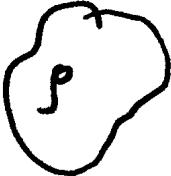


We've already looked at physical dipoles (notes 3-34-35)

I argued $\begin{array}{c} +q \\ \cdot \\ -q \end{array}$ dipole has "dipole moment" $\vec{p} = q \vec{d}$

(and in general)  has "dipole moment" $\vec{p} = \iiint p \vec{r} dV$

What if you have a bunch, like

(In E_{ext} , the torque lines them up (mostly) + the bulk material

is polarized, (Sometimes it even happens spontaneously!)

we could think of total polarization = $\sum \vec{p}_i$ in some region

But better to think of average polarization in a unit volume:

$$\vec{P} = \frac{1}{\text{Volume}} \sum_i \vec{p}_i \equiv \text{"Polarization"} \quad \leftarrow \text{Definition!}$$

Small fluctuations in local \vec{p}_i directions wash out in this averaging,

so if volume is small (but large on atomic scale)!

$\vec{P}(\vec{r})$ is smooth, it's the "dipole moment" locally.

It's analogous to $\rho(r) = \frac{\text{charge}}{\text{m}^3}$!

very simple case!

(If have N dipoles volume, each with $\vec{p} = q \vec{d}$, then $\vec{P} = Nq \vec{d}$)

3310 4-2 Dielectric!

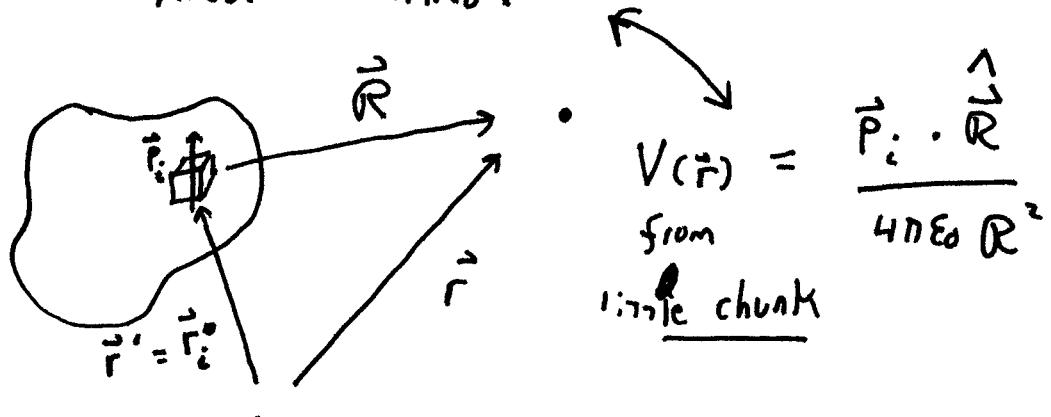
Neutral material (e.g. chunk of plastic) likely has $\rho(r^2) = 0$

So, if interested in the \vec{E} -field it produces, our "multipole expansion" says, after $V = \frac{Q_{701}}{4\pi\epsilon_0 r}$ which vanishes,

the leading contribution comes from dipole moments!

Remember $V(r^2) = \frac{Q_{701}}{4\pi\epsilon_0 r} + \frac{\vec{P} \cdot \hat{r}}{4\pi\epsilon_0 r^2} + \text{higher order terms}$.

So if we have neutral, polarized chunk
(so polarization term leads)



$$V(\vec{r}) = \frac{\vec{P}_i \cdot \hat{R}}{4\pi\epsilon_0 R^2}$$

from little chunk

But we just said $\vec{P}_i = \text{"dipole moment of small chunk"} \text{ of volume } d\tau'$

$$= P(\vec{r}') d\tau'$$

Do you see this? $P = \frac{\text{"dipole moment"}}{\text{unit volume}} = \frac{\vec{P}}{\text{Volume}}$

$$\text{so } \vec{P} = P \cdot d\tau$$

$$\text{Thus } V_{\text{total}}(\vec{r}) = \iiint \frac{\vec{P}(\vec{r}') \cdot \hat{R} d\tau'}{4\pi\epsilon_0 R^2}$$

(This allows for possibility that \vec{P} is not same everywhere)

Griffiths now proves the following: this $V(\vec{r})$ can be re-written in a simple way, looks familiar!

