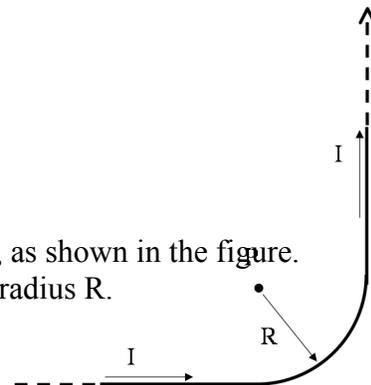
Question 3. Magnetic field

An infinitely long wire has been bent into a right angle turn, as shown in the figure.

The "curvey part" where it bends is a perfect quarter circle, radius R .

Point P is exactly at the center of that quarter circle.

A steady current I flows through this wire.



Find the magnetic field at point P (magnitude and direction)

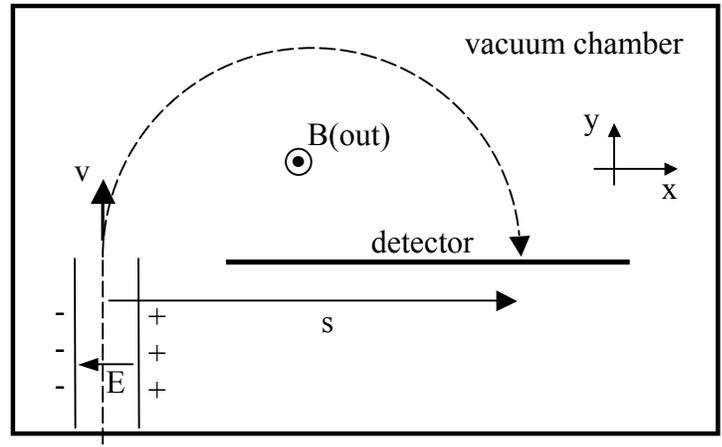
Assigned in SP08 (average score: 87.06%)

Assigned in FA08

Instructor notes: Students who started to go astray (by doing Biot-Savart) were generally set straight by other students.

Question 4.

In a mass spectrometer, a particle of known charge q and unknown mass m enters a vacuum chamber filled with constant uniform magnetic field $\vec{B} = B\hat{z}$. It passes between the closely spaced plates of a capacitor with electric field $\vec{E} = -E\hat{x}$ (with the same B-field as elsewhere in the chamber) and emerges with velocity $\vec{v} = v\hat{y}$. Its semicircular trajectory then carries to a point on the detector a distance s from the exit of the capacitor, as shown.



- A) The capacitor acts as a *velocity selector*. Explain how it works and relate the speed v of the emerging particle to E and B .
- B) How is the mass m related to the known quantities q , E , B , and s ? (Gravity can be ignored since it is very weak compared to the forces due to E and B .) Given the geometry shown, will this spectrometer work for $+$ charges only, $-$ charges only, or both $+$ or $-$ charges?

Assigned in FA08

Question 5.

A conducting ball bearing (radius R) is coated with a dielectric paint (dielectric constant ϵ_r), which extends out to a radius R' . (The paint is thick, at least on an atomic scale, so R' is measurably larger than R) This object is sitting in a uniform external electric field, \mathbf{E}_0 . Find the electric field *in* the paint region.

This is a slightly nasty little problem, which lets you work out for yourself the various methods invoked in Example 4.7 of Griffith, but with different boundary conditions. I recommend you do NOT try to apply the boundary condition on dV/dr at the inner boundary - you will get a paradox (it may look like $E=0$ everywhere in the paint region, but that's not right). This is another one of those cases where you have to be careful making assumptions about D : even though $Q_{\text{free}}(\text{enclosed})$ is 0, you can't assume D is purely radial in the paint region! But that's ok, you simply don't need this boundary condition anyway!

Assigned in SP08 (extra credit)

Question 6. Helmholtz coils

Pollack and Stump, 8-10 pg. 299

Two circular coils of radius a , each carrying current I in the same sense, are parallel with the xy plane with their centers at $(0, 0, \pm s/2)$. On the z axis the magnetic field is $B = B(z)\hat{k}$, in the z direction; and at $z = 0$, ie., halfway between the coils, $\partial B / \partial z$ is 0.

