

# Transformed E&M I homework

## Separation of Variables in Cartesian and spherical (Griffiths Chapter 3)

### Separation of Variables

#### Question 1. Separation of variables- Cartesian 2D

A square rectangular pipe (sides of length  $a$ ) runs parallel to the  $z$ -axis (from  $-\infty$  to  $+\infty$ )

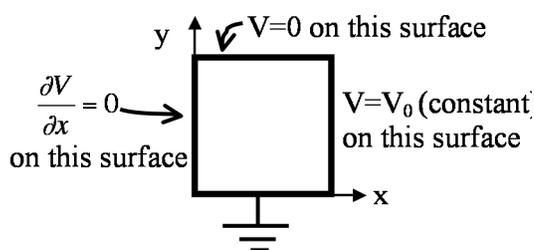
The 4 sides are maintained with boundary conditions given in the figure. (Each of the 4 sides is insulated from the others at the corners)

i) Find the potential  $V(x,y,z)$  at all points in this pipe.

ii) Sketch the E-field lines and equipotential contours inside the pipe.

(Also, state in words what the boundary condition on the left wall means - what does it tell you? Is the left wall a conductor?)

iii) Find the charge density  $\sigma(x,y=0,z)$  everywhere on the bottom conducting wall ( $y=0$ ).



Assigned in SP08

Assigned in FA08

Instructor notes: Students easily managed to get to the point where they had  $X(x) = C \cdot \cosh(n \pi x/a)$  and  $Y(y) = A \cdot \sin(n \pi y/a)$ . But there is a mixture of student ability to proceed from there, how to apply the last boundary condition, whether Fourier's trick is familiar or not. Among those who are familiar with Fourier's Trick, however, some still do not quite understand orthogonality of the sin functions. The sketch is challenging for students. I saw several who had gotten the E field sketch right just by thinking about where a charge would go when placed on one face, but then struggled with the equipotentials, finally figuring it had to be perpendicular to the field lines but still having a bit of trouble from there.

#### Question 2. Separation of variables- Cartesian 3D

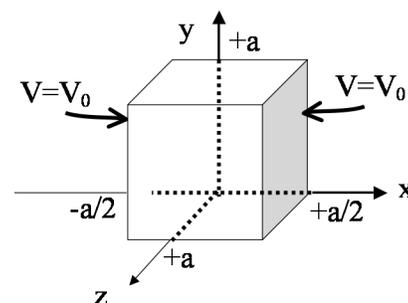
You have a cubical box (sides all of length  $a$ ) made of 6 metal plates which are insulated from each other.

The left wall is located at  $x=-a/2$ , the right wall is at  $x=+a/2$ .

Both left and right walls are held at constant potential  $V=V_0$ .

All four other walls are grounded.

(Note that I've set up the geometry so the cube runs from  $y=0$  to  $y=a$ , and from  $z=0$  to  $z=a$ , but from  $x=-a/2$  to  $x=+a/2$ . This should actually make the math work out a little easier!)



Find the potential  $V(x,y,z)$  everywhere inside the box.

(Also, is  $V=0$  at the center of this cube? Is  $E=0$  there? Why, or why not?)

**Assigned in SP08**

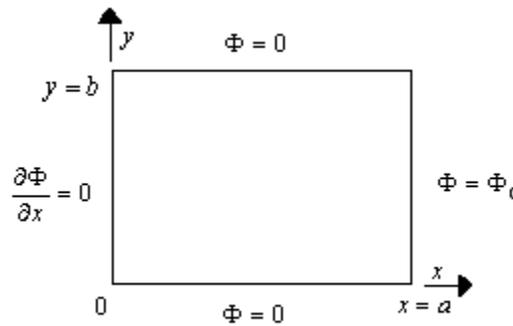
**Assigned in FA08**

Instructor notes: Most students have the procedure right, but don't know they should use different summation index to distinguish  $\sin(\pi n_y y/a)$  and  $\sin(\pi n_x z/a)$ , so there is only one summation in their results instead of two.

