

# 6-10

In general, inside any material, you'll have free currents (basically, wires running through it, or flowing ions...) and, as a result,  $\vec{B}$  fields appear which further polarize the material, adding in these bound currents. (which in turn alter the field even more!)

All together, in "equilibrium"

$$\vec{J} = \vec{J}_{\text{free}} + \vec{J}_{\text{bound}}$$

$\uparrow$   
what you  
"inject"

$\uparrow$   
what the material  
responds with

This  $\vec{J}$  is real, it creates the total, real  $\vec{B}$  field via Ampere's law,  $\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{J} \cdot d\vec{A}$

$$\underbrace{\vec{\nabla} \times \vec{B}}_{\text{no exceptions in magnetostatics, this is Ampere's law}} = \mu_0 \vec{J} = \mu_0 (\vec{J}_f + \vec{J}_{\text{bound}}) = \mu_0 (J_f + \vec{\nabla} \times \vec{M})$$

no exceptions in magnetostatics, this is Ampere's law

$$\text{so } \vec{\nabla} \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f$$

[ See notes p.4-11  
the electric stuff  
was very similar,  
leading to  $\vec{D}$  ]

$$\text{so we Define } \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

usually easy to measure  
this is the current in wires, normally!

$$\text{with } \vec{\nabla} \times \vec{H} = \vec{J}_f \quad \text{or} \quad \oint \vec{H} \cdot d\vec{l} = I_f, \text{ through}$$

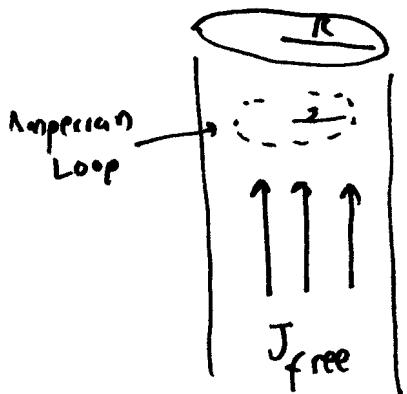
[ Units of  $H$  are Amps/m, not Teslas! ]

[ Not really sure what to call  $H$ ,  
it's just  $H$ ! ]

Example A long Al rod (Radius R) carries uniform  $J_{\text{free}}$ , total current  $I = J_f \cdot \pi R^2$  in +z direction.

Find H and B everywhere.

Like Griff ex 6.2, but Al is paramagnet, his example was Cu = diamagnet, so compare orthis with his example!



I expect  $\vec{B} = \text{circle}$  by Ampere's law  
(just like always, current is up,  $\vec{B}$  circulates)

I expect  $\vec{M}$  will be parallel to  $\vec{B}$  because  
inside

it's a paramagnet so  $\vec{M} = \text{circle}^{700}$

of course, outside is vacuum, so  $M_{\text{outside}} = 0$ .

Since  $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ , these "cancel", but I know  $\vec{M}$  is

going to be small/weak for real materials, so I'm sure  
 $\vec{H}$  will still (also) go  $\text{circle}$  this way. Indeed,

$\oint \vec{H} \cdot d\vec{l} = I_f$ , through proves that  $\vec{H}$  "looks" like  $\vec{B}$ ,  
Loop inside, shown dashed in direction at least

$$\text{so } \oint H \cdot d\ell = I_{\text{f}, \text{through}} \Rightarrow H \cdot 2\pi s = J_f \cdot \pi s^2$$

$$\vec{H} = \frac{J_f s}{2} \hat{\phi} = \frac{I}{2\pi R^2} s \hat{\phi} \quad (\text{since } I = J_f / \pi r^2)$$

This is same as Griff, dia or para, makes no diff!

outside  $\vec{H} = \frac{I}{2\pi s} \hat{\phi}$  (same as Griff, doesn't matter)  
what material is

outside  $\vec{M} = 0$ , so  $\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$  ← usual old  
infinite wire  
result,  
material independent!

Inside, we know direction of  $\vec{M}$ , + know it's less than  $\frac{\vec{B}}{\mu_0}$ ,

but we're stuck without knowing how it magnetizes.

We'll get to that soon!

