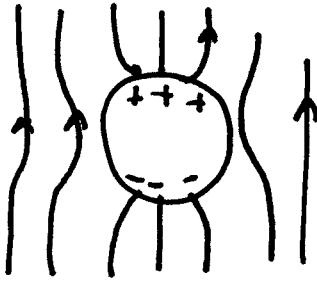


4-21 a

Example: Remember the "classic" conductor in  $\vec{E}$  field.



We solved this in ch. 3 by following logic:

$$\textcircled{1} \rightarrow V(\text{outside}) \text{ solves } \nabla^2 V = 0 \Rightarrow V = \sum_{l=0}^{\infty} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(\cos\theta)$$

$$\textcircled{2} \rightarrow V(\text{far away}) = -E_0 z = -E_0 r \cos\theta$$

$$\textcircled{3} \rightarrow V(\text{on surface}) = 0$$

$$\textcircled{3} \text{ told us } A_2 R^2 = -B_2 / R^{3+1}$$

$$\textcircled{2} \text{ told us all } l \neq 1 \text{ terms vanish, and } A_1 = -E_0$$

The  $E_{ext}$  was  $\uparrow \uparrow \uparrow \uparrow$ , and the sphere polarized,

cancelling out  $\vec{E}$  entirely inside.

$$\text{In end, we found } \left. \frac{\partial V}{\partial r} \right|_{\text{out}} - \left. \frac{\partial V}{\partial r} \right|_{\text{in}}^0 = -\frac{\sigma(\theta)}{\epsilon_0}$$

$$\text{gave us } \sigma = 3\epsilon_0 E_0 \cos\theta \leftarrow \text{"perfect polarization"}$$

this is just the  $\sigma$  you need to create (by itself)

( $\vec{E} = -E_0 \hat{z}$  inside, i.e. cancelling  $E_{ext}$ )

4-21 b

Let's do the same thing with dielectric sphere!

Put it into  $E_{ext} = \uparrow \uparrow \uparrow \uparrow$ .

It will polarize... but not perfectly, it's dielectric, not conductor!

Here again:  $V_{out} = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$

But  $V_{out}(r \rightarrow \infty) = -E_0 \cos\theta$  tells us

all  $A_{l \neq 1}$  must vanish, and  $A_1 = -E_0$

so  $V_{out} = -E_0 \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

Inside, cannot assume  $\vec{E} = 0$  !

$V_{in}(r, \theta) = \sum_{l=0}^{\infty} \tilde{A}_l r^l P_l(\cos\theta)$

- $\hookrightarrow$  No  $\tilde{B}_l$  terms to keep  $V(r)$  finite
- $\tilde{A}_l$ 's are different than  $A_l$ 's !

we need Boundary Conditions.

As always  $V(in) \Big|_R = V(out) \Big|_R$

(But the other B.C. is new, it comes from p. 15 ideas)

4-22

The "New" B.C. When Dielectrics are present:

$$D_{\perp}^{\text{above}} - D_{\perp}^{\text{below}} = \sigma_F \quad \leftarrow \text{from } \oint \vec{D} \cdot d\vec{\Lambda} = Q_{\text{free, enc.}}$$

For linear Dielectrics, this means

$$\sum_{\text{above}} E_{\perp}^{\text{above}} - \sum_{\text{below}} E_{\perp}^{\text{below}} = \sigma_F$$

or

$$E_{\text{above}} \frac{\partial V}{\partial n} \Big|_{\text{above}} - E_{\text{below}} \frac{\partial V}{\partial n} \Big|_{\text{below}} = -\sigma_F$$

(Because  $\vec{E} = -\vec{\nabla}V$  still, as always!)

Very much like Ch. 3, just bear in mind  $E$  might change at a boundary, if dielectric material ends!

In our problem



$$\epsilon_0 \frac{\partial V^{\text{out}}}{\partial r} \Big|_{\text{at } R} - \epsilon \frac{\partial V^{\text{in}}}{\partial r} \Big|_R = -\sigma_F, \quad \text{but we have no free charges in this problem!}$$

So use this...  $\epsilon_0 \frac{\partial V^{\text{out}}}{\partial r} = \epsilon \frac{\partial V^{\text{in}}}{\partial r}$  in this problem

4-23

$$\begin{aligned} \varepsilon_0 (-E_0 \cos\theta) + \varepsilon_0 \sum_{l=0}^{\infty} \frac{B_l}{R^{l+2}} (-)(l+1) P_l(\cos\theta) \\ = \varepsilon \sum_{l=0}^{\infty} l \tilde{A}_l R^{l-1} P_l(\cos\theta) \end{aligned}$$

and  $V_{in} I_R = V_{out} I_R$

$$\Rightarrow \sum_{l=0}^{\infty} \tilde{A}_l R^l P_l = -E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l$$

↓

this tells me  $\sum \left( \tilde{A}_l R^l - \frac{B_l}{R^{l+1}} \right) P_l = -E_0 R \cos\theta$   
 → must vanish for all  $l \neq 1$ !

$$\text{so } \tilde{A}_1 = \frac{B_1}{R^{2l+1}} \text{ for } \underline{\text{all }} l \neq 1 \quad (\text{i})$$

$$\text{and } \tilde{A}_1 R - \frac{B_1}{R^2} = -E_0 R \quad (\text{ii})$$

The upper eq'n tells me

$$\sum_{l=0}^{\infty} \left( l \varepsilon \tilde{A}_l R^{l-1} + (l+1) \frac{\varepsilon_0 B_l}{R^{l+2}} \right) P_l \theta = -\varepsilon_0 E_0 \overbrace{\cos\theta}^{P_1}$$

so here,

$$l \varepsilon \tilde{A}_l R^{l-1} + (l+1) \frac{\varepsilon_0 B_l}{R^{l+2}} = 0 \text{ for } \underline{\text{all }} l \neq 1 \quad (\text{iii})$$

$$\text{and } \varepsilon \tilde{A}_1 + \frac{2\varepsilon_0 B_1}{R^3} = -\varepsilon_0 E_0 \quad (\text{iv})$$

4-24

(c) and (iii) tell me  $\tilde{A}_z = B_2 = 0$  for all  $l \neq 1$ .

ii + iv tell me

$$\frac{2\epsilon_0}{R} \left( \tilde{A}_1 R - \frac{B_1}{R^2} \right) = \frac{2\epsilon_0}{R} (E_0 R)$$

$$\epsilon \tilde{A}_1 + \frac{2\epsilon_0}{R^3} B_1 = -\epsilon_0 E_0$$

$$\tilde{A}_1 (\alpha \epsilon_0 + \epsilon) = -\epsilon_0 E_0 (\alpha + 1)$$

$$\text{or } \tilde{A}_1 = -\frac{3\epsilon_0 E_0}{(\epsilon + 2\epsilon_0)} = -\frac{3E_0}{(\alpha + \epsilon_r)}$$

$$\text{and } B_1 = (\tilde{A}_1 + E_0) R^3 = R^3 \left( E_0 \left( 1 - \frac{3}{\alpha + \epsilon_r} \right) \right) = E_0 R^3 \left( \frac{\alpha - 1 + \epsilon_r}{\alpha + \epsilon_r} \right)$$

$$\text{so } V_{in} = \tilde{A}_1 r P_{\theta_1} = -\frac{3E_0}{\alpha + \epsilon_r} r \cos \theta = -\frac{3E_0}{\alpha + \epsilon_r} z$$

$$\text{so } \vec{E}_{in} = -\frac{\partial V_{in}}{\partial z} \hat{z} = +\frac{3E_0}{\alpha + \epsilon_r} \hat{z}$$

This is uniform, and  $\frac{3}{\alpha + \epsilon_r} \propto \epsilon_{ext}$ .

$\epsilon_r = 1 \Rightarrow$  Nothing there at all, vacuum everywhere,  $E_{in} = \epsilon_{ext}$

$\epsilon_r \rightarrow \infty \Rightarrow \epsilon_{in} = 0$ , perfect conductor ( $\propto$  dielectric)

$E_{in}$  is damped but not zero.

$$\frac{4-23}{24} \text{ alt. (quicker)}$$

Look, seems clear (with  $\cos\theta$  term) we're going to

only get  $V_{in} = -C_1 r \cos\theta$

$$V_{out} = -E_0 r \cos\theta + \frac{C_2 R^3}{r^2} \cos\theta$$

continuity  $\Rightarrow C_1 = E_0 + C_2$

$$-\epsilon \frac{\partial V_m}{\partial r} \Big|_R = -\epsilon_0 \frac{\partial V_{out}}{\partial r} \Big|_R \Rightarrow \epsilon C_1 = \epsilon_0 E_0 + 2\epsilon_0 C_2$$

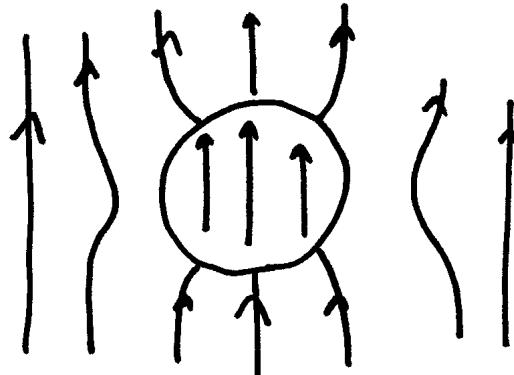
$$\text{or } \epsilon_r C_1 = E_0 + 2C_2$$

$$\text{so } C_1(1-\epsilon_r) = C_2(-3)$$

$$\text{and thus } C_1 = E_0 + C_1 \left(\frac{1-\epsilon_r}{3}\right) \Rightarrow C_1 \cdot \left(\frac{3-1+\epsilon_r}{3}\right) = E_0$$

$$C_1 = \frac{3}{2+\epsilon_r} E_0$$

$$C_2 = \frac{\epsilon_r-1}{2+\epsilon_r} E_0$$



4-25

$$\vec{P}_{in} = \epsilon_0 \chi_e \vec{E}_{in} = \epsilon_0 \frac{(\epsilon_r - 1) \cdot 3}{(\epsilon_r + 2)} E_0 \hat{z}$$

(always  $\approx \epsilon_r - 1$ )

$$\text{so } \vec{\sigma}_{\text{BOUND}} = \vec{P} \cdot \hat{r} = 3\epsilon_0 \left( \frac{\epsilon_r - 1}{2 + \epsilon_r} \right) E_0 \cos\theta.$$

so this is again  $\propto \cos\theta$ , it's "dipole", ~~so~~

Vacuum  $\Rightarrow \epsilon_r \rightarrow 1 \Rightarrow \sigma_B = 0$  ✓

conductor  $\Rightarrow \epsilon_r \rightarrow \infty \Rightarrow \sigma_B = 3\epsilon_0 E_0$  ✓ (see p. 21a!)

---

OUTSIDE, it's also just a dipole field (+ orig),

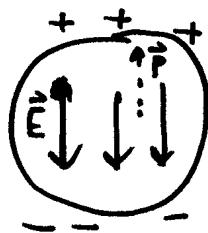
but dipole is "weaker" than the ideal conductor would give.

A pure polarized sphere

makes internal  $\vec{E}$  that

partly, but not totally,

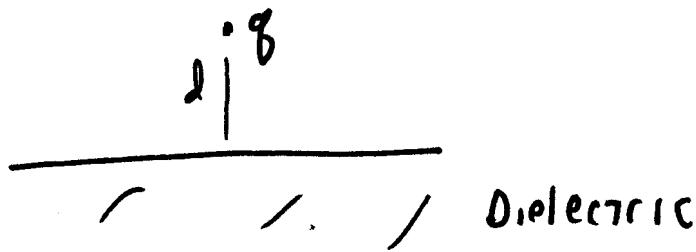
"fights" the  $\vec{E}_{ext}$  that Polarized you in the first place



4-26

One last example :

Like "image charge",  
but w. Dielectric. →



Can we use method of images?

- No free charges on surface, so  ~~$D_z |_{\text{above}}$~~   $D_z |_{\text{above}} = D_z |_{\text{below}}$   
 $\oint \vec{D} \cdot d\vec{n} = Q_{\text{free, enc}}$  →

$$\text{thus } \epsilon_0 E_z |_{\text{above}} = \epsilon E_z |_{\text{below}}$$

$$\text{or } \epsilon_0 \frac{\partial V}{\partial z} |_{z=0} = \epsilon \frac{\partial V}{\partial z} |_{z=0}$$

Could we use an image  $Q'$  at  $z=-d$ ? Well, let's try!

$$\text{Above, } V(x, y, z) = \frac{kQ}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{kQ'}{\sqrt{x^2 + y^2 + (z+d)^2}}$$

below ... well, there really isn't a  $Q'$  below, it's all  $\sigma_B$  on surface. seen from below, that surface charge looks like a  $Q'$  above!

$$V_{\text{below}}(x, y, z) = \frac{kQ}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{kQ'}{\sqrt{x^2 + y^2 + (z+d)^2}}$$

N.B. if  $Q' = -Q$   
 $V_{\text{below}} = 0$   
as it was  
for conductor

$$\text{so } \left. \epsilon_0 \frac{\partial V_{\text{above}}}{\partial z} \right|_{z=0} = \epsilon_0 K \left[ \frac{-(z-d) Q}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{-(z+d) Q'}{(x^2+y^2+(z+d)^2)^{3/2}} \right]_{z=0}$$

$$= \frac{\epsilon_0 K}{(x^2+y^2+d^2)^{3/2}} [dQ - dQ']$$

$$\begin{aligned} \epsilon \left. \frac{\partial V_{\text{below}}}{\partial z} \right|_{z=0} &= \epsilon K \left[ \frac{-(z-d)(Q+Q')}{(x^2+y^2+(z-d)^2)^{3/2}} \right]_{z=0} \\ &= \frac{\epsilon K d(Q+Q')}{(x^2+y^2+d^2)^{3/2}} \end{aligned}$$

$$\text{so B.C. requires } \epsilon_0 (Q-Q') = \epsilon (Q+Q')$$

[Continuity of  $V \Rightarrow Q+Q' = Q+Q'$  doesn't help, it's a symmetric.]

$$\text{or } Q(\epsilon_0 - \epsilon) = Q'(\epsilon + \epsilon_0)$$

$$\text{or } Q' = Q \left( \frac{1-\epsilon_r}{1+\epsilon_r} \right)$$

[Note: conductor  $\Rightarrow \epsilon_r \rightarrow \infty$   
 $Q' = -Q \checkmark$

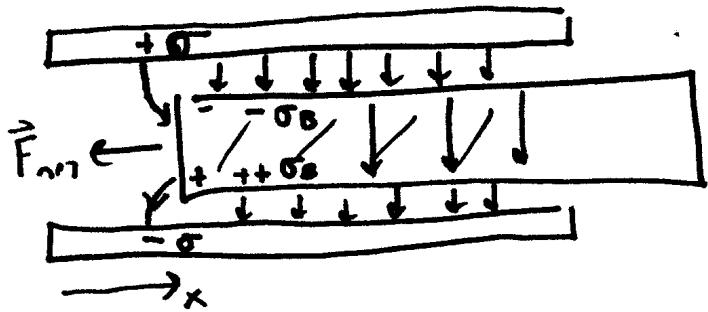
Now we know  $V$  everywhere.

$\rightarrow$  Vanishes as  $\infty$

$\rightarrow$  Continuous, right B.C., at  $z=0$

[Note:  $\epsilon_r \rightarrow 0 \Rightarrow$  No image needed!]

Could find  $\sigma_b$  from  $\vec{P} \cdot \hat{z} = \epsilon_0 \chi_e E_z^{\text{below}} = \epsilon_0 \chi_e \left( -\frac{\partial V_{\text{below}}}{\partial z} \right) \Big|_{z=0}$ .



Fringe fields : pull dielectric in!

Very complicated fields. Could we possibly hope to figure out  $\vec{E}$ 's, then  $\vec{F}$ 's? Not directly!

Use energy argument:

$$\vec{F}_{\text{ext}} = \frac{dW}{dx} = -\vec{F}_{\text{field}}$$

← Better, then no battery worry about! keep  $Q$  fixed!

and  $W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$

Can find  $C(x)$  ← exercise!

+ thus deduce  $\vec{F}$ .

~~As~~ As Dielectric enters,  $E \downarrow$ ,  $C \uparrow$ ,  $W \downarrow$ .  
 Shielding,  $\epsilon_r > 1$ ,  $\frac{Q^2}{C} \uparrow$  or  $\int E^2$   
 so, gets "sucked in" (by fringe field!)