**Question 3. Potential of rectangle, with equipotentials**

SEP OF VARIABLES; SKETCH (Heald and Marion, 3-7 pg. 120)

Obtain the potential for any point within a two-dimensional rectangle, subject to the boundary condition given in the figure. (The system is invariant in the  $z$  dimension.) Sketch some equipotentials and field-lines.



**Question 4. Parallel plate capacitor and Legendre polynomials**

CONCEPTUALIZING THE COEFFICIENTS; SKETCH (Heald and Marion, 3-14 pg. 122)

A parallel-plate capacitor produces the uniform field described by the  $A_1$  term in the Legendre-polynomial expansion,  $V(r,\theta) = \sum_{l=0}^{\infty} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(\cos\theta)$ . What configuration of the electrodes would produce a region of space in which the potential is described by the  $A_2$  term alone? Sketch some equipotentials and field-lines.

**Question 5. Equipotentials for sphere in E field**

VISUALIZATION (Marion and Heald, 3-15 pg. 122)

Sketch some of the equipotentials lines for the cases discussed in Examples 3.3(a) and 3.3(b) [the conducting and dielectric sphere placed in a uniform E field]

**Question 6. Charged sphere in E field**

CALCULATION (Marion and Heald, 3-16 pg. 122)

Modify Example 3.3(a) [conducting sphere in E field; Griffiths example 3.8] by assuming that the conducting sphere carries a net charge  $Q$ . Find the potential exterior to the sphere.

**Question 7. Concentric spherical surfaces with theta-dependent V**

CALCULATION (Marion and Heald, 3-19 pg. 122)

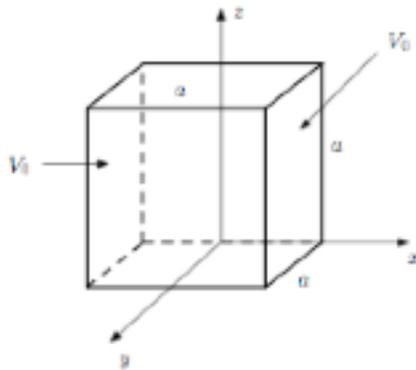
Two concentric spherical surfaces have radii of  $a$  and  $b$ . If the potential on the inner surface is given by  $\Phi_a P_3(\cos\theta)$  and the potential on the outer surface is given by  $\Phi_b P_3(\cos\theta)$ , find the potential in the region between the two surfaces ( $a < r < b$ ).

**Question 8. Potential of square box 1**

SEPARATION OF VAR (CARTESIAN, 3D)

Consider a square box with length  $a$  on each side, where the two sides of the box that lie in the  $y$ - $z$  plane are held at a constant potential  $V_0$  and the other four sides are grounded. Find the potential inside the box. Note that one can solve this using the result from class and superposition, but it is cleaner to start over. Place the origin as shown in the figure below to exploit the symmetry. The boundary conditions may be stated as

$$\begin{aligned} V(-a/2, y, z) &= V_0 & V(-x, y, z) &= V(z, y, z) \\ V(x, 0, z) &= 0 & V(0, a, z) &= 0 \\ V(x, y, 0) &= 0 & V(x, y, a) &= 0. \end{aligned}$$



**Question 9. Potential of square box 2**

SEP OF VAR, CARTESIAN; CALCULATION, EXTENSION (From Pollack and Stump, Electromagnetism, Problem 5.30)

Consider a cube made of 6 conducting plates of size  $a \times a$ , that encloses the space  $-a/2 \leq x, y, z \leq a/2$ . The plates at  $z = \pm a/2$  are held at potentials  $\pm V_0$ , respectively, while the other four sides are all grounded ( $V = 0$ ).

- Solve the boundary value problem for  $V(x, y, z)$  inside the cube.
- The potential is obviously zero at the center of the cube. What is the electric field there?

