Transformed E&M I homework

Laplace's Equation
(Griffiths Chapter 3)

Laplace's equation and uniqueness

Question 1. Laplace solver
COMPUTATION (Munsat)
Laplace Solver
http://plasma.colorado.edu/mathcad/ES_Laplace.mcd
This exercise shows how successive approximations are used to obtain solutions to Laplace's equation on a 2-d grid.
Physics/math content: teaches that a characteristic of Laplace's equation is that the potential at a grid point is the average of the values at the neighboring grid points.
Mathcad content: program loop, if function, relax function, matrices, matrix transpose.
Mathcad graphics: Contour plots

Question 2. Poisson's equation and Debye Shielding
COMPUTATION (Munsat)
Poisson's Equation and Debye Shielding
http://plasma.colorado.edu/mathcad/ES_Poisson.mcd
This exercise shows how successive approximations are used to obtain solutions to Poisson's equation in one-dimension. It uses electron and ion densities determined by the Boltzmann factor.
Physics content: shows that the potential around an electrode placed in plasma decays exponentially with distance. Demonstrates the utility of dimensionless units.
Math content: shows a numerical instability and how to stop it.
Mathcad content: Program loop.
Mathcad graphics: Simple x,y plots.

Question 3. Poisson's equation in cylindrical geometry
COMPUTATION (Munsat)
Poisson's equation in cylindrical geometry
http://plasma.colorado.edu/mathcad/ES_Poisson_Cylindrical.mcd
The finite-difference form of Poisson's equation in cylindrical geometry is solved by the relaxation method, first assuming no z dependence and then allowing a z dependence. A cylinder of charge is assumed to be within a cylindrical container at zero potential. This extends the previous exercise to cylindrical coordinates.

Question 4. The ionosphere
COMPUTATION (Munsat)

The ionosphere: Separation of ions by gravity (new in 2007)
http://plasma.colorado.edu/mathcad/ES_Separation_Of_Ions.mcd
The ionosphere contains ions of both oxygen, O+, and hydrogen, H+. The proportion of H+ increases with altitude because the H+ is lighter. The electrons are the lightest species, but are constrained by quasineutrality to be at the same altitude as the ions. In this exercise we use the relaxation methodsfor Poisson's equation to apply the quasineutrality constraint and find the altitude dependence of the densities of O+, H+, and electrons.

Question 5. Taylor series and Laplace
(Purcell; 3.29 pg. 119)
Let $\varphi(x, y, z)$ be any function that can be expanded in a power series around a point (x_0, y_0, z_0). Write a Taylor series expansion for the value of φ at each of the six points
$(x_0+\delta, y_0, z_0),(x_0-\delta, y_0, z_0),(x_0, y_0+\delta, z_0),(x_0, y_0-\delta, z_0),(x_0, y_0, z_0+\delta),(x_0, y_0, z_0-\delta)$
, which symmetrically surround the point (x_0, y_0, z_0) at a distance δ. Show that if φ satisfies Laplace's equation, the average of these six values equal to $\varphi(x_0, y_0, z_0)$ through terms of the third order in δ.

Question 6. Uniqueness theorem
Griffiths section 3.1.6 is a discussion and proof of the "Second uniqueness theorem". Read through that theorem (and his proof), make sense of it! Then for this homework problem, prove it yourself, using a slightly *different* method than what Griffiths does (though you may find some common "pieces" are involved!) Do it like this:
Go back to Green's Identity (stated in Problem 1.60c, p. 56, you may recognize this from our HW#2) This identity is true for ANY choice of T and U, so let the functions T and U in that identity both be the SAME function: specifically, you should set them both equal to $V_3=V_1-V_2$. Then, Green's Identity (along with some arguments about what happens at the boundaries, rather like Griffith's uses in *his* proof) should let you quickly show that E_3 (which is defined to be the negative gradient of V_3, as usual) must vanish everywhere throughout the volume. QED.
Understand the game. We are checking if there are two different potential functions, V_1 and V_2, each of which satisfies Laplace's equation throughout the region we're considering. You construct (define) V_3 to be the difference of these, and you prove that V_3 (or in this case, $E_3=-\nabla V_3$) must vanish everywhere in the region. Which means there really is only one unique E-field throughout the region after all! This is another one of those "formal manipulation" problems, giving you a chance to practice with the divergence theorem and think about boundary conditions...