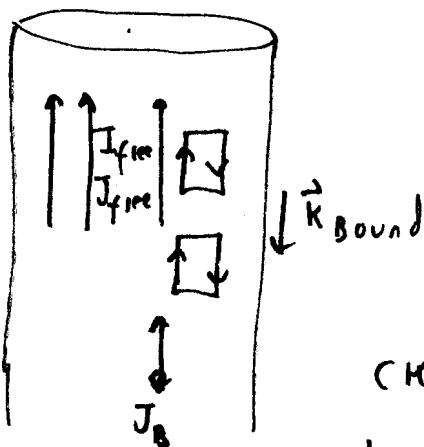
But I can argue qualitatively that if  $\vec{M}$  looks like 

then  $\vec{M} \times \hat{n}$  will run down the outside of cylinder (parallel to  $\vec{J}_f$ )

(That's opposite Griff example!)

and  $\vec{B} \times \vec{M} = J_B$  points up the cylinder (~~parallel to~~ <sup>parallel to</sup>  $\vec{J}_f$ ,  
which is consistent with paramagnetism, (and opposite Griff ex))

so



But to compute  $\vec{J}_B$  and  $\vec{H}_B$ , we need  $M_{\text{inside}}$ , and I can't get that from my (known)  $H_{\text{inside}}$  without knowing how  $A_E$  magnetizes. So...

### Linear Materials

Many (common) materials magnetize proportional to  $\vec{B}$

Recall electric polarization was  $\vec{P} = \epsilon_0 \chi_E \vec{E}_{\text{tot}}$  (linear dielectrics)

we define  $\chi_m$  for magnetic materials as

$\vec{M} = \chi_m \vec{H}$  note the lack of "symmetry", in the case of dielectrics,  $\chi_E$  is defined looking at  $\vec{E}$ , but for magnetic materials, we use  $\vec{H}$ , not  $\vec{B}$ . Why? Because  $\vec{H}$  is easy to compute in many cases!  $\oint \vec{H} \cdot d\vec{l} = I_{\text{free}}$ ,  $\leftarrow$  that's what you measure + control, not  $I_f + I_b$

$\chi_m$  is "magnetic susceptibility"  
Unless, typically small.

$\chi_m$  is + for paramagnets,  $\vec{M}$  lines up with  $\vec{H}$

$\chi_m$  is - for diamagnets,  $\vec{M}$  opposes  $\vec{H}$ .

of course,  $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$  so  $\vec{B} = \mu_0(\vec{H} + \vec{M})$   
 $= \mu_0 \vec{H} (1 + \chi_m)$

so  $\vec{H}$  and  $\vec{B}$  point in the same direction, if  $\chi_m > 0$   
 and even if  $\chi_m < 0$ , as long as it's  $< 1$ .

For normal materials,  $|\chi_m|$  is like  $10^{-9}$  to  $10^{-95}$

For superconductors,  $\chi_m = -1$ . (which means  $\vec{B} = 0$  inside.)  
 total "shielding"

Summary

$\vec{M} = \chi_m \vec{H}$	$\rightarrow$	"permeability"
$\vec{B} = \mu_0(1 + \chi_m) \vec{H}$	$\equiv$	$\mu \vec{H}$
$\vec{M} = \frac{\chi_m}{\mu} \vec{B}$		

Note: free space  $\vec{B} = \mu_0 \vec{H}$ , so  $\mu_0 = 4\pi \cdot 10^{-7} \frac{N}{A^2}$  is  
 "permeability of free space"

Back to our Al rod example:  $\chi_{Al} = +2 \cdot 10^{-5}$   
 $\uparrow$   
paramagnetic

we had found (p. 6-12)  $\vec{H}_{\text{inside}} = \frac{I}{2\pi R^2} s \hat{g}$

so  $\vec{M}_{\text{inside}} = \chi_m \frac{I}{2\pi R^2} s \hat{g}$  very small

$\vec{B}_{\text{inside}} = \mu \frac{I}{2\pi R^2} s \hat{g}$  which is almost identical to

what we got in ch. 5 without knowing about magnetization,

since  $\mu = \mu_0(1 + \chi_m)$ . However,  $\mu > \mu_0$  (a bit)  
 $\uparrow$   
tiny! so  $\vec{B}_{\text{inside}}$  is "enhanced" a bit.

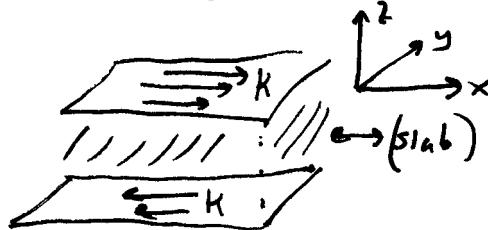
That's paramagnetism. A copper wire has  $\chi_m \approx 10^{-5}$ ,

so  $\vec{B}_{\text{inside}}$  is ever so slightly reduced  
(outside, none of this matters.)

Example : TWO ~~SHEETS~~ CARRY CURRENT  $I$ ,

TOP IN  $+x$  direction

BOTTOM IN  $-x$  "

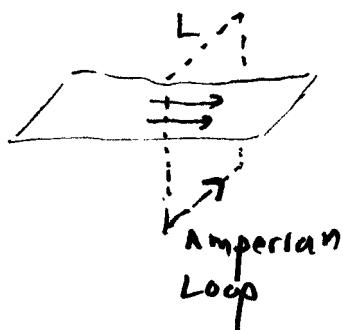


Both are same  $|K|$ .

Between sheets is "linear material" (Slab with susceptibility  $\chi_m$ )

what's  $\vec{B}$ ,  $\vec{H}$ ,  $\vec{M}$  in slab? First,  $\oint H \cdot dL = I_{\text{free}}$  ← what's what "K" gives!

By symmetry, I know  $\vec{H}$  will point into page ( $+y^1$ ) between slabs, + will cancel to zero outside (Think back to this same example in free space with no slab!)



so this AMPERIAN LOOP gives  $\oint H \cdot dL = H \cdot L$

$$I_{\text{free}} = K \cdot L$$

$$\text{so } \vec{H} = K \hat{y}^1 \text{ in between.}$$

$$= 0 \text{ outside}$$

$$\text{so } \vec{M}_{\text{inside}} = \chi_m \vec{H} = \chi_m K \hat{y}^1$$

$$\vec{B}_{\text{inside}} = \mu \vec{H} = \mu K \hat{y}^1 = \mu_0 (1 + \chi_m) K \hat{y}^1$$

Pretty much same as free space, just slight ↑ enhancement for paramag reduction for diamag.

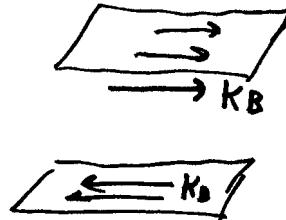
What do Bound currents look like in this example?

Inside  $\vec{\nabla} \times \vec{M} = 0$  (since  $\vec{M}$  is uniform)

So, no Bound  $\vec{J}_B$ .

But  $\vec{M} \times \hat{n} = +\chi_m K \hat{x}$  at top so

$-\chi_m K \hat{x}$  at bottom



This is parallel to  $K_{free}$ , but very small, if  $\chi_m > 0$

Makes sense, that's the mechanism to enhance  $\vec{B}$  inside

If  $\chi_m < 0$ , this opposes  $K_{free}$ , reducing  $\vec{B}$  inside a little.

### Boundary conditions

Since  $\oint H \cdot d\ell = I_{free}$ ,  $H''$  can "jump" at boundaries.

$$H''_{\text{above}} \cdot L - H''_{\text{below}} \cdot L = K_f \cdot L$$



or, in vector notation (Since  $H''$  in fact has two possible components we should really write)

$$\vec{H}_{\text{above}}'' - \vec{H}_{\text{below}}'' = \vec{K}_f \times \hat{n}$$

convince yourself this gives right answer for "steker" example above,

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Meanwhile,  $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$

so  $\nabla \cdot \vec{H} = - \nabla \cdot \vec{M}$  which says



$$H_{\perp}^{\text{above}} - H_{\perp}^{\text{below}} = - (M_{\perp}^{\text{above}} - M_{\perp}^{\text{below}})$$

This will vanish if  $\vec{M}$  is continuous. Since  $\vec{M} = \frac{\chi_m}{\mu} \vec{B}$  for linear materials, and  $\vec{B}_{\perp}$  is always continuous by  $\nabla \cdot \vec{B} = 0$ ,