**Question 10. Potential of square box 3**

SEP OF VAR, CARTESIAN (deGrand)

Find the potential inside a metal box which is a cube of sides of length  $a$ . At  $z = 0$   $V = -V_0$ , at  $z = a$   $V = +V_0$ , and  $V = 0$  at  $x = 0$ ,  $x = a$ ,  $y = 0$ , and  $y = a$ .

**Question 11. Energy of square box**

EXTENSION OF IDEAS, PROOF (U WASHINGTON)

3<sup>†</sup> In class I considered the potential inside a box ( $0 \leq x \leq a$ ), ( $0 \leq y \leq b$ ) with  $V(0, y) = V(a, y) = V(x, 0) = 0$  and found the general solution

$$V(x, y) = \sum_{n=1} F_n \left(\frac{2}{a}\right)^{1/2} \sin(n\pi x/a) \frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}$$

- a) From the above, write down an expression for the electric field.  
 b) The energy of the system is

$$U = \frac{\epsilon_0}{2} \int_0^b dy \int_0^a dx E^2.$$

The integral over  $x$  can be carried out easily when you recall that  $(2/a)^{1/2} \sin(n\pi x/a)$  and  $(2/a)^{1/2} \cos(n\pi x/a)$  are orthonormal functions in the interval  $0 \leq x \leq a$ . To do the integral over  $y$ , recall the identity  $\sinh^2 z + \cosh^2 z = \cosh(2z)$ , Use this and the identity  $\sinh 2z = 2 \sinh z \cosh z$  to show that

$$U = \frac{\epsilon_0}{2} \sum_n (F_n)^2 \frac{n\pi}{a} \coth(n\pi b/a)$$

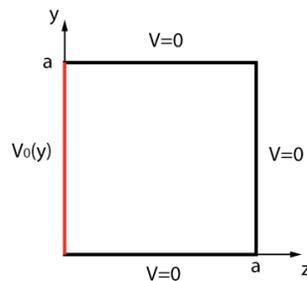
### Question 12. Potential of square tube

SEP OF VAR (CARTESIAN, 2D) (Downloaded from Reed).

(a) Consider a square rectangular tube as shown below. Three walls are grounded and the wall at  $x=0$  is at a potential  $V_0(y)$  given by

$V_0(y) = V_0$  (when  $y < a/2$ ), and  $-V_0$  (when  $y > a/2$ ).

Find the potential  $V(x,y)$  everywhere in the box and make a plot of  $V(x,y)$  using the few non-zero Fourier modes.



(b) Consider the same situation except this time the potential at  $x=0$  is given by

$$V_0(y) = V_0 \sin\left(\frac{\pi \cdot y}{a}\right)$$

Find the potential  $V(x,y)$  inside the box, the electric field  $\mathbf{E}(x,y)$  inside the box, and the charge density  $\sigma$  on the conducting wall at  $x=a$ . Make sure that  $\mathbf{E}$  is perpendicular to this wall as required.

### Question 13. Potential of infinite slot

SEP OF VAR, CARTESIAN (deGrand)

For the infinite slot example ( $V = 0$  at  $y = 0$  and  $y = Y$ ,  $V \rightarrow 0$  as  $x \rightarrow \infty$ ), find  $V(x,y)$  if  $V(0,y) = +V_0$  for  $0 < y < Y/2$ ,  $V(0,y) = -V_0$  for  $Y/2 < y < Y$ . You can begin with a formula like Eq. 3.26.

### Question 14. Potential of charged metal sphere in E field

SEP OF VAR, SPHERICAL (deGrand)

A metal sphere is given a charge  $Q$  and placed in an otherwise uniform E-field. Find the potential everywhere.

