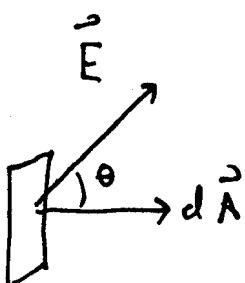


Gauss' Law

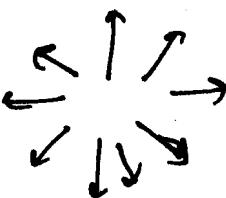
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

} • Experimental, a law of nature.
• Can "derive" from Coulomb's law in electrostatics *

This is flux

$$\vec{E} \cdot d\vec{A} = E dA \cos \theta = E_{\perp} \cdot \text{Area}$$

That's the idea of flux: Proportional to Area + the amount of E "poking through".

If draw field lines  then $|E|$ is represented by density of lines
(in 3-D! 2-D pictures can fool you)

so # of lines $\leftrightarrow |E|$, which means flux is represented by how many lines "poke" through area.

* (Griffiths shows that Gauss' law follows from Coulomb's law, if the surface is a sphere centered around one q.)

Can we show Gauss' law for ugly surface, \vec{q} off center?

$$\theta = \frac{S}{RL}$$

"angle"

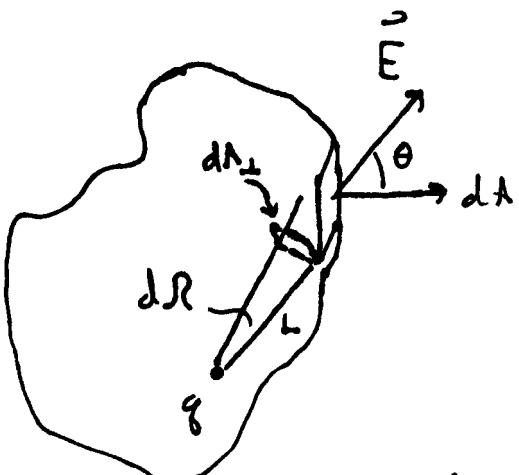
$$\phi = \frac{A}{L^2}$$

"solid angle"

For small Areas,



$$dR = \frac{\text{Perpendicular Area}}{L^2} = \frac{dA_{\perp}}{L^2}$$



$$\vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta \cdot dA}{L^2}$$

$$\text{But note } dR = \frac{dA_{\perp}}{L^2} = \frac{dA \cdot \cos\theta}{L^2}$$

$$\text{so } \oint \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0} \iint dR = \frac{q}{\epsilon_0} !$$

If add more q 's, by superposition

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{tot}}}{\epsilon_0}, \text{ enclosed.}$$

Figure out for yourself why q 's outside do not contribute!

(1-2) -16.

Some vector calculus (at last!) [see Griff 1.56]

Divergence theorem (= Gauss' theorem, or Green's theorem)

$$\iiint_{\text{volume}} \vec{\nabla} \cdot \vec{F} dV = \oint_{\text{closed } S} \vec{F} \cdot d\vec{A} \quad \text{for any function } \vec{F}.$$

This is a "3-D version" of $\underbrace{\int_a^b \frac{df}{dx} dx}_{\text{integral of a deriv}} = \underbrace{f(b) - f(a)}_{\text{function on boundary}}$

Meaning: If $\vec{\nabla} \cdot \vec{F}$ is any kind of "flow", then

$$\vec{F} \cdot d\vec{A} = \text{flux exiting } dA$$

$\vec{\nabla} \cdot \vec{F}$ = "divergence", it's the spread from a point,
or the "creation" of arrows

so $\iiint \vec{\nabla} \cdot \vec{F} dV = \text{total spread created at all points}$

$$\oint \vec{F} \cdot d\vec{A} = \text{total outflow.}$$

what goes out ↑ must have originated from sources inside

always true, so also for \vec{E} fields.



(1-2) -17

$$\oint \vec{E} \cdot d\vec{\lambda} = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint \rho \cdot d\tau \quad \leftarrow \text{physics!}$$

$$\oint \vec{E} \cdot d\vec{\lambda} = \iiint \vec{\nabla} \cdot \vec{E} \, d\tau \quad \leftarrow \text{math!}$$

True for any and all volumes, so integrands must agree

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \quad \text{at all points.}$$

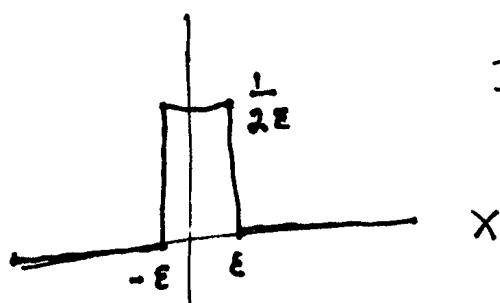
Suppose you have a point charge, so $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$

- what should $\rho(\vec{r})$ look like??
- what's $\vec{\nabla} \cdot \vec{E}$?

Need a math INTERLUDE: δ functions.

In 1-D, $\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x=0 \end{cases}$ such that $\int \delta(x) dx = 1$

It's a limit of sensible functions, like



It's "Tall + skinny", with area 1.

Let it get very skinny, that's $\delta(x)$

Note: $\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0)$ ← (this defines the δ fn, actually)

$$\text{In 3-D, } \iiint \delta^{(3)}(\vec{r}) f(\vec{r}) d\tau = f(0).$$

(Same deal, really!)

So a point charge has

- No charge density away from origin
- $\propto \dots \text{ at origin}$

$$\text{But } Q = \iiint \rho d\tau \text{ is finite.}$$

$\rho(\vec{r}) = Q \underbrace{\delta^{(3)}(\vec{r})}_{\text{very "concentrated" at one point, but finite integral.}}$ is exactly that.

$$\text{Side Note: } \int_{-\infty}^{\infty} f(x) \delta(ax+b) dx \quad \text{Let } u = ax+b \\ du = a dx$$

$$= \int_{-\infty}^{\infty} f\left(\frac{u-b}{a}\right) \delta(u) \cdot \frac{du}{a} = \frac{1}{a} \cdot f \left(\begin{array}{l} \text{the } x \text{ for which} \\ \text{the } \delta\text{-fn argument} \\ \text{vanished,} \end{array} \right)$$

namely $-b/a$

Except if $a < 0$, then integration

limits are flipped, + you get $-\frac{1}{a} f(-b/a)$

$$\text{i.e. } \delta(ax) = \frac{1}{|a|} \delta(x).$$

$$\text{Back to } \vec{\nabla} \cdot \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}. \quad \text{focus on} \quad \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^3} \right)$$

Method 1: Front flyleaf! $\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r)$

$$\text{Here, } \underline{\underline{Vr}} = \frac{1}{r^2}, \text{ so } \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot \frac{1}{r^2}) = 0.$$

$$\text{Method 2: } \vec{\nabla} \cdot \left(\frac{1}{r^3} \vec{r} \right) = \underbrace{\left(\nabla \frac{1}{r^3} \right) \cdot \vec{r}}_{\left(-\frac{3}{r^4} \hat{r} \right) \cdot \vec{r}} + \frac{1}{r^3} \vec{\nabla} \cdot \vec{r}$$

$$\left(-\frac{3}{r^4} \hat{r} \right) \cdot \vec{r} + \frac{1}{r^3} (3) = 0$$

Looks like $\vec{\nabla} \cdot \vec{E} = 0$ for point charge.

It is, except maybe at $r=0$, where all those formulas above involve r^2/r^2 , not well defined.

so look again at Gauss: $\int \vec{\nabla} \cdot \vec{E} d\tau = \oint \vec{E} \cdot d\vec{x}$

Imagine a tiny surface. RHS = $\iiint \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \cdot d\hat{A}$
around origin



$$= \frac{q}{4\pi\epsilon_0 r^2} \int dA = \frac{q}{\epsilon_0}$$

so $\int \vec{D} \cdot \vec{E} d\gamma = 8/\epsilon_0$. No matter what.

$$\vec{\nabla} \cdot \vec{E} \text{ must be a } \delta \text{ fn!} \quad \vec{\nabla} \cdot \vec{E} = \frac{q}{\epsilon_0} \cdot \delta^{(3)}(\vec{r})$$

(1-2) -20.

so $\rho(\vec{r}) = q \delta^{(3)}(\vec{r})$

And we've just learned $\vec{\nabla} \cdot \frac{\hat{q}}{4\pi\epsilon_0} \frac{\vec{r}}{r^2} = \frac{\hat{q}}{\epsilon_0} \delta^{(3)}(\vec{r})$

so $\vec{\nabla} \cdot \frac{\hat{q}}{r^2} = 4\pi \delta^{(3)}(\vec{r})$ // \leftarrow (see Griff. 1.100)

Summary: Point charges can be described by $\rho(\vec{r})$

Origin  $\rho(\vec{r}) = q \delta^{(3)}(\vec{r} - \vec{r}')$, and $\vec{E}(r) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$ as usual.

And then $\vec{\nabla} \cdot \vec{E} = \frac{q}{4\pi\epsilon_0} \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$

$$\begin{aligned} &= \frac{q}{4\pi\epsilon_0} \cdot 4\pi \delta^{(3)}(\vec{r} - \vec{r}') \\ &= \rho(\vec{r}) \text{ as it must be.} \end{aligned}$$

work it out! $\vec{\nabla} \cdot = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z})$

• $\frac{\vec{r} - \vec{r}'}{1}^3 = \frac{(x-x')\hat{x} + (y-y')\hat{y} + \dots}{[(x-x')^2 + (y-y')^2 + \dots]^{3/2}}$

For $\vec{r} \neq \vec{r}'$, just do it!

You'll get zero :)

(1-2) -21

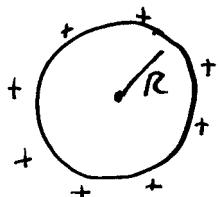
Applying Gauss

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \quad (\text{or} \quad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0})$$

• Gauss' law is always true.

• Only useful to find E if symmetry lets you "pull it out"

Ex 1: Remember the horrid hw problem, uniform σ on sphere of radius R ?



$$\vec{E}(\vec{r}) = ?$$

• Symmetry says \vec{E} is radial.

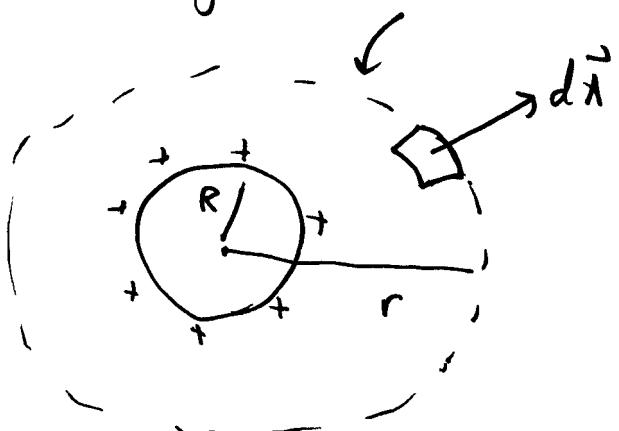
what other direction could it be?

• Symmetry says $|E(\vec{r})|$ depends only on r ,
not θ , or ϕ .

so $\vec{E} \cdot d\vec{A} = E dA \quad \leftarrow \text{dot product} \Rightarrow \cos \theta = 1$.

wxir! Because \vec{E} is radial, + so is $d\vec{A}$

• What is my $d\vec{A}$? what surface? It's not the sphere, it's
imaginary ("Gaussian") sphere. I made it up!



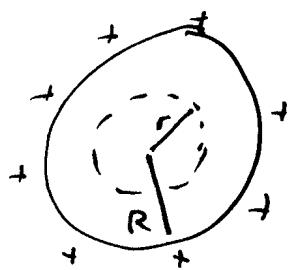
$$\begin{aligned} \text{so } \iint \vec{E} \cdot d\vec{A} &= \iint \vec{E} dA \\ &\stackrel{\text{img sphere}}{=} E \iint dA \end{aligned}$$

why? Because $r = \underline{\text{constant}}$, so $E = \underline{\text{const}}$,
everywhere on that sphere.

$$\text{so } E \cdot 4\pi r^2 = \frac{Q_{\text{enc}}}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2} \quad (\text{with } Q = 4\pi R^2 \sigma)$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}, \text{ like a point charge @ origin.}$$

What about inside? Same arguments!



\vec{E} is radial, E depends only on R ,

$$\text{so } \oint \vec{E} \cdot d\vec{\lambda} = \iint E d\lambda = E \iint d\lambda = E \cdot 4\pi r^2$$

But Q_{enc} is now zero. So $E = 0$!

What if the sphere had uniform ρ , rather than just σ ?

Outside \Rightarrow no difference, you can't tell! Only Q_{ext} matters

Inside $\Rightarrow Q_{\text{enc}} \neq 0$ anymore. What is it?

$$Q_{\text{enc}} = \iiint_{V_{\text{inside}}} \rho d\tau = \rho \cdot \iiint d\tau = \rho \cdot \frac{4}{3}\pi r^3$$

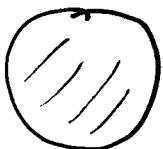
Gauss surf

$$\text{so } E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \rho \cdot \frac{4}{3}\pi r^3 \Rightarrow \vec{E} = \left(\frac{\rho}{3\epsilon_0} r \right) \hat{r}$$

Vanishes only at origin!

Basically 3 cases that are symmetric:

spherical geometry



(\mathbf{Q} 's uniform in θ, ϕ ,
but maybe not in r !)



Draw a spherical
Gaussian surface.

cylindrical



\mathbf{Q} 's uniform in θ, z ,
But maybe not in r !

(Bettter be infinite in z ,
then!)



Draw a cylindrical
"beer can" surface

planar



\mathbf{Q} 's uniform in x, y ,
Maybe not in z

Better be ∞ in
 x, y though.

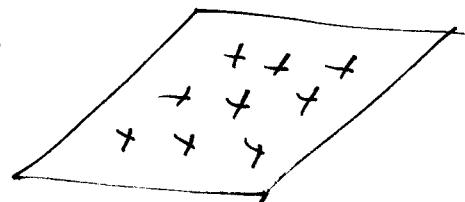


Draw a
"pillbox" *

* Let's do one planar case.

we'll take a sheet with uniform \mathbf{D}
(in x, y plane)

what's $\vec{\mathbf{E}}(x, y, z)$?



This is clearly an idealization, but any flat surface will
look like this nearby, so it'll be quite useful!

(1-2) - 24

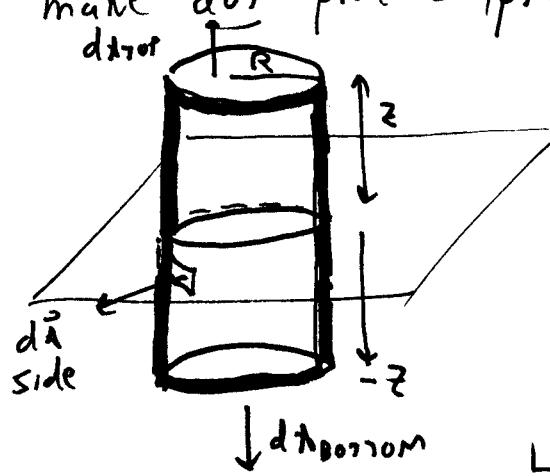
What imaginary surface do we pick?

- Symmetry says $\vec{E}(x, y, z)$ can only point in $\pm \hat{z}$ direction
Do you see why? Don't take my word for it!

- Symmetry says $\vec{E}(x, y, z) = E(z) \hat{z}$

This guides me: Draw a pillbox with faces \perp or $\parallel z$

make dot prod simple, and so E is uniform along surfaces.



There are other choices (e.g. 2.22 of Griffiths)

But look:

$$\oint \vec{E} \cdot d\vec{x} = Q_{enc} / \epsilon_0$$

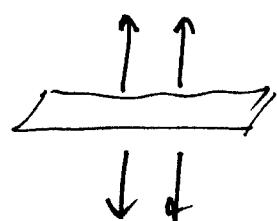
Do you see that?!

$$\iint_{\text{TOP}} \vec{E} \cdot d\vec{x} + \iint_{\text{curvy side}} \vec{E} \cdot d\vec{x} + \iint_{\text{Bottom}} \vec{E} \cdot d\vec{x} = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma \cdot \pi R^2}{\epsilon_0}$$

$$E(z) \cdot \pi R^2 + 0 + E(-z) \pi R^2 (-1)$$
$$\underbrace{\vec{E} \cdot d\vec{x} = 0}_{\text{because } d\vec{x} \text{ points down}}$$

do you see
why?

But $E(z) = -E(-z)$, by symmetry



(1-2) - 25

$$\text{so } 2E(z)\pi R^2 = \frac{\sigma}{\epsilon_0} \pi R^2$$

↑
This is critical + subtle. L.H.S. is total flux = flux through top and through bottom. But both are equal, so sum is twice the flux thru top!

$$\vec{E}(z) = \frac{\sigma}{2\epsilon_0} \hat{k} \quad \text{if } z > 0$$

$$\vec{E}(z) = -\frac{\sigma}{2\epsilon_0} \hat{k} \quad \text{if } z < 0$$

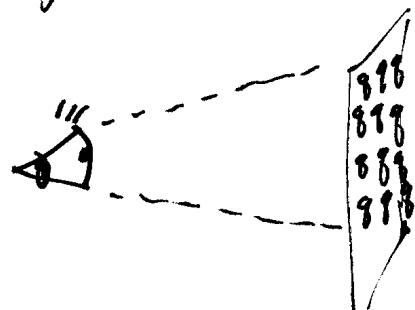
(you must think about that factor of 2!)

Notes: \vec{E} is discontinuous at the sheet.

$\Delta E = \frac{\sigma}{\epsilon_0} \hat{k}$ as move across. This turns out to be "deep" + universal!

→ $E(z)$ is constant! Moving away from a sheet has no effect.

(How could you tell how far you are?)



As move away, $E_{\text{from each } g} \propto \frac{1}{r^2}$

But your "see" an area $\propto r^2$
so effect of both cancels...

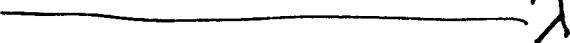
→ R of my imaginary surface dropped out. Good!!

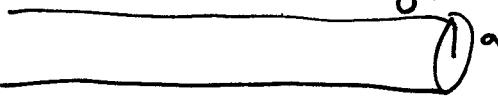
(1-2) - 26.

- There aren't many more such "easily" solved problems,

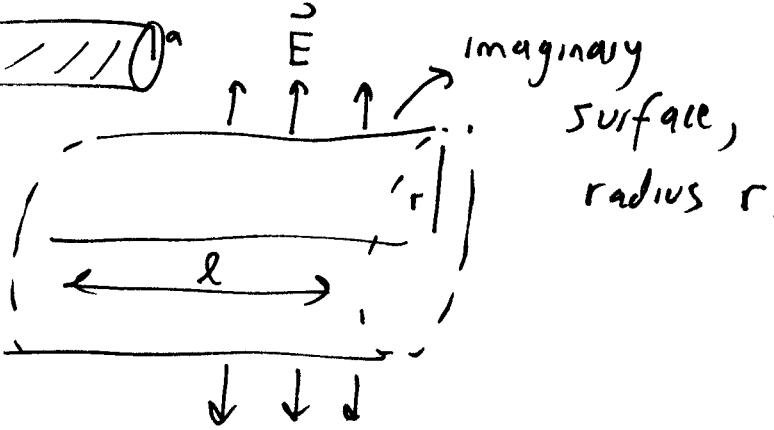
except: can use superposition

e.g.  \vec{E} everywhere is \vec{E} due to σ_1 ,
 $+ \vec{E}$ due to σ_2 .
 (etc)

For line: 

or rod: 
coated

or rod: 
filled

\vec{E} outside is easy! 

$$\iint E \cdot dA = \frac{Q_{enc}}{\epsilon_0}$$

$$E \iint dA = \frac{Q_{enc}}{\epsilon_0} = \frac{\pi \cdot l}{\epsilon_0} \left(\text{or } \frac{\sigma \cdot 2\pi r l}{\epsilon_0}, \text{ or } \frac{\rho \cdot \pi r^2 l}{\epsilon_0} \right)$$

Beer can

$$E \cdot 2\pi r l = \dots \quad (\text{l cancels, good! } E \sim \frac{1}{r}, \text{ not } \frac{1}{r^2} \text{ uniform})$$