(a) Determine s such that $\partial^2 B / \partial z^2$ is 0 at $z = 0$ on the z axis. This configuration is called Helmholtz coils, and it produces a very uniform magnetic field in a neighborhood of the origin. Show that for this configuration the third derivative with respect to z is also 0 at $z = 0$.

(b) Use a computer program to plot the magnetic field $B(z)$ as a function of position z along the axis of the Helmholtz coils.

Helmholtz coils are often used in laboratories to cancel out the Earth's magnetic field in a small region of space. However, if a large magnetic-field-free region is needed, say room sized, the necessary coils would be impractically large, as you can see from your answer to (b). In practice, to produce a field-free volume one surrounds it with a material with high magnetic susceptibility, e.g., Mumetal, which concentrates the magnetic field in the walls.

Question 7. B at center of square loop and regular polygon

Pollack and Stump, 8-11 pg. 299

Find the magnetic field at the center of a square loop of size $2a \times 2a$ carrying current I . Repeat the calculation for a regular polygon with n sides, letting the perpendicular distance from the center to any side be a . Show that the result approaches the field at the center of a circular loop of radius a in the limit $n \rightarrow \infty$.

Question 8. B of square loop, and limiting cases

Pollack and Stump, 8-12 pg. 300

A square wire loop of size $2a \times 2a$ lies in the xy plane with its center at the origin and sides parallel to the x and y axes. A counterclockwise current I runs around the loop.

(a) Find the magnetic field on the z axis.

[Answer: $[B_z(z) = 2\mu_0 I a^2 / [\pi(a^2 + z^2)(2a^2 + z^2)^{\frac{1}{2}}]]$]

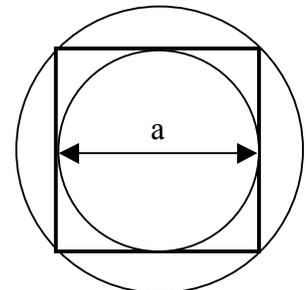
(b) Show that for $z/a \gg 1$ the field becomes that of a magnetic dipole, and find the magnetic moment. (See (8.78) and (8.80).)

(c) Compare the field at the center of this square loop with that at the center of a circular loop of diameter $2a$.

Question 9. Square loop

A) Find the magnetic field at the center of a square current loop with current I and edge length a .

B) If I had such a loop in my lab and wanted the B field at the center, I might do the above calculation, but if I was planning an experiment and just wanted a *rough estimate* of the B -field, I might "assume a spherical cow": assume the square was really a circle. We've done that problem (**B** at center of a circular loop - it's much simpler than the square. You don't have to rederive it, but do think back to how we got that result, and why it turned out to be a relatively easy application of Biot-Savart.) But what radius circle would you use, to estimate B ? Find the B fields for the inscribed



and circumscribed circles and then average. How good an approximation does that turn out to be?

C) Go back to your result for part A, and take the limit $z \gg a$, finding an approximate simple formula for $B_z(0,0, \text{large } z)$. Do the same for a circular current loop of radius a , and compare (The exact expression is derived in Griffiths, Eq. 5.38, you don't have to rederive that, just consider the limit)

Notice your two expressions differ only by the constant out front, which should go like $m = I \times (\text{area of loop})$ This is called the magnetic dipole moment.

Assigned in SP08 (average score: a) 9.78, b) 4.61)

Assigned in FA08

Instructor notes: We didn't observe any student difficulties on this question.

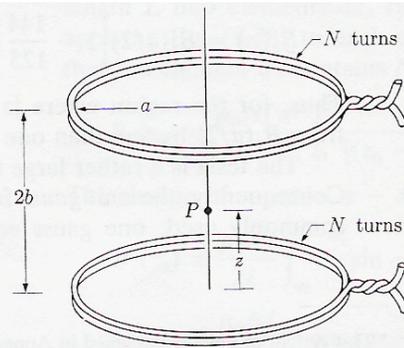
Question 10. Axial magnetic field of a Helmholtz coil

Reitz, Milford, Christy, Example 8-3 pg. 201

Figure 8-6 shows the configuration of a Helmholtz coil. Find the magnetic field at a point on the axis of the coil.