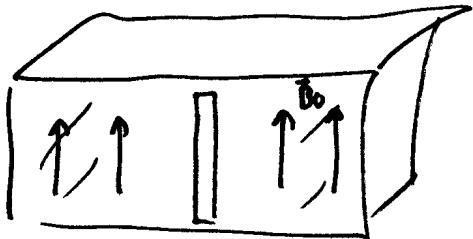
it means  $H_{\perp}$  is always continuous everywhere except

where  $\chi_m$  suddenly changes (edge of a material)

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Consequence:  $\oint \vec{H} \cdot d\ell = I_{\text{free}}$  looks simple, and for cases of "high symmetry" lets you deduce  $\vec{H}$  easily (a la Ampere's law). But if symmetry is not high, beware.

E.g. if  $I_{\text{free}} = 0$  everywhere, you cannot conclude  $\vec{H} = 0$  everywhere! Think of a toy magnet! Just because  $\nabla \times \vec{H} = 0$  everywhere does not mean  $\vec{H} = 0$  everywhere (unless some symmetry argument can be invoked  $\oplus$ , like our infinite sheet example)

Example :

Big chunk of material  
with uniform  $\vec{B}_0$  (up)  
throughout.

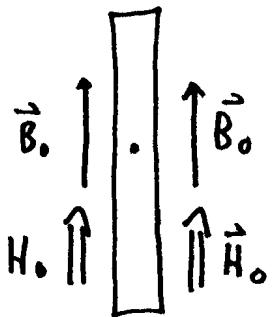
Thus  $\vec{B}_0$  is the total  $\vec{B}_0$  field, arising (perhaps) from external  $\vec{B}$  and the magnetization of the material, superposed.

so that material has  $\vec{M}_0 = \frac{\chi_m}{\mu} \vec{B}_0$  (uniform)

[and  $\vec{H}_0 = \frac{1}{\mu_0} \vec{B}_0 - \vec{M}_0$ , of course]

Then I dig a "needle shaped" hole, as shown. What's  $\vec{B}, \vec{H}$

inside the hole? (clearly  $\vec{M} = 0$  in there, it's a hole!)  
(at the center)



There are no free currents at this boundary,

$$\text{so } H^{\text{above}} - H^{\text{below}} = 0$$

which says  $H = H_0 \hat{z}$  inside the hole

But that means  $\vec{B} = \mu_0 \vec{H} = \mu_0 H_0 \hat{z}$  inside the hole, at centers.

In terms of  $B_0 + M_0$ , this is  $\mu_0 H_0 = B_0 - \mu_0 M_0 = \frac{B_0}{1 + \chi_m}$

so  $\vec{B}$  is reduced in there, if  $\chi_m > 0$  ] Due to Bound currents  
 "metal" has high  $\chi_m$ , shields  $\vec{B}$  inside!  $\vec{B}$  is enhanced if  $\chi_m < 0$ . ] on walls, presumably!  
 Picture is like solenoid...

### 6.19 supp.

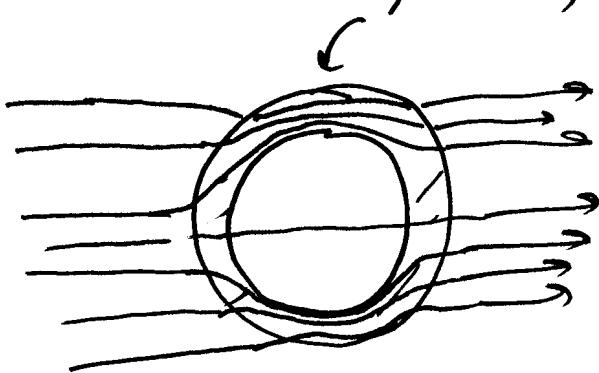
" $\mu$  metal" =  $\text{Ni}_{0.77} \text{Fe}_{0.16} \text{Cu}_{0.05} \text{Cr}_{0.02}$  has

$$\frac{\mu}{\mu_0} = 10^5 \quad (\chi = +10^5, \text{ unlike } \underline{\text{most}} \text{ paramags with } \cancel{\mu} \\ \text{typical } \chi \approx 10^{-3})$$

inside  $\mu$ -metal,  $B$  likes to be high, it's a super paramagn.

