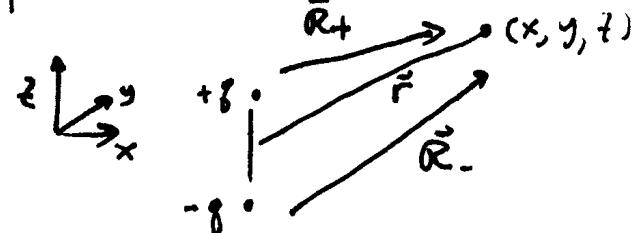


Dipoles & Multipoles

$\vec{d} \uparrow \uparrow \vec{p}$ This is an electric dipole: 0 net charge.
 $+q$
 $-q$ we define $\vec{p} = q \vec{d}$ = "the dipole moment". (We'll see why right away)

- Consider $V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R_+} - \frac{1}{R_-} \right]$

with $R_{\pm} = \sqrt{x^2 + y^2 + (z \mp \frac{d}{2})^2} = \sqrt{r^2 + dz^2/4}$



If $d \ll r$, $R \approx r \sqrt{1 \mp \frac{dz^2}{r^2}} \approx r(1 \mp \frac{1}{2} \frac{dz^2}{r^2})$

so $V(\vec{r}) \approx \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r} \left(1 \mp \frac{1}{2} \frac{dz^2}{r^2} \right) - \frac{1}{r} \left(1 \mp \frac{1}{2} \frac{dz^2}{r^2} \right) \right] \approx \frac{q dz}{4\pi\epsilon_0 r^3}$

But note that $\vec{p} \cdot \vec{r} = p_z \cdot r_z = (qd)(z)$ so.

$V(\vec{r}) \approx \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3} = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}$ Not Coulomb, it drops like $\frac{1}{r^2}$, not $\frac{1}{r}$.
 → Note that this is "coordinate free", we can now choose any axes we like!

- Consider $\vec{E}(\vec{r}) = -\vec{\nabla} V$

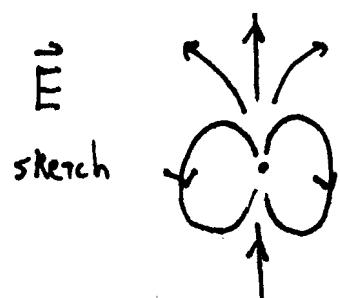
Moving to spherical coords $V(\vec{r}) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$

$$E_r = -\frac{\partial V}{\partial r} = +\frac{2p \cos \theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = \frac{-1}{r} \frac{\partial V}{\partial \theta} = +\frac{p \sin \theta}{4\pi\epsilon_0 r^3}$$

so $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3} \left(\cos \theta \hat{r} + \sin \theta \hat{\theta} \right)$

dies fast.

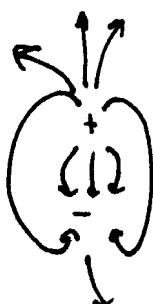


Note: $\theta = \pi/2 \Rightarrow \vec{E} = \frac{\rho}{4\pi\epsilon_0 r^3} (\hat{\theta})$ points "down"

$$\theta = 0 \Rightarrow \vec{E} = \frac{\rho}{4\pi\epsilon_0 r^3} \cdot 2r \quad " \text{ up.}$$

It's an ideal or pointlike dipole. ($d \rightarrow 0$, but $\rho = qd$ is finite)

Real Dipole



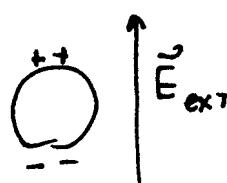
Looks like \oplus above,
(+ looks a lot like it far away!)
(But not right up close)
(Need more terms in that expansion!)

Where do Dipoles come from?

