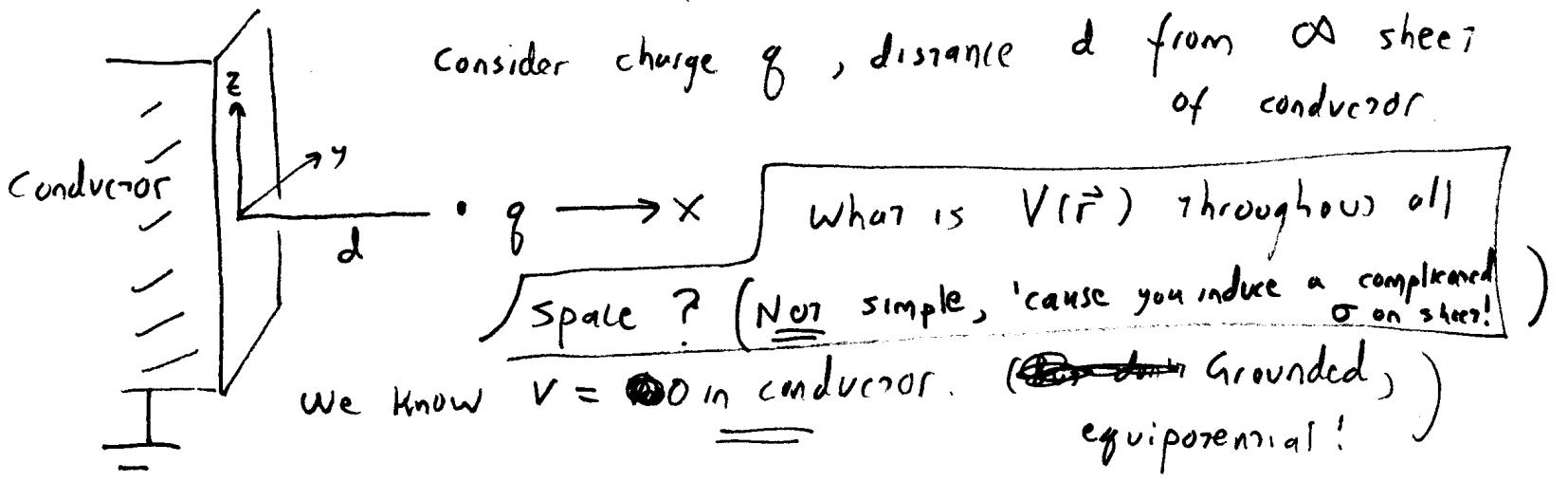


Method of Images:

- Remember, if  $\nabla^2 V = \rho / \epsilon_0$  in a region, and if we can guess a  $V$  which obeys  $\nabla^2 V = \rho / \epsilon_0$ , and which is correct at boundary of our region, we're done, we have  $V(\vec{r})$ !



We know  $V = 0$  off at  $\infty$ .

so we know  $V = 0$  on giant boundary (wall on left, everywhere else)

and we want  $V$  in rest of space. (the "right half" of universe)

We will use a trick. I can create an imaginary scenario

where  $V = 0$  everywhere on  $x=0$  plane

$V = 0$  off at  $\infty$

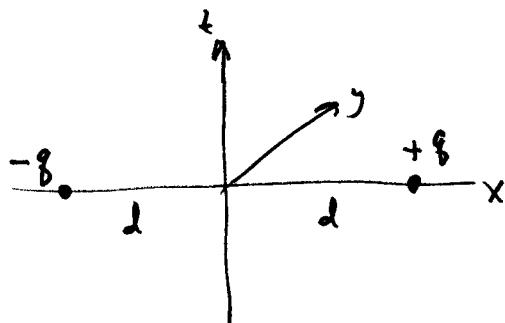
and  $\nabla^2 V = \rho / \epsilon_0$  in "right half", and  $V(r)$  is easy to compute.

If I do this, we're done. By uniqueness, that is the unique sol'n

It satisfies all our boundary conditions and Poisson's eq'n.

(And, we won't need to know the complicated  $\sigma$  on the sheet which nature generates to cancel  $\vec{E}$  out to 0 inside conductor)

The trick: Consider this problem:



No conductor.

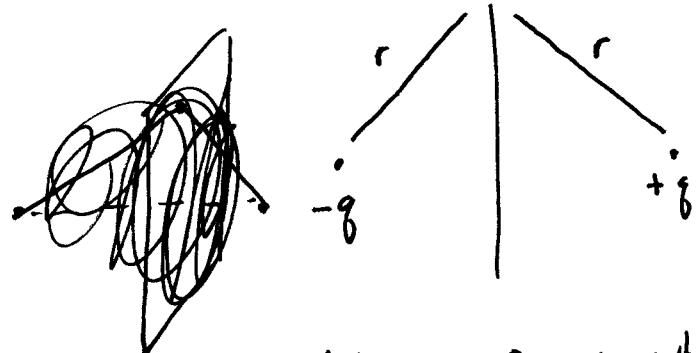
Nothing but a new (made up)  
"image charge" at  $x = -d$ .

1)  $V(r)$  here is just  $\frac{1}{4\pi\epsilon_0} \left( \frac{q}{|\vec{r}-d\hat{x}|} + \frac{-q}{|\vec{r}+d\hat{x}|} \right)$  by inspection

2)  $V(\infty) = 0$ . Good!

3)  $V$  (on  $yz$  plane) = 0.

Distance is same to  $\pm q$ !



$\nabla^2 V(r) = \rho/\epsilon_0$ , for right half of universe, (this is 0 everywhere except  $q \delta^{(3)}(\vec{r}-d\hat{x})$ , it's perfectly fine)

So for right half of universe our  $V$  has correct B.C.  
+ solves Poisson. So we're done.

That's it!

$$V(x, y, z) \Big|_{x>0} = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}} \right)$$

Important comments.

- ① This  $V$  is dead wrong for  $x < 0$  (where  $V = 0$ , right?!)
- ② There are only a limited # of situations where this method works (e.g. conducting sphere + conducting sheet + cond. cylinder)   
 But important for people who think about antennas!
- ③ The method is: make up a new situation, where you add image charge(s) in special spots (with special  $g$ 's) such that
  - Images are not located in the region of space you want to know  $V(r)$  for (!)
  - $V$  at boundaries of region you're interested in is ~~same~~ as your (real) situation

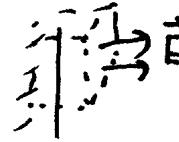
And, done!  $V = \sum_i \frac{1}{4\pi\epsilon_0} \frac{g_i}{|\vec{r} - \vec{r}_{il}|}$  easy to write down!  
 ✓ Some  $g$ 's are real  
Some are fictitious.

only valid in the region you were considering, not all space.

- So, this method works if have bunch of  $g$ 's outside sheet (just add bunch of matching image charges!)

By the way, since we know  $V(x>0, y, z)$ , we can easily

Figure out  $\sigma$  on the sheet!



Remember,  $\vec{E} = \frac{\sigma(y, z)}{\epsilon_0} \hat{x}$  just outside the conductor

$$\text{But } \vec{E} = -\nabla V, \text{ so } \frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial x} \Big|_{x=0}$$

Griffiths:

$$\frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial n} \Big|_{x=0}$$

This derivative is not so hard,  $\left[ \frac{\partial}{\partial x} ( )^{-1/2} = -\frac{1}{2} \cdot ( )^{-3/2} \cdot \frac{\partial}{\partial x} (\text{inside}) \right]$

$$\frac{\sigma(y, z)}{\epsilon_0} = -\frac{q}{4\pi\epsilon_0} \left( \frac{-(x-d)}{(x-d)^2 + y^2 + z^2}^{3/2} + \frac{(x+d)}{(x+d)^2 + y^2 + z^2}^{3/2} \right) \Big|_{x=0}$$

$$= \frac{-q}{4\pi\epsilon_0} \left( \frac{2}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

- It peaks as  $y=z=0$  & then "fades out" ✓
- It's negative everywhere ✓ (It would've been hard to guess!)
- Griffiths shows  $\iint \sigma(y, z) dy dz = -q$ . ✓

So our fictional image charge,  $-q$ , is physically manifested by  $-q$  "smeared" appropriately on surface of conductor.

(There really is not any " $-q$ " at  $x=-d$  !!)

(• oh, one more thing If you try e.g.  $E_y = -\frac{\partial V}{\partial y}$ , you'll get 0 at  $x=0$ , check for yourself!)

~~Work + Energy:~~

$$W_{2 \text{ charges in}} = \frac{1}{4\pi\epsilon_0} \frac{(+q)(-q)}{(2d)}$$

"fictional" situation

$$W_{\text{real situation, } +q \text{ outside grounded plane}} = \frac{1}{2} \epsilon_0 \iint \overbrace{E^2 dx}^{\text{only right half of universe contributes!}} = \frac{1}{2} W_{\text{fictional world}}$$

Why not same? See Griffiths... but here's my explanation

- (i) In "fictional world", you bring  $-q$  to  $x = -D$  (no work!) + then bring  $+q$  to  $+D$ , doing (-) work all the way.

- (ii) In "real world", you bring  $+q$  to  $+D$ , and it "sees" an image charge which is always further away than what it would be in scenario i, with image fixed at  $-d$ .

(Because image is always at " $-x$ ")

- Note that, when  $q$  is at  $x = +d$ , the force on it (at that spot) is just  $\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} (-\hat{x})$ . Because  $\vec{E}(x)$  is same whether in real world or "image charge" world, since  $\vec{E} = -\vec{\nabla}V$ .