

5.22.

Ampere's Law: + "Gauss for Magnets"

There are two of Maxwell's eq'ns, tells you the connection between current + B field. It's the magnetic analogy of

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0 \quad \longleftrightarrow \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \longleftrightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (\text{Ampere's Law})$$

(All of these are for electro/magneto statics, + will be fleshed out later with time dependence!)

Could say "these are exper'tal facts" + go from there!

But it's nice to see consistency / connection back to Biot-Savart

I claim: From Biot-Savart  $\rightarrow$  you can show Ampere's Law \*  
" Ampere's Law  $\rightarrow$  " " " Biot-Savart \*\*

Much like From Coulomb  $\rightarrow$  you can show Gauss' law  
From Gauss' Law  $\rightarrow$  " " " Coulomb's

+ Like there, the "Maxwell Form" turns out to be deeper + broader.  
" " , both may prove useful in different circumstances!

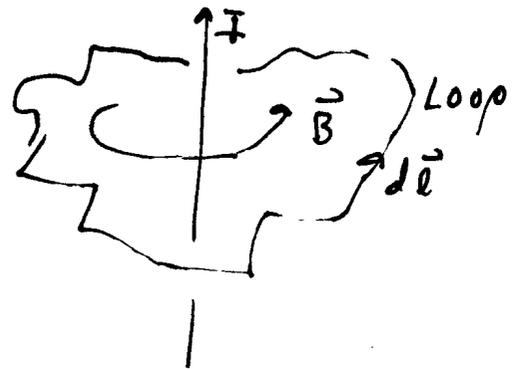
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\* is hard - see Griff 5.3.2

\*\* is harder!

Plausibility argument:

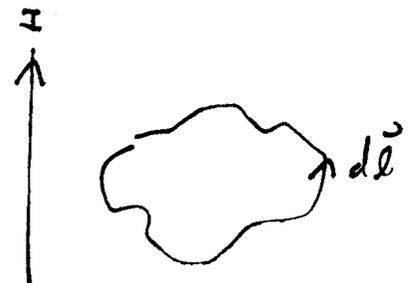
we know of one case (a wire) we can solve for  $\vec{B}$  with Biot-Savart,  
 we did this + got  $\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi r} \hat{\phi}$



In this case,  $\oint \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{r} \hat{\phi} \cdot d\vec{l}$   
 Any loop of any shape around the wire  $\int d\phi$

$$= \frac{\mu_0 I}{2\pi} \cdot \oint d\phi = \mu_0 I.$$

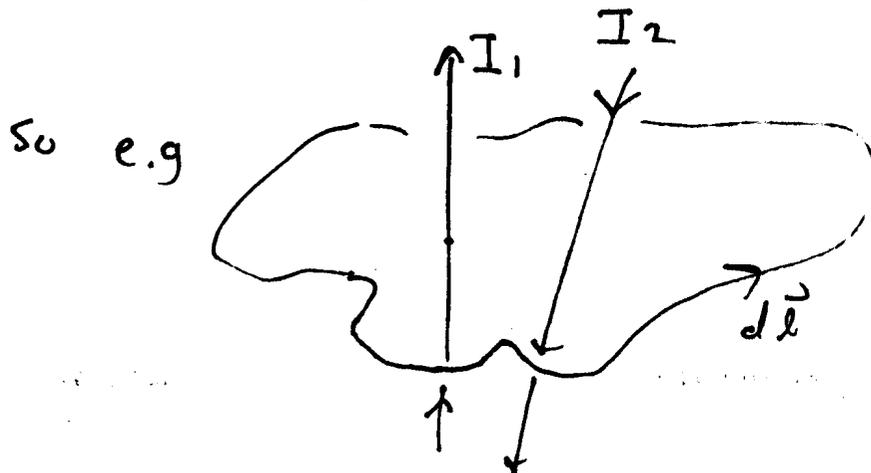
If the loop had not gone around I, then  $\int d\phi = 0!$   
 like this:



so here  $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$

By superposition, with multiple currents,  $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{total}}$  poking thru loop.

- Note: If loop is opposite direction ( $d\vec{l}$  goes other way)  $\Rightarrow$  - sign,  
 which means I is + if it "pokes" through loop in RH sense  
 I is - " " " " " " " " LH sense



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (|I_1| - |I_2|)$$

here.