FIGURE 8-6

Axial field of a Helmholtz coil



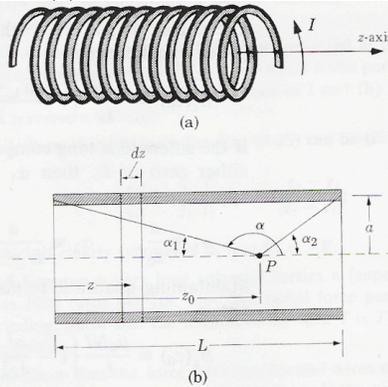
Question 11. Magnetic field of a solenoid

Reitz, Milford, Christy, Example 8-4 pg. 203

Another device to which Eq. (8-38) can be applied is the solenoid. A solenoid may be described as N turns uniformly wound on a cylindrical form of radius a and length L . Such a configuration is shown in Fig. 8-7 (a). Find B on the axis of the solenoid.

FIGURE 8-7

(a) A solenoid; (b) axial magnetic field of a solenoid



Question 12. Plot B for solenoid

Reitz, Milford, Christy, 8-10 pg. 215

A solenoid 15 cm long is wound in two layers. Each layer contains 100 turns; the first layer is 2 cm in radius, the second 2.05 cm. (a) If the winding carries a current of 3 A, find the magnetic induction at various points along the axis of the solenoid. (b) Make a plot of the axial magnetic induction as a function of distance, from the center to one end of the solenoid.

Question 13. Magnetic field of half-infinite wire

Purcell, 6-8 pg. 246

A wire carrying current I runs down the y axis to the origin, then out to infinity along the positive x axis. Show that the magnetic field in the quadrant $x > 0, y > 0$ of the xy plane is given by

$$B_z = \frac{I}{c} \left(\frac{1}{x} + \frac{1}{y} + \frac{x}{y\sqrt{x^2 + y^2}} + \frac{y}{x\sqrt{x^2 + y^2}} \right)$$

Question 14. Magnetic field of square loop

CALCULATION, LIMITS (deGrand, similar to Griffiths ___)

- (a) Find the B field at the center of and in the plane of a square loop of current; each side is length a .
- (b) Now assuming the loop lies in the $x - y$ plane, find the field along the z axis above the center of the loop.
- (c) If $z \gg a$ find an appropriate simple formula for B_z . Now consider the problem of a circular loop of radius a . We derived the exact expression (eq. 5.34) in class. Again let $z \gg a$ and find a simple limiting form for B_z . Notice your two expressions differ only in that they involve the quantity $m = I \times$ area of the loop. This is called the magnetic dipole moment.

Question 15. Polygonal electromagnets

INTEGRATION; LIMITS (From Reitz, Milford and Christy, Foundations of Electromagnetic Theory, 4th Ed., Problems 7-7)

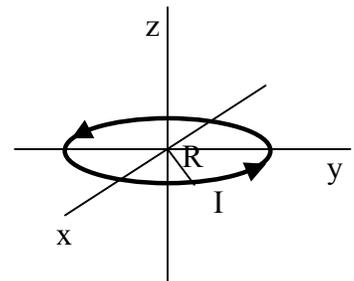
Polygonal electromagnets:

- a. A square loop of wire, with size $2a \times 2a$ lies in the xy -plane centered at the origin. Calculate the field everywhere on the z -axis produced by a current I through this loop.
- b. What is the line integral of the field along the z -axis from $z = -\infty$ to $z = \infty$? Could you have expected this result based on Ampere's law? Explain.
- c. Now find the magnetic field at the center of a n -sided regular polygon, letting the perpendicular distance from the center to any side be a . Show that the result approaches that for a circular loop of radius a in the limit $n \rightarrow \infty$.