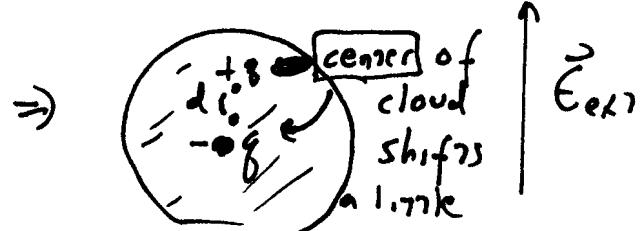
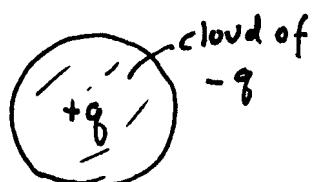
1) Natural (polar molecules) e.g. $+H \quad H+$ \approx $+ \quad -$
 $H_2O :$

(Typical "d" = ~Angstroms $\sim 10^{-10} m$
"q" = e, so $\rho \approx 10^{-26} Cm$)

2) INDUCED : put neutral object in an \vec{E} field,



think of atom



How much shift? $P = \frac{q^2}{4\pi\epsilon_0 d^2}$ attraction = $q\vec{E}_{ext}$ "repulsive"

so, at equilibrium, pull on +q to center = push from \vec{E}_{ext}

$\underline{\text{N.B.}} \frac{q^2}{4\pi\epsilon_0 d^2}$! Need field inside "sphere of charge", which by Gauss'

law is $E = \frac{q}{4\pi\epsilon_0 R^3}$ ← remember this? If not, work it out!

$$\text{so } \frac{q}{4\pi\epsilon_0 R^3} = E_{\text{ext}}$$

which means $P = qd = 4\pi\epsilon_0 R^3 E_{\text{ext}}$.

• Polarization is linear in E_{ext} !

• • Estimate "polarizability" $\equiv P/E \equiv \alpha \simeq 4\pi\epsilon_0 R^3$

"size of atom",
about 10^{-10} m .

works pretty well, see Griffiths p. 161!

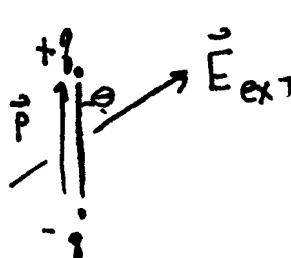
Once you have a dipole: It creates \vec{E} field

" has nonzero $V(r) \Rightarrow$ potential energy associated

It can create torque on other dipoles!

Even though it's neutral, and "tiny".

It feels a torque if put it
in an E-field, + force too!

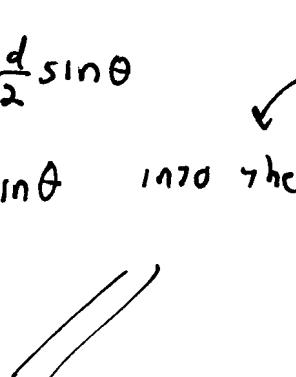


$$\vec{r} = \sum \vec{r} \times \vec{F} = 2 q E_{\text{ext}} \frac{d}{2} \sin\theta$$

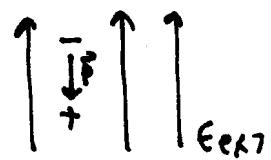
about
origin

$$= qd E_{\text{ext}} \sin\theta \quad \text{into the page}$$

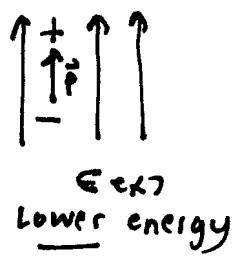
$$= \vec{p} \times \vec{E}$$



It has energy associate with orientation \vec{p} in external field



High energy



Lower energy

$$U = -\underbrace{\vec{p} \cdot \vec{E}}$$

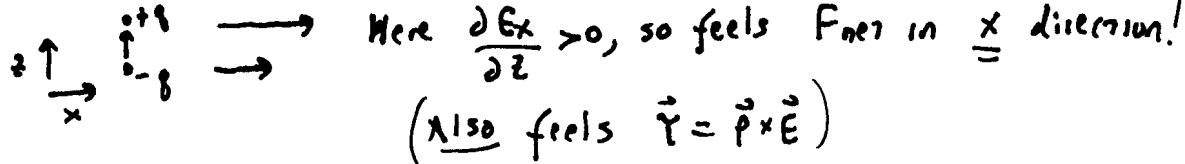
lowest when \vec{p} points in \vec{E} direction