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Also,  $I_{\text{enc}} = \iint_{\text{where this covers the loop area}} \vec{J} \cdot d\vec{A}$  ← Just def of  $\vec{J}$ !

$$\text{so } \oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{J} \cdot d\vec{A}$$

Stokes ↓

$$\iint (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \mu_0 \iint \vec{J} \cdot d\vec{A} \quad \text{true for any / every loop}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}} \quad \text{which is Ampere's Law.}$$

This is not a proof, it assumed  $\infty$  wires ... but it shows consistency.

Proof involves starting with Biot-Savart:  $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2} d\vec{r}'$

then formally showing  $\vec{\nabla} \cdot \vec{B} = 0$

and  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r})$  By taking  $\vec{\nabla} \cdot$  or  $\vec{\nabla} \times$  this ↑

\* "massaging". It's a lovely exercise in vector calc.

\* Puzzle: "I<sub>through</sub>" =  $\iint \vec{J} \cdot d\vec{A}$ . But there are many surfaces sharing the same boundary loop. Which do you use?

(A: It's I<sub>through</sub>! Any / All OK)

## 5.25

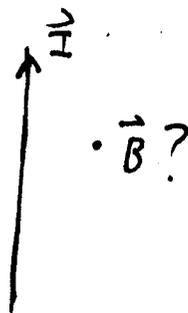
Applying Ampere's Law: For steady currents

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}} \quad \text{or} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

↑ any loop      ↑ I "poking through" that loop, (in the R.H. sense)

Like Gauss, Ampere's law lets us find  $\vec{B}$  if we have symmetry + can argue up front that  $B$  is constant +/or can be pulled out of the integral. (Just like Gauss)

Ex #1: The Infinite wire:



• Claim:  $\vec{B}$  cannot be radial, this would violate  $\vec{\nabla} \cdot \vec{B} = 0$  or  $\iint \vec{B} \cdot d\vec{\lambda} = 0$   
(consider a gaussian "can" around the wire)

• Claim:  $\vec{B}$  cannot have any  $B_z$  ~~dependence~~ dependence, or  $\phi$  dependence (symmetry!)

• Claim:  $\vec{B}$  " " "  $\hat{z}$  component (why?) ← (Biot-Savart!)

• So  $\vec{B} = B(s) \hat{\phi}$ , just by symmetry.

$$\text{then } \oint \vec{B} \cdot d\vec{\ell} = B \oint d\ell = B \cdot 2\pi s = \mu_0 I$$

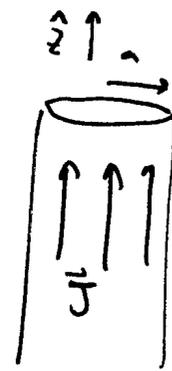
Pick a circular loop

$$\text{so } B = \frac{\mu_0 I}{2\pi s}$$

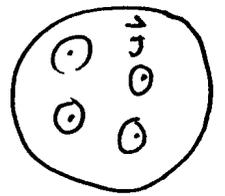
(which we knew from a horrible integration of Biot-Savart, this is much easier!)

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Example II: A thick wire  
 radius  $a$ ,  
 uniform  $\vec{J} = J_0 \hat{z}$  for  $s \leq a$   
 $= 0$  for  $s > a$



side



Top view

Draw Amperian Loop, circle centered around origin.

Claim: Like thin wire,  $\vec{B} = B(s) \hat{\phi}$  still. we haven't introduced  
 any  $z$  dependence by thickening wire! This is the crucial part,  
 once we know this,

$$\oint_{\text{around loop}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{thin loop}} \quad \text{both sides are simple}$$

$$\text{L.H.S. } \oint \vec{B} \cdot d\vec{l} = \oint B(s) dl = B(s) \cdot 2\pi s$$

$$\text{R.H.S. } I_{\text{thin loop}} = \iint \vec{J} \cdot d\vec{A} \quad \text{If } s \leq a, \text{ this gives}$$

$$\iint J_0 dA = J_0 \cdot \pi s^2$$

$$\text{so } s \leq a \Rightarrow \vec{B}(s) = \mu_0 J_0 \frac{s}{2} \hat{\phi}$$