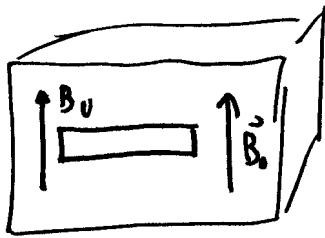
Inside the "hole",  $B$  is quite weak.

This is used on equipment where you want to eliminate external  $B$  fields

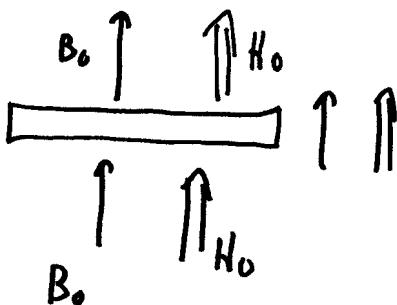


passively.

Let's do that for a wafer-like cavity:



now



This time, I can argue  $\vec{B}_\perp^{\text{above}} = \vec{B}_\perp^{\text{below}}$ , so  $\vec{B} = B_0 \hat{z}$   
at center

so this time  $\vec{B}$  is not modified!

But then  $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{B_0}{\mu_0} \hat{z}$  is not  $H_0$ !! It's  $\frac{\mu}{\mu_0} H_0$ !

$$\text{so } \vec{H}_{\text{center}} = \frac{\mu}{\mu_0} H_0 = \frac{B_0}{\mu_0} = H_0 + M_0$$

for paramag material,  $H_{\text{center}}$  is enhanced ( $\mu > 1$ )  
for " "  $H_{\text{center}}$  is reduced. ( $\mu < 1$ )

(Remember,  $H_\perp$  can jump at boundary, if  $M_\perp$  suddenly changes!)

Here, it's like we've "superposed" a disk of  $\downarrow \downarrow \downarrow \downarrow M_0$

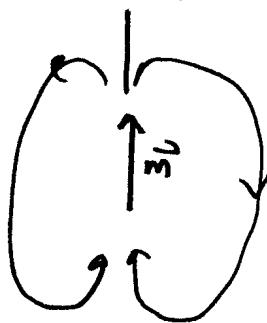
$B_{\text{center}}^{\text{superposed}} \sim \frac{1}{R}$  will be small if disk has big  $R$ , hence  $\vec{B}$  unchanged.

In prev example

we superposed a solenoid which does change  $\vec{B}$  inside!

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Ferromagnetism: won't spend a ton of time on this, it's pretty complicated microscopically, but some materials have a unique property, due to QM electron-pair interactions, that favors spin aligned electrons.



I can easily see why spins above & below like to align, but not why spins on sides would. It's not a classical effect!

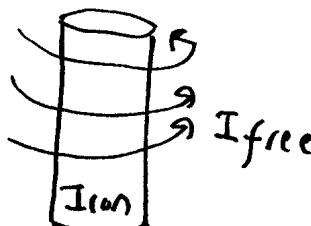
This means  $\vec{M}$  "locks in", it's not a response to  $\vec{B}_{ext}$ , so you cannot define  $\chi_m$  or  $\mu$  for ferromagnets. It's not a linear material!

It's a local effect, & you find regions of parallel spins  $\Rightarrow \vec{m}$ , called domains. See fig 6.26 in Griffiths. In your fridge, you can align those domains in presence of  $\vec{B}_{ext}$  (Kitchen magnet) making strong pair like effect  $\Rightarrow$  attraction.

When pull magnet away, thermal relaxation randomizes domains quickly, so fridge doesn't remain magnetized. ("Magnet" does =) it's a very special ferromagnet!)

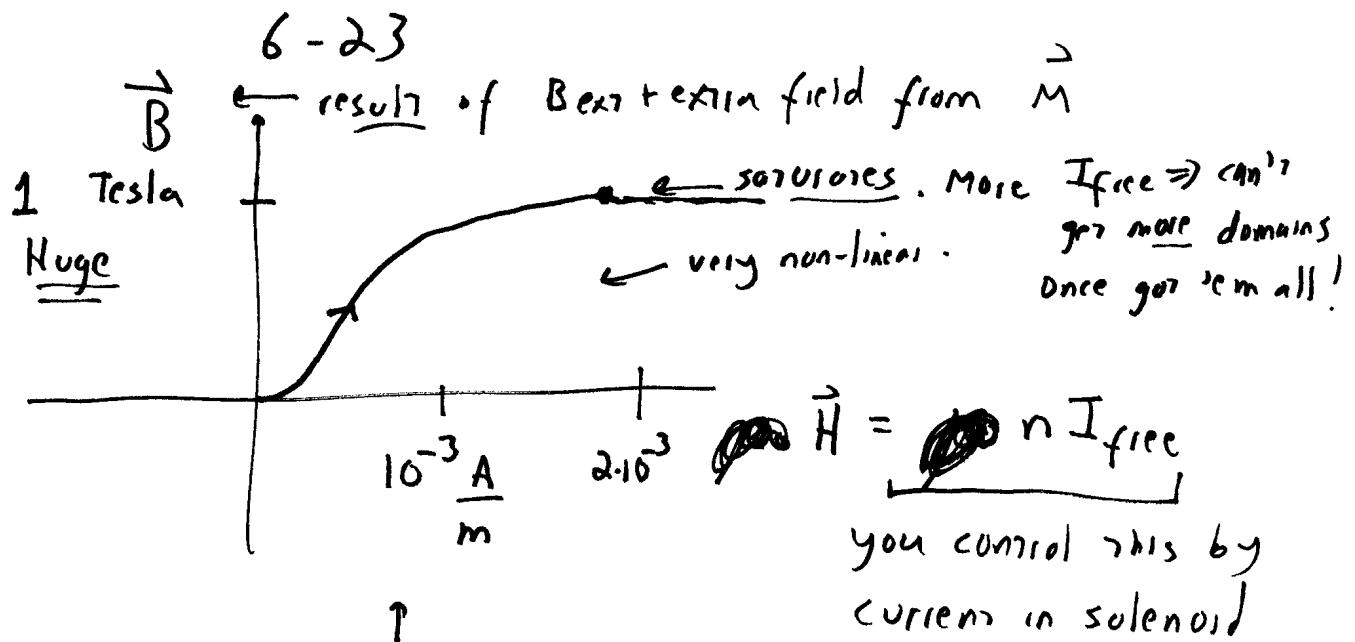
- Fe, Co, + Ni are about the only materials like this
- At high T, even permanent mags will randomize ("Curie Temp" is the critical value, above which they "melt their alignment")
- Typical Fe domains are  $\sim \text{mm}$  on a side.
- " Magnetization in a domain is  $M \sim 10^6 \frac{\text{A}}{\text{m}}$   
which yields typical  $\vec{B} = \underbrace{\frac{2}{3}\mu_0 M}_{\text{for spherical domain}} \sim 1 \text{ T}$ . So that's about the most you're likely to get from normal magnets
- Applying  $\vec{B}$  to ferromagnets merely re-aligns domains. So small  $\vec{B}_{\text{ext}}$  can radically (non-linearly!) alter  $\vec{B}_{\text{tot}}$ .

For Metal inside Solenoid  
(Iron, e.g.)



the  $I_{\text{free}} \Rightarrow \vec{H}_{\text{inside}}$   
in usual way,  
by Ampere:  $\vec{H}_{\text{inside}} = n \vec{I}_{\text{free}}$

But  $\vec{B}_{\text{inside}}$  will be  $\mu_0(\vec{H} + \vec{M})$ , +M can be huge (+ depends on  $\vec{H}$ , more field  $\Rightarrow$  more domains join in, influencing each other too)



Note that, by itself, this solenoid would only generate

$$\vec{B}_{\text{vacuum}} = \mu_0 \vec{H} = 4\pi \cdot 10^{-7} \frac{T \cdot m}{A} < 10^{-3} \frac{A}{m} \approx 4\pi \cdot 10^{-10} T$$

so we're ~~getting~~ getting billion fold enhancement!

If you back off the current,  $B$  decreases, but some alignment remains! This is "hysteresis", memory.