It feels force if \vec{E} bigger on one side than other,

$$\vec{F}_{\text{net, on}} = (\vec{p} \cdot \vec{\nabla}) \vec{E} \Rightarrow \text{For a Dipole } \underbrace{\begin{array}{c} \uparrow \\ \hline \end{array}}_{\text{in } z \text{ direction}}$$

$$\vec{F}_{\text{net, on}} = \underbrace{\vec{p} \frac{\partial}{\partial z} \vec{E}}_{\text{dipole}} = \left(p \frac{\partial E_x}{\partial z}, p \frac{\partial E_y}{\partial z}, p \frac{\partial E_z}{\partial z} \right) \quad (\text{yuck!})$$

Ex:



Let's go back to dipole as source:

We found, $V(\vec{r}) = \underbrace{\vec{p} \cdot \hat{r}}_{\text{dipole}} \frac{1}{4\pi\epsilon_0 r^2} \quad \leftarrow \text{dies like } \frac{1}{r^2} \quad \begin{bmatrix} \text{Simple "pattern",} \\ \vec{p} \cdot \hat{r} = p \cos\theta \text{ for} \\ \vec{p} \uparrow = \hat{p}_z, \text{ e.g.} \end{bmatrix}$

Recall $V(\vec{r}) = \frac{q}{8\pi\epsilon_0 r} \quad \leftarrow \text{dies like } \frac{1}{r} \cdot \begin{bmatrix} \text{No "pattern",} \\ \text{same at all } \theta \end{bmatrix}$

$+q$

= "Monopole".

$V \sim 1/r$

(only exactly true if q is pointlike & spherically symm) $+q$ $-q$ = "Dipole"

$V \sim 1/r^2 \approx$ (simple angular dependence)

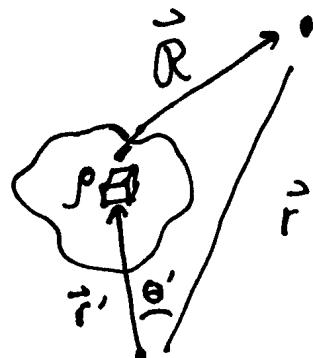
only exactly true if $+q$ $-q$ are very close

Leads us to a method to estimate/approximate $V(r)$ when r is far from a bunch of charges.

[If $q_{\text{net}} \neq 0$, $V \sim 1/r$ "dominates".]

Multipole Expansion.

Go back to $V(r) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(r') dV'}{r'}$



If $\vec{r} \gg$ all \vec{R} 's, "far away",

this complex $\rho(r')$ looks simple.

(like a point charge! or, if $q_{\text{net}} = 0$, like a dipole.)

or, if e.g. $\begin{array}{c} + \\ - \end{array}$ like this, two canceling dipoles,
it's a "quadrupole", and $V \sim \frac{1}{r^3}$)

we can be more explicit:

$$R^2 = r^2 + r'^2 - 2rr' \cos\theta' \approx r^2 \left(1 + \frac{r'^2}{r^2} - \frac{2r' \cos\theta'}{r}\right)$$

pull this out, it's big, (or will be ...)

$$R \approx r\sqrt{1+\epsilon} \quad \text{with } \epsilon = \frac{r'^2}{r^2} - \frac{2r' \cos\theta}{r} \quad (\text{will be small})$$