Assigned in SP08 (average score: n/a)
Assigned in FA08
Instructor notes: Some student confusion about why you can pull V3 out of the integrals. There was some confusion in following the proof in the book. Why del-dot-E3 was zero was confusing until we noticed the words "in the space between the conductors." Most students seemed to

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get the proofs OK, though they were intimidated by them.  Overall, good
practice.
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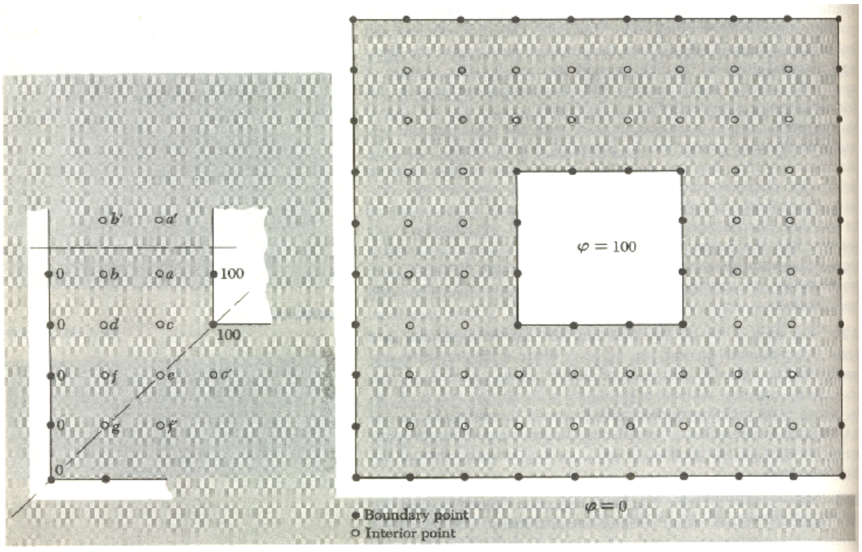
Question 7. Relaxation method for solving Laplace's equation

(Purcell; 3.30 pg. 119-120)

Here's how to solve Laplace's equation approximately, for given boundary values, using nothing but arithmetic. The method is the relaxation method mentioned in Section 3.8, and it is based on the result of Problem 3.29. For simplicity we take a two-dimensional example. In the figure there are two square equipotential boundaries, one inside the other. This might be across section through a capacitor made of two sizes of square metal tubing. The problem is to find, for an array of discrete points, numbers which will be a good approximation to the values at those points of the exact two-dimensional potential function $\phi(x, y)$. For this exercise, we'll make the array rather coarse, to keep the labor within bounds. Let us assign, arbitrarily, potential 100 to the inner boundary and zero to the outer. All points on these boundaries retain those values. You could start with any values at the interior points, but time will be saved by a little judicious guesswork. We know the correct values must lie between 0 and 100, and we expect that points closer to the inner boundary will have higher values than those closer to the outer boundary. Some reasonable starting values are suggested in the figure. Obviously, you should take advantage of the symmetry of the configuration: Only seven different interior values need to be computed. Now you simply go over these seven interior lattice points in some systematic manner replacing the value at each interior point by the average of its four neighbors. Repeat until all charges resulting from a sweep over the array are acceptably small. For this exercise, let us agree that it will be time to quit when no charge larger in absolute magnitude than one unit occurs in the course of the sweep. The relaxation for the values toward an eventually unchanging distribution is closely related to the physical phenomenon of *diffusion*. If you start with much too high a value at one point, it will "spread" to is nearest neighbors, then to its next nearest neighbors, and so on, until the bump is smoothed out. Enter your final values on the array, and sketch the approximate course two equipotentials, for $\phi = 25$ and $\phi = 50$, would have in the actual continuous $\phi(x, y)$.