### Question 15. Potential of charged disk

SEP OF VAR, SPHERICAL (deGrand; similar to a Griffiths problem)

A disk of radius  $R$  has a uniform surface charge density  $\sigma$ . (a) [5 points] Find the potential along the axis of the disk. (b) [10 points] Find the potential away from the axis, for distance  $r > R$ , using the result of (a) and fiddling with the Legendre formula. (c) [10 points] Now repeat for  $r < R$ . Caution: you must treat the top and bottom sides of the disc separately.

### Question 16. Potential of line charge, expansion

EXPANSION (C. Gwinn, UC Santa Barbara)

a) Consider a segment of line charge, extending from  $z = 0$  to  $z = -d$ . The charge density along the line is  $\lambda = q/d$ . Find the potential  $V(z)$  along the positive  $z$ -axis.

b) Expand your potential  $V(z)$  from part a in powers of  $1/z$  along the  $z$ -axis. (You may use Taylor series, or use Mathematica or Maple, or look it up in a table; but please explain how you got your result).

c) Use your expansion from part b to evaluate coefficients  $b$  in a general solution for Laplace's Equation at large  $z$ :

$$V(r, \theta) = \sum_{\ell} b_{\ell} r^{-(\ell+1)} P_{\ell}(\cos \theta)$$

Find the region where this series converges. What's the problem at small, positive  $z$ ?

For fun: Find a series that will converge, for the region at small  $z$  where your result from part c fails.

### Question 17. Potential of non-uniformly charged sphere

SEP OF VAR, SPHERICAL; CALCULATION (From Pollack and Stump, Electromagnetism, Problem 5.33)

a. Suppose that on the surface of a sphere of radius  $R$  there is a surface charge density  $\sigma(\theta) = \sigma_0 \cos^2 \theta$ , where  $\sigma_0$  is a constant. What is  $V(r, \theta)$  inside and outside the sphere? What is the electric field  $E(r, \theta)$  inside and outside the sphere? Confirm that the field is

discontinuous at the surface of the sphere in accord with the given charge distribution.  
 b. Now repeat the above exercise for the surface charge density  $\sigma(\theta) = \sigma_0 \sin^2 \theta$ . Note that in this case:

$$V(r, \theta) = \begin{cases} \frac{2}{3} \frac{R\sigma_0}{\epsilon_0} - \frac{2R\sigma_0}{15\epsilon_0} \left(\frac{r}{R}\right)^2 P_2(\cos \theta) & r < R \\ \frac{2}{3} \frac{R\sigma_0}{\epsilon_0} - \frac{2R\sigma_0}{15\epsilon_0} \left(\frac{R}{r}\right)^3 P_2(\cos \theta) & r \geq R \end{cases}$$

### Question 18. Spherical

The potential on the surface of a sphere (radius R) is given by  $V = V_0 \cos(2\theta)$ .

(Assume  $V(r=\infty)=0$ , as usual. Also, assume there is no charge inside or outside, it's ALL on the surface!)

i) Find the potential inside and outside this sphere.

(Hint: Can you express  $\cos(2\theta)$  as a simple linear combination of some Legendre polynomials? )

ii) Find the charge density  $\sigma(\theta)$  on the sphere.

**Assigned in SP08**

**Assigned in FA08**

Instructor notes: Overall students seemed to do well with this. Some students seemed totally unfamiliar with the Gauss' Law trick to determine the discontinuity in E at a surface charge, which is a trick that could use emphasis.

### Question 19. Spherical sigma

The surface charge density on a sphere (radius R) is a constant,  $\sigma_0$

(As usual, assume  $V(r=\infty)=0$ , and there is no charge anywhere inside or outside, it's ALL on the surface!)

i) *Using the methods of section 3.3.2* (i.e. explicitly using separation of variables in spherical coordinates), find the electrical potential inside and outside this sphere.

ii) Discuss your answer, explain how you might have just "written it down" without doing all that work! (Be explicit - what about all the specific coefficients you got in i?)

iii) Now, suppose the surface charge density is  $+\sigma_0$  on the entire northern hemisphere, but  $-\sigma_0$  on the entire southern hemisphere. Again, find voltage inside and outside. (This time, you will in principle need an infinite sum of terms - but for this problem, just work out explicitly what the first *two* nonzero terms are. (In both cases, for  $V(r<R)$ , and  $V(r>R)$ )

Note: some terms you might have expected to be present will vanish. Explain physically or mathematically why the first "zero" term really *should* be zero.