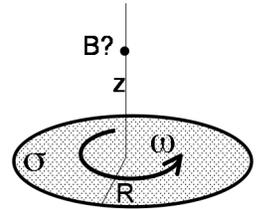
Question 16. Magnetic field of rotating disk

CALCULATION (deGrand)

A) Compute the magnetic field $\mathbf{B}(0,0,z)$ on the axis of a circular ring



of radius R carrying a current I . The ring is in the xy plane and is centered on the origin.
 B) Last week we had a problem with a CD of radius R with a fixed, constant, uniform surface electric charge density σ everywhere on its top surface. It was spinning at angular velocity $\omega \hat{z}$ about its center (the origin). You found the current density \mathbf{K} at a distance r from the center. Use that result to find the magnetic field $\mathbf{B}(0,0,z)$ at any distance z directly above the origin. Does your answer seem reasonable? Please check its limiting behaviors (e.g. what do you expect if $R \rightarrow 0$? $z \rightarrow \infty$? $\omega \rightarrow 0$? Slightly less obvious, but also worth checking/thinking about, what about $z \rightarrow 0$?)



Assigned in SP08 (average score: 85.29%)

Assigned in FA08

Instructor notes: Students can do this problem, but lack confidence. There was some difficulty with visualization of the problem, figuring out if $\mathbf{J} \times \mathbf{r}$ involved a sine of some angle or not. Some wanted to use the "theta" between the wire and script- \mathbf{r} . Many rederived the \mathbf{B} field from a ring, which had been done in class and in the book. Some difficulty on the far field approximation ($z \gg R$), with students answering that $\mathbf{B} = 0$ when $z \gg R$, which is the trivial term in the expansion.

Question 17. Magnetic field of current-carrying loop

CYLINDRICAL COORDINATES, APPROXIMATION (deGrand)

The magnetic induction at a point on the axis of a circular loop of wire of radius a carrying a current I is given by eq. 5.38 (is that the right formula?). Use the fact that $\nabla \cdot \mathbf{B} = 0$ to obtain an approximate expression for B_r , the radial component of the magnetic field in the cylindrical coordinates, valid for points near the axis.

Question 18. Magnetic field of thin sheet of current

CALCULATION; ARGUMENT, LIMITS (OSU Paradigms, Vector Fields HW 11)

- Use the Biot-Savart Law to find the magnetic field at a point a distance z above the center of an infinitesimally thin, square sheet of current. The current is parallel to one of the square sides. (Obviously, since the current cannot just begin and end in the middle of nowhere, this current is just the building block for some larger current.) You may use any symmetry arguments you like, but do **not** use Ampere's Law. Simplify as much as possible: How do you know when to quit?
- Consider your previous answer in the limit that the square becomes infinitely large. Discuss your answer in the light of the Ampere's Law example from class.

Question 19. Magnetic field of parallel wires

COMPUTATION (Munsat)

Magnetic Field of Parallel Wires

http://plasma.colorado.edu/mathcad/Bfield_Wires.mcd

This exercise plots the field of a single wire, two wires with parallel currents, and two wires with antiparallel currents. A magnetic “x point” is illustrated. The vector fields are added.

Physics content: The azimuthal field of a wire is projected onto the x and y directions.

Mathcad graphics: Plots a vector field in two dimensions.

Question 20. Plotting field lines

COMPUTATION (Munsat)

Plotting Field Lines

http://plasma.colorado.edu/mathcad/Bfield_plotting.mcd

This exercise shows how to integrate with Runge Kutta a unit vector to obtain a curve in 3-d space. The components of B_q are projected onto the x and y directions.

Mathcad content: Demonstrates the Runge-Kutta solution of a differential equation using rkfixed.

Mathcad graphics: illustrates a 3-d scatter plot that can be tilted by the user.

Question 21. Field of a loop

COMPUTATION (Munsat)

Field of a Loop

http://plasma.colorado.edu/mathcad/Bfield_of_a_loop.mcd

This exercise plots the field of a loop of wire. The expression for B involves the complete elliptic integrals $E(k)$ and $K(k)$. These integrals are defined at the outset. The unit vector is found in the r,z plane and integrated. The result is B looping around the minor axis for points close to this axis. In a second plot, a constant field is added which makes an x point.

Mathcad content: integration of a unit vector as in the field line plotting exercise.

Evaluation of elliptic integrals. TOL is used to decrease calculation time.

Mathcad graphics: 2-d plots of lines.

Question 22. Magnetic field of Tokamak

COMPUTATION (Munsat)

B Field of the tokamak

http://plasma.colorado.edu/mathcad/Bfield_tokamak.mcd

This exercise plots the magnetic surface mapped by a magnetic field line of a tokamak. It follows a magnetic field line created by a current loop perpendicular to the z axis and a wire along the z axis. The concepts of magnetic surface and aspect ratio are discussed. The content is similar to that in Field of a Loop.

Question 23. The TOP trap

CALCULATION; MAGNETIC DIPOLES, REAL WORLD (Berkeley; Stamper-Kurn)

3) The TOP trap: Losses from spherical quadrupole traps made evaporative cooling of

atoms in magnetic traps ineffective until some manner was employed to avoid these losses. As you found above, the problem becomes ever worse as the temperature of the gas gets lower, thus precluding the ultra-low temperatures needed to reach Bose-Einstein condensation. One solution, employed by the team of Cornell and Wieman in Colorado for the first achievement of Bose-Einstein condensation in a gas, was the TOP (time-averaged orbiting potential) trap. Consider that a spherical quadrupole trap is formed, and then an additional bias field is applied of the form

$$B \text{ TOP } (x, y, z) = B_0 \cos \omega t \hat{x} + B_0 \sin \omega t \hat{y} \quad (2)$$

Atoms held in a time-varying magnetic trap of this type will experience a time-averaged potential at every point so long as (1) the field rotates at a rate faster than the motional time-scale of atoms so that an atom experiences a local average of the magnitude of the field, and (2) the field rotates at a rate slower than the Larmor precession frequency so that atomic magnetic moment remains anti-aligned with the magnetic field.

a) Assume that these conditions are met. What is the time-averaged magnitude of the magnetic field near the origin? You should find that this magnitude has a minimum at the origin, and varies quadratically away from the origin with a fixed ratio of trap curvatures in the different directions.

b) Now put in some numbers. Suppose on top of the 120 G/cm radial gradient spherical quadrupole field, the time-varying TOP field has a magnitude of 10 G and rotates at a rate of $\omega = 2\pi \times 7.5$ kHz. What are the oscillation frequencies of a 87 Rb atom in this trap? (Note: if a one dimensional potential has the form $U(x) = (1/2)m\Omega^2 x^2$, then a particle of mass m will oscillate harmonically in this potential with oscillation angular frequency Ω). Do the field settings chosen satisfy the two conditions for adiabaticity described above?

c) Finally, the critical temperature T_c for Bose-Einstein condensation for atoms in a harmonic trap is determined by the geometric mean of trapping frequencies $\bar{\omega}$ and the number of atoms N by the relation

$$N = \zeta(3) \left(\frac{k_B T_c}{\hbar \bar{\omega}} \right)^3 \quad (3)$$

where k_B is the Boltzmann constant and ζ is the Riemann Zeta function ($\zeta(3) \approx 1.202$).

This equation can be interpreted as stating that once the number of quantum states available to a gas at a temperature T_c roughly equals the number of particles, multiple occupancy of quantum states occurs and quantum statistical effects such as Bose-Einstein condensation ensue. Consider the above TOP trap with 2×10^4 atoms. What is the Bose-Einstein condensation transition temperature?

References for magnetic trapping and Bose-Einstein condensation include the following:

- *Pethick and Smith, Bose-Einstein condensation in Dilute Gases (Cambridge University Press, New York, 2002).*
- *Metcalf and van der Straten, Laser Cooling and Trapping (Springer, New York, 1999).*
- *M.H. Anderson, et al., "Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor," Science 269, 198 (1995).*

Question 24. Magnetic trapping of ultracold atoms

CALCULATION; MAGNETIC DIPOLES, REAL WORLD (Berkeley; Stamper-Kurn)
Magnetic trapping of ultracold atoms

The next set of problems concern the magnetic trapping of ultracold atoms, specifically

atoms of ^{87}Rb (one of the rubidium isotopes). In one of their internal states (the $F = 1$ hyperfine state), such atoms have a magnetic dipole moment of $m = m_B / 2$ where $m_B = e \hbar / 2m_e$ is the Bohr magneton. Magnetic trapping is achieved under the condition that the atomic magnetic moment remains anti-aligned with the magnetic field as the atom moves around in space, thus producing a trapping potential of the form $U(r) = m |B|$.

1) First, let's consider a simple kind of magnetic trap known as the spherical quadrupole magnetic trap.

a) Consider two electromagnet coils sharing the same axis and separated by d , each of radius a and N turns, each coil carrying current I running in the opposite direction as that of the other coil. Placing the origin at the midpoint between the two coils, calculate the magnetic field $B(x, y, z)$ in the region near the origin, keeping just the lowest order terms. Note that to determine the field for radial displacements (i.e. in the x or y directions), you can consider the symmetry of the problem and apply the condition of zero-divergence for the magnetic field.

b) You should find a spherical quadrupole field with field gradients in the radial and axial directions determined by properties of the coils which create the field. A spherical quadrupole field can be written as

$$B(x, y, z) = B (\hat{x}x + \hat{y}y - 2\hat{z}z) \quad (1)$$

where B is the radial field gradient. Now put in some real numbers. Consider coils of radius $a = 2.5$ cm separated by $d = 2a/\sqrt{3}$ (the optimal separation), each of 20 turns running $I = 92$ amperes. What are the axial and radial gradients (G/cm)?

2) Majorana losses: In the spherical quadrupole magnetic field described above, the potential $U(r) = m |B|$ is clearly minimized at a point, and, thus, particles with their magnetic moment anti-aligned with the field will be trapped by such a field. However, a problem with the spherical quadrupole trap is the fact that it is always lossy. Magnetic trapping requires that the atomic magnetic moment always follow the local orientation of the magnetic field. If the field is changing its direction faster than an atom's orientation can keep up, the atomic orientation may change, causing the atom to be ejected from a magnetic trap. This type of loss is given the name Majorana loss.

a) Consider a ^{87}Rb atom moving with velocity v and impact parameter b in the $x - y$ plane of a spherical quadrupole trap. Give an estimate for the critical impact parameter b_0 below which the atom has a high probability of being lost, and above which the atom has a low probability of being lost. To do this, consider the maximal rate at which the orientation of the field is changing – this quantity has units of s^{-1} . Compare this with the Larmor precession frequency of the atom in the local magnetic field (this frequency is zero at the origin and larger elsewhere).

b) Now we will estimate the Majorana loss rate for atoms at a certain temperature trapped in a spherical quadrupole trap. To do this, consider that the treatment above determines an ellipsoidal volume within which atoms at a given velocity may be lost from the trap. A loss rate can be calculated as the product $n\sigma v$ where n is the density of the atomic gas, σ is the area of the ellipsoid (which differs slightly if approached from different directions; just take some characteristic area), and v is the typical velocity of atoms. To estimate the volume occupied by the gas (you'll need this to determine the density n), consider how large is the volume in which the trap potential $U(r)$ is smaller

than $k_B T$ where k_B is the Boltzmann constant. Similarly, to obtain a typical velocity, consider the velocity at which the kinetic energy is $k_B T$. Having made all these estimates, find the loss rate per atom given the temperature T and the radial field gradient B .

c) Again, let's put in some numbers. Consider a gas of ^{87}Rb trapped in a spherical quadrupole trap with a radial gradient of 120 G/cm . What is the Majorana loss rate per atom at temperatures of 1 , 10 , and $100\ \mu\text{K}$?