PROBLEM 3.30

Replace value at an interior point by ¼ × sum of its four neighbors: $c \to \frac{1}{4}(100 + a + d + e)$; keep $a' = a$, $b' = b$, $c' = c$, and $f' = f$. Suggested starting values:

$a = 50$	$e = 50$
$b = 25$	$f = 25$
$c = 50$	$g = 25$
$d = 25$	

Question 8. Holomorphisms

ANALYTIC FUNCTIONS, EXTENSION OF IDEAS TO NEW TOPICS, PROOF (C. Gwinn, UC Santa Barbara)

In complex analysis, a holomorphism, (often called an analytical function) is a function $\tilde{f}$ that can be expressed as a convergent power series, such as:

$$\tilde{f}(\tilde{z}) = \sum_{n=0}^{\infty} \tilde{a}_n \tilde{z}^n$$

where $\tilde{f}$, $\tilde{z}$ and $\tilde{a}_n$ may be complex. Usually, a function is holomorphic only over some specific "open set" in the complex plane. (Note: Here I use the "tilde" accent: ˜ to denote a complex value.)

a) Prove that holomorphisms are solutions to Laplace's Equation. More precisely, if $\tilde{f}(\tilde{z})$ is a holomorphism, then let $\tilde{z} = x + iy$, and show that

$$\nabla^2 \mathrm{Re}[\tilde{f}(x,y)] = \nabla^2 \mathrm{Im}[\tilde{f}(x,y)] = 0.$$

One approach to this proof is to express $\tilde{z}$ in polar coordinates (s, φ):

$$\tilde{z} = se^{i\phi} = s\cos\phi + is\sin\phi.$$

Then, one can express the real and imaginary parts of a single term, Re[$\tilde{a}_n \tilde{z}^n$] and Im[$\tilde{a}_n \tilde{z}^n$], in polar coordinates. These terms satisfy Laplace's Equation in polar coordinates, where ∇^2 in cylindrical coordinates is given inside the front cover of the text (ignore the spurious third dimension, z , in cylindricals). The fact that ∇^2 is a linear operator allows completion of the proof.

b) Holomorphisms are differentiable. Use this fact, and the chain rule, to show that if $\tilde{f}$ and $\tilde{g}$ are holomorphisms, then the real and imaginary parts of $\tilde{f}(\tilde{g}(\tilde{z}))$ are solutions of Laplace's Equation. If you know one solution to Laplace Equation in polar coordinates, this fact can be used to find others.

c) For fun: Can you use this fact to solve problem 3.10 in Griffiths? Discuss your answer in a few sentences.

This problem is an application of the one above (also by C. Gwinn, UC Santa Barbara)

a) Suppose that $\tilde{z} = x + iy$ represents coordinates on the (x, y) plane. Show that the function:

$$f(x,y) = \mathrm{Re}\left[\tilde{f}(\tilde{z})\right] = \mathrm{Re}\left[\sum_{n=1,\ n\ \mathrm{odd}}^{\infty} \frac{1}{n}\tilde{z}^n\right]$$

is a solution to Laplace's Equation for $|\tilde{z}| < 1$. Prove that

$$g(x,y) = \mathrm{Re}\left[\tilde{g}(\tilde{z})\right] = \mathrm{Re}\left[\sum_{n=1,\ n\ \mathrm{odd}}^{\infty} \frac{1}{n}\frac{1}{\tilde{z}^n}\right]$$

is also a solution to Laplace's Equation for $1 < |\tilde{z}|$.

b) Show that, along the y-axis, f = g = 0.

c) Show that f and g are equal on the circle x2 + y2 = 1, wherever they exist. Where do they not exist? Useful fact:

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$$

Why are two series necessary to describe this potential? Are two series necessary?

For fun: For what values of α are f^α and g^α solutions of Laplace's Equation? Is charge conserved by this transformation?