Let's call $\epsilon = \frac{r'}{r}$, which will be small when we're far from the charge distribution

$$R = r \sqrt{1 - 2\epsilon \cos\theta' + \epsilon^2} \quad (\text{and we'll need } \frac{1}{R})$$

$$\text{Recall } (1+\eta)^{-1/2} = 1 + \frac{(-1/2)}{1!} \eta + \frac{(-1/2)(-1/2-1)}{2!} \frac{\eta^2}{2!} + \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!} \frac{\eta^3}{3!} + \dots$$

$$\begin{aligned} \text{so } \frac{1}{R} &\stackrel{?}{=} \frac{1}{r} (1 - 2\epsilon \cos\theta' + \epsilon^2)^{-1/2} \\ &\approx \frac{1}{r} \left(1 - \frac{1}{2} [-2\epsilon \cos\theta' + \epsilon^2] \right) + \left(\frac{3}{8} \right) [-2\epsilon \cos\theta' + \epsilon^2]^2 + \dots \\ &\text{collect all terms with } \left(\epsilon, \epsilon^2, \text{etc} \right) \\ &\approx \frac{1}{r} \left(1 + \epsilon \cos\theta' - \frac{1}{2} \epsilon^2 + \frac{3}{8} \cdot 4 \cos^2\theta' \epsilon^2 + \mathcal{O}(\epsilon^3) \right) \\ &\approx \frac{1}{r} \left(1 + \epsilon \cos\theta' + \left(-\frac{1}{2} + \frac{3}{2} \cos^2\theta' \right) \epsilon^2 + \dots \right) \end{aligned}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(r') dr'}{r} \left(1 + \epsilon \cos\theta' + \left(-\frac{1}{2} + \frac{3}{2} \cos^2\theta' \right) \epsilon^2 + \dots \right)$$

↑ ↑ ↑
Look at the terms one leading smaller smaller still
at a time

$$\begin{aligned} \text{Leading: } V(r) &\approx \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r} \underbrace{\iiint \rho(r') dr'}_{\substack{\uparrow \\ \text{came out} \\ \text{of integral!}}} \\ &\qquad \qquad \qquad \text{This is JUST Q/r.} \end{aligned}$$

so $V(r) \approx \frac{Q}{4\pi\epsilon_0 r}$, the "monopole term", just potential of point charge.

Next term: $\frac{1}{4\pi\epsilon_0} \iiint \rho(r') \frac{d\tau'}{r} \left(\frac{r'}{r} \cos\theta' \right)$

$$= \frac{1}{4\pi\epsilon_0 r^2} \underbrace{\iiint \rho(r') r' d\tau' \cos\theta'}_{\text{drops off faster.}}$$

Note: This integral is a number, independent of \vec{r} . It's a property of the charge distribution! It's called "the dipole moment".

Note: $\cos\theta' = P_1(\cos\theta')$ is the first Legendre polynomial.

Next term: $\frac{1}{4\pi\epsilon_0} \iiint \rho(r') \frac{d\tau'}{r} \cdot \frac{r'^2}{r^2} \left(\frac{3}{2} \cos^2\theta' - \frac{1}{2} \right)$

$$= \frac{1}{4\pi\epsilon_0 r^3} \underbrace{\iiint \rho(r') d\tau' r'^2 \cdot P_2(\cos\theta')}_{\text{the "quadrupole moment"}}$$

And so it goes...

$$V(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{Q_{20}}{r} + \frac{\text{Dipole moment}}{r^2} + \frac{\text{Quad moment}}{r^3} + \dots \right)$$

- When r is big, leading term dominates.

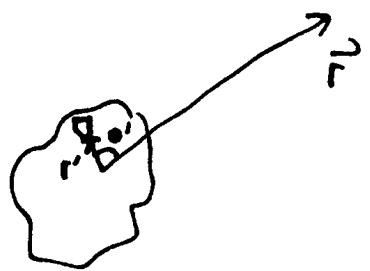
e.g. If $Q_{20} = 0$, Dipole term ".