$$s > a \Rightarrow \vec{B}(s) = \mu_0 \frac{J_0 a^2}{2s} \hat{\phi}$$

$$\text{If } s > a, \text{ this gives}$$

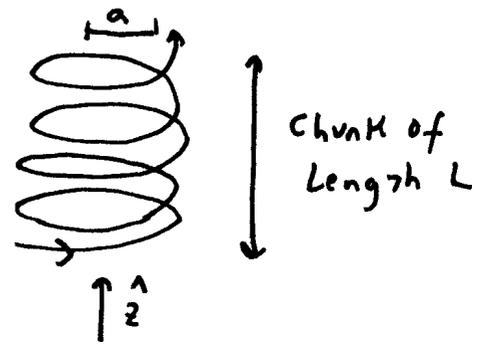
$$\iint_{\text{out to } s=a} J_0 da = J_0 \cdot \pi a^2 (= I_{\text{tot}})$$

Note: No surface current  $\Rightarrow \vec{B}$  is continuous ✓  
 $\vec{B}(0) = 0$ , makes sense by symmetry!

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Example III: Infinite Solenoid

- Radius  $a$
- assume  $n$  windings / meter.



(so  $nL = 4$  in my little sketch!)

Outside:  $\vec{B}$  better not have  $\hat{s}$  component  $\rightarrow$  that would make  $\oint \vec{B} \cdot d\vec{\lambda}$  non zero, violating  $\vec{\nabla} \cdot \vec{B} = 0$ . (in or out!)

How about a  $\hat{\phi}$  component? Tempting! (what a wire gave)

But no, draw a loop,  $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$  through of radius  $s$

$$\underbrace{\oint \vec{B} \cdot d\vec{l}}_{B(s) \cdot 2\pi s} = \mu_0 \cdot 0 \leftarrow \text{!! convince yourself!}$$

current is not "poking through", it circles around! well....

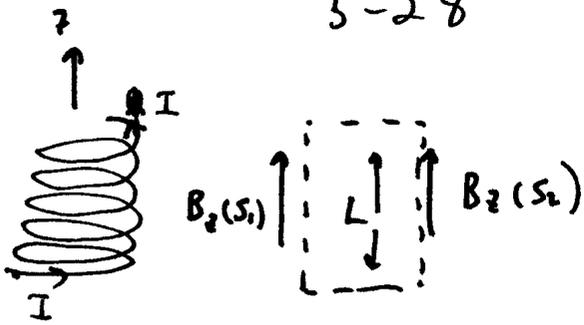
(N.B. I'm assuming we really have  $\vec{K} \Rightarrow$  Purely in  $\hat{\phi}$  direction)

In reality,  $I$  does flow up page  $\Rightarrow$  Through this loop, so there is the usual  $\mu_0 I$  on RHS. But you'll see, this is small + not so important compared to the huge  $B$  inside we're going to get

So maybe  $\vec{B} = B_z(s) \hat{z}$ . Really all we could have.

- Claim:  $B_z(\infty) = 0$ .
- Experimental fact
  - could convince yourself from Biot-Savart
  - $B$  must vanish far from line currents...

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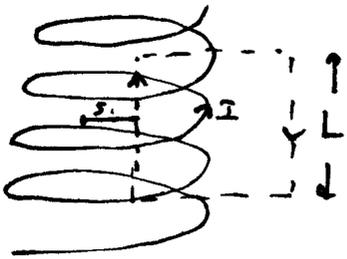
$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{thru}}$$

$$(B(s_1) - B(s_2))L = 0$$

so  $B(s_1) = B(s_2)$

But  $B(\infty) = 0$ , so  $B = 0$  everywhere outside !!

Surprising, important result!



Now use this loop

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{through}}$$

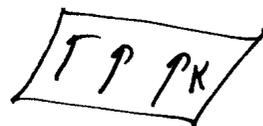
$$B(s_1) \cdot L + 0 = \mu_0 \underbrace{I}_{\text{this current through each wind}} \cdot \underbrace{nL}_{\text{this many windings}}$$

$$\text{so } \vec{B} = \mu_0 n I \hat{z} \quad \text{if } s < a$$

$$= 0 \quad \text{if } s > a.$$

Sort of "Capacitor-Like": uniform  $B$  inside  
 $0$   $B$  outside.

See Griff for more examples:



Sheet

Sheet +

Torus are both good ones!