Griffiths solves a generic example problem, for which part i above is a simple special case (and for that matter, so is part iii). But, please work through the details on your own - you're welcome to use Griffiths to guide you if/whenever you need it, but in the end, solve the whole problem yourself and show your work!

**Assigned in SP08 (average score: i) 9.29, ii) 3.29, iii) 7.76)**

Instructor notes: Many questions on this one.

Students were confused about the simpleness of parts (i) and (ii) and didn't understand why they had to go through the whole procedure when they already knew the answer. And even when they DID start going through the separation of variables game, it wasn't clear to them when they could start taking advantage of the simplifications that occur

once you know  $\sigma$  is just a pure  $P_0$  (constant). I told everyone to treat the problem as though  $\sigma$  was GENERAL/unknown, get as FAR as they could, and only then see what happens when you realize that  $\sigma$  is pure  $P_0$ . This seemed to help clarify.

Many students struggled with the +  $\sigma$  and -  $\sigma$  part of the problem. They weren't sure how to take that into account. Some broke up the integral at the wrong point (for example, before they solved for their  $A_l$ 's using Fourier's trick, they tried to just solve for the  $A_l$ 's in one region or the other). I tended to probe them by writing  $\sigma$  as  $\sigma(\theta)$  instead, so that when they set up their integral in Fourier's trick they noticed that  $\sigma$  was a function of  $\theta$  and they had to have two separate integrals. Most saw it at that point. Lot of students were a little stumped in part iii, when  $\sigma$  can not "by inspection" be written as one (or a couple) Legendre terms. They asked surprisingly elementary questions, like whether the integral from zero to  $\pi$  "could be" split up into two integrals, one from 0 to  $\pi/2$  the other from  $\pi/2$  to  $\pi$ . (I was just surprised they had to ask me if that was legit.). Several expressed confusion as to why every other term was vanishing, and a couple were frustrated by the fact that they had to actually look up and integrate  $P_3$ !

### Question 20. Dipole moment of a sphere of charge

The surface charge density on a sphere of radius  $R$  is  $+\sigma_0$  on the entire northern hemisphere, and  $-\sigma_0$  on the entire southern hemisphere, where  $\sigma_0$  is a positive constant. There is no other charges present inside or outside the sphere.

A) Use the methods of section 3.3.2 (explicitly using separation of variables in spherical coordinates) to find the electrical potential inside and outside this sphere. In principle, you will need an infinite sum of terms, but for this problem, just work out explicitly what the first *nonzero* term is, for both  $V(r < R)$ , and  $V(r > R)$ .

Explain physically why the first "zero term" really should be zero. This first non-zero term potential should look familiar. What is its name?

*Griffiths solves a generic example problem, but please work through the details on your own. You are welcome to use Griffiths to guide you if/whenever you need it, but in the end, solve the problem yourself and show your work!*

B) Compute the dipole moment of this sphere (with the +z-axis up through the pole of the positive hemisphere). Begin with the definition of a dipole moment,  $\vec{p} = \int \rho \vec{r} d\tau$ , which, in this case, becomes  $\vec{p} = \int \sigma \vec{r} da = \hat{x} \int da \sigma x + \hat{y} \int da \sigma y + \hat{z} \int da \sigma z$ . Two of the three components are zero. Working in spherical coordinates, show why those components are zero. (Show this explicitly. Simply stating that they are zero "by symmetry" is not enough.) Write your final answer for the dipole moment in terms of the charge  $Q$  on the upper hemisphere. Does your answer make sense? How does your answer to part A relate to this part?

**Assigned in FA08**

### Question 21. Concentric spheres

Two concentric spherical surfaces have radii of  $a$  and  $b$ . If the potential on the inner surface, at  $r = a$ , is a nonzero constant  $V_{in}$  and the potential on the outer surface is given by  $V(b, \theta) = V_{out} P_1(\cos \theta) = V_{out} \cos \theta$ , where  $V_{out}$  is a constant, find the potential in the region *between* the two surfaces ( $a < r < b$ ). Check explicitly that your final answer gives the correct results for  $r = a$  and  $r = b$ .

**Assigned in SP08 (average score: 7.7)**

**Assigned in FA08**

Instructor notes: There was a lot of difficulty in setting up the BC's for this. They got stuck thinking that we knew the potential AT the boundary and didn't see how to use this to match BC's. They didn't see that the potential had a functional form on either SIDE of the boundary, and that was what they needed to use and then match for the coefficients. Many people (on this and on the previous) didn't quite seem to recognize that if we had, for example, a term that went as  $1/r$  on one side of the equation, then the coefficient of that term HAS to match the term that goes as  $1/r$  on the other side of the equation. There can be a tendency to jump the gun, writing down the boundary condition for JUST the  $l=0$  term at the inner boundary, and the  $l=1$  term at the outer boundary, not realizing that there is information about ALL terms at EACH boundary. I also thought at first I didn't have enough information/boundary conditions to solve the problem.

### Question 22. Multipole and Separation of Variables - Disk

A disk of radius  $R$  has a uniform surface charge density  $\sigma_0$ . Way back on Set #2 you found the E-field along the axis of the disk (and on the midterm, you again solved a very similar (but *harder*) version of this where  $\sigma$  was not uniform). You can check for yourself by direct integration, (but don't have to): I claim that along the  $z$  axis, (i.e.  $\theta=0$ ),

$$V(r, \theta = 0) = \frac{\sigma_0}{2\epsilon_0} \left( \sqrt{r^2 + R^2} - r \right)$$

i) Find the potential *away* from the axis (i.e. nonzero  $\theta$ ), for distances  $r > R$ , by using the result above and fiddling with the Legendre formula, Griffiths' 3.72 on page 140. You will in principle need an infinite sum of terms here - but for this problem, just work out explicitly what the first *two* \*non-zero\* terms are.

*(It might help to remember that  $P_l(1)$  is always equal to 1, and you will have to think mathematically about how the formula above behaves for  $r \gg R$ )*

ii) Griffiths Chapter 3.4 talks about the "multipole expansion". Look at your answer to part i, and compare it to what Griffiths says it *should* look like (generically) on page 148. Discuss - does your answer make some physical sense? Note that there is a "missing term" - why is that?

**Assigned in SP08 (average score: i) 8.57, ii) 3.33)**

**Assigned in FA08**

Instructor notes: Some students, not surprisingly, struggle with the "theta" dependence in the Legendre formula and want to delete the theta-dependent term, not recognizing that in this case  $\theta=0$  and  $P_l(1)=1$ . The idea of matching terms with the same  $(1/r)$  dependence is a logical step that many don't immediately recognize. Many do not have Taylor series memorized.

In part (ii), many students saw that the dipole was missing, but had no ability to argue WHY that should be the case. They were not making the math/physics connection. Some argue that the absence of negative charge results in the lack of a dipole moment.

**Question 23. Charged metal sphere**

You have a conducting metal sphere (radius  $R$ ), with a net charge  $+Q$  on it. It is placed into a pre-existing uniform external field  $\mathbf{E}_0$  which points in the  $z$  direction. (So, this is *exactly like* Griffiths Example 3.8, except the sphere is not neutral to start with.) Find the potential everywhere inside and outside this sphere. Please explain clearly *where* you are setting the zero of your potential. Do you have any freedom in this matter? Briefly, explain.

**Assigned in SP08 (extra credit)**

**Assigned in FA08 (extra credit)**