- Note that $\cos\theta'$ appears, which does involve relative direction of $\vec{r} + \vec{r}'$, so these moments do depend on location/orientation of ρ and \vec{r} (but not on $|r|$)

Nice: we can approximate $V(r)$ far away without fussing about the details, one number will likely tell us how $V(r)$ behave far away.

Let's look at the dipole term. (e.g., imagine $\underline{Q_{tot} = 0}$)

$$V(r) \approx \frac{1}{4\pi\epsilon_0 r^2} \iiint \rho(r') r' \cos\theta' d\tau'$$



$$\approx \frac{1}{4\pi\epsilon_0 r^2} \iiint \rho(r') \vec{r}' \cdot \hat{r} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0 r^2} \left(\iiint \rho(r') \vec{r}' d\tau' \right) \cdot \hat{r}$$

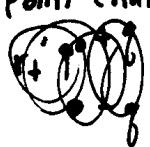
Note: $r' \cos\theta' = \vec{r}' \cdot \hat{r}$

recall $V_{\text{pure dip}} = \frac{1}{4\pi\epsilon_0 r^2} \vec{p} \cdot \hat{r}$

P.6

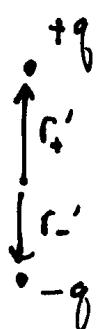
In general, $\vec{p} = \text{"dipole moment"} \equiv \iiint \rho(\vec{r}') \vec{r}' d\tau'$

For point charges:



$$\iiint g \delta^{(3)}(\vec{r}_+) \vec{r}' d\tau' + \iiint -g \delta^{(3)}(\vec{r}_-) \vec{r}' d\tau'$$

$$= g \vec{r}_+ + (-g) \vec{r}_-$$

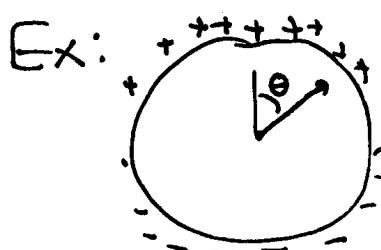


$$\vec{p} = \sum_i g_i \vec{r}_i$$

In our case, pure dipole, $g \frac{d}{2} \hat{z} + -g \left(-\frac{d}{2} \hat{z}\right) = g d \hat{z} = \vec{p}$ ✓

Bottom line: The dipole term, for a dipole, is everything! ☺

The dipole moment of any distribution can be computed.



shell, $\rho = \sigma_0 \delta(r-R) \cos\theta$ has a dipole moment:

$$P_z = \iiint \rho(r') z' d\tau' = \iint \sigma_0 \delta(r'-R) \cos\theta' r' \cos\theta' d\tau'$$

$$= \sigma_0 \cdot R^3 \frac{\cos^3 \theta}{3} \Big|_0^\pi = \frac{2}{3} \sigma_0 R^3$$

(Convince yourself, the integral kills off P_x and P_y .)

Summary: Finite charge distributions



can be characterized by "numbers, "moments".

(which depend on direction you're "looking from" ~~at~~)

$$\text{Then } V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{\text{Quadrupole term}}{r^3} + \dots \right]$$

↑ ↑
 Monopole Quadrupole
 term term

These moments are easy to compute, given p .

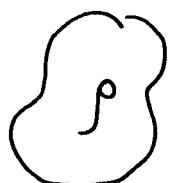
The dipole moment \vec{p} is just what it should be, $\sum_i q_i \vec{r}_i$

We talked about "dipole moments" of neutral molecules earlier,

so the $V(\vec{r})$ will depend on $\frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$ + higher order terms that drop off faster

so \vec{p} , a "property of the dipole" (like the charge, but a vector) is the key information we need to know $V(r)$ (thus \vec{E} , and forces, etc)

In general:



has a dipole moment

$$\vec{p} = \iiint p(\vec{r}') \vec{r}' d\tau'$$

(It depends on your choice of origin ... unless $g_{tot} = 0$,

then it turns out, (like $\vec{p} = g \vec{d}$) to be "universal".)