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\text{Surf}} \frac{\sigma_b(r') dA'}{R} + \frac{1}{4\pi\epsilon_0} \iiint_{\text{Vol}} \frac{\rho_b(\vec{r}') d\tau'}{R}$$

wish

$$\left[\begin{array}{l} \sigma_b \equiv \vec{P} \cdot \hat{n} \quad = \text{Normal component of polarization at surface} \\ \rho_b \equiv -\nabla \cdot \vec{P} \quad = \text{Divergence of } P(\vec{r}) \end{array} \right]$$

So there are these ("imaginary") bound charges; a polarized object looks to outside world like an ordinary charged object (with these charge distributions) + thus our old ch. 2 approaches to find V will work! Just integrate...

Let's understand the meaning / origin of these 2 terms!

3310 4-4

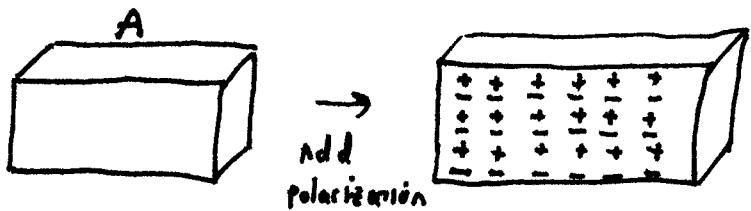
Start with chunk of material (dielectric)

Keep it simple: neutral,

unpolarized at first... but

then we polarize it (somehow!)

Eg, put it in a big capacitor

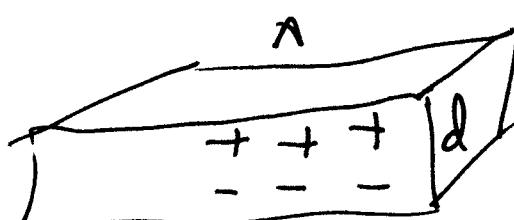


Each little molecule became a dipole, with $\vec{p} = q \vec{d}$

Note that although it's still neutral, the top face has a $+σ$ on it, the bottom has a $-σ$.

$$\text{How big? Well... } σ_{\text{top}} = \frac{\text{charge on top}}{\text{Area}} = \left(\frac{\# \text{ atoms} \times q}{\cancel{\text{area}}} \right) g \times \cancel{\frac{\text{area}}{\text{Area}}}$$

$$= \frac{\# \text{ atoms}}{\text{Volume}} \times \text{volume} \times g$$



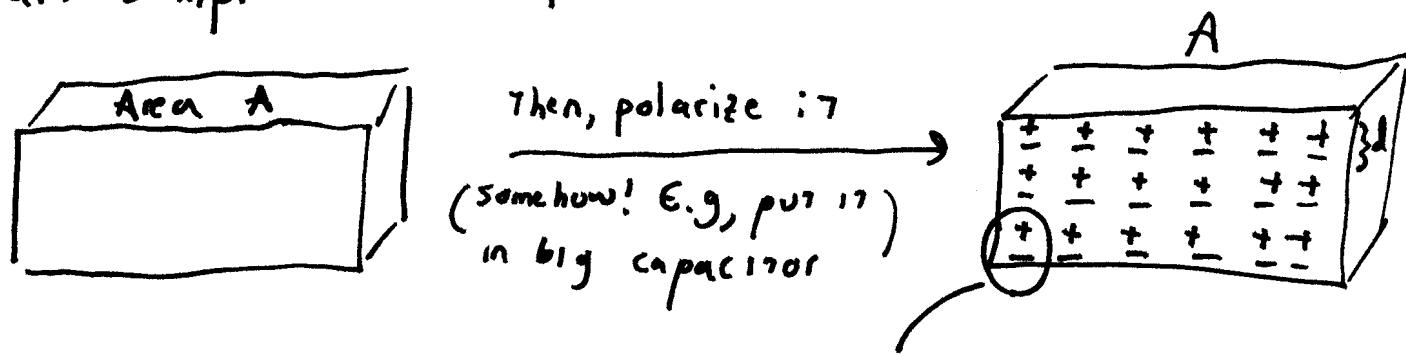
$$σ = \frac{\text{charge}}{\text{area}} \approx \frac{\text{charge}}{\text{Volume}} \times d$$

$$=$$

$$\text{Total charge} = \left(N \left[\frac{\text{atoms}}{\text{Volume}} \right] \times \text{this volume} \right) \times g \text{ on each atom}$$

$$= N \times A \cdot d \times g$$

Start simple: chunk of dielectric. Neutral.



Each little atom is a dipole, with $\vec{p}_i = q \vec{d}$, [with N atoms/volume]

Still neutral, but look, there is now $+5$ on very top face and -5 on bottom. How big?

Look at top layer

$$Q_{\text{on top}} = \# \text{ atoms} * q \text{ each}$$

Q has "passed thru" the middle line + landed on top surface

$$= (N A d) * q$$

($\frac{\# \text{ atoms}}{\text{volume}}$ volume)

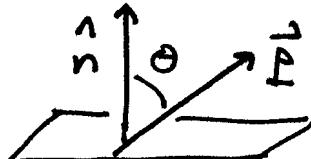
so $\underline{\sigma_{\text{top}}} = \frac{N d q \cdot A}{\text{Area}} = N q d$

But remember, $\underline{P} = \frac{\text{polarization}}{\text{Volume}} = \frac{(N \cdot \text{Volume}) (q d)}{\text{Volume}}$

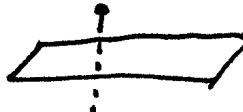
$$= N q d \quad \leftarrow \text{sec p. 4-1, bottom}$$

so here $\underline{\sigma} = \underline{P}$. (Interp: σ = amount of charge crossing area A when polarize material.)

what if \vec{p} was not parallel to my area?



Amount of charge passing through Area will now be $P \cos \theta = \vec{P} \cdot \hat{n}$

Throughout the bulk, every area  gets Q passing through it due to polarization, but $Q_{\text{leaving}}(\text{up}) = Q_{\text{entering}} \text{ from below}$ to this has no impact anywhere except surface,

where $\sigma_b = \vec{P} \cdot \hat{n}$ = "bound surface charge"

It's physical, there really is an excess $+\sigma_b$ on top, $-\sigma_b$ on bottom.
These charges create external \vec{E} fields like any charges

If P is uniform, that's it, simple, all we get.

But if $P(r)$ varies, something else happens:

locally, charge entering some region may not equal charge leaving!



we just argued $\frac{Q}{\text{area}} = \vec{P} \cdot \hat{n}$

this is Q that leaves, passes ~~out~~ through this dA

so total charge leaving Volume

$$\text{is } \oint_S \vec{P} \cdot \hat{n} dA = \iiint_V \vec{D} \cdot \vec{P} dV$$

leaving!

!!
Divergence
Theorem

so after polarizing, (creating $\vec{P}(\vec{r})$) you find

$$\text{Volumes have } \underline{\text{lost}} \quad Q = \iiint \vec{E} \cdot \vec{P} dV$$

i.e. dV has a negative charge, $Q_{\text{remaining}} = -\nabla \cdot \vec{P} dV$

which means $\rho_B = \frac{Q \text{ in } dV}{\text{volume } dV} = -\nabla \cdot \vec{P}$.

only present if \vec{P}
non-uniform, of course!

It's real, there is a true net charge in dV !

Bottom line: Polarizing a dielectric $\Rightarrow \vec{P}(\vec{r}) = \frac{\text{dipole moment}}{\text{volume}}$

this results in tiny shifts of charge everywhere, which "piles up" on edges. If more polarization in some places in bulk than others, may also get local charging throughout volume.

$$\sigma_B = \vec{P} \cdot \hat{n} = \text{charge density on surface}$$

$$\rho_B = -\nabla \cdot \vec{P} = \text{volume charge density inside}$$

These are real, physical "bound charges", + create ordinary voltage + \vec{E} field as usual.

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Formal proof: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{Vol}} \frac{\vec{P}(\vec{r}') \hat{R}}{R^2} d\tau' \quad \underline{\text{P. 4.2}}$

Neutral dielectric \rightarrow

Math trick: $\vec{\nabla} \frac{1}{|R|} = -\frac{1}{R^2} \hat{R}$ with $\vec{R} = \vec{r} - \vec{r}'$
 $|R| = |\vec{r} - \vec{r}'|$

$$\vec{\nabla}' \frac{1}{|R|} = +\frac{1}{R^2} \hat{R}$$

[See notes ch 1-2 p. 28
or Griff p. 15]

Doesn't matter if take $\frac{\partial}{\partial x'} \text{ or } \frac{\partial}{\partial x}$
the function is symmetric
 $\vec{r} - \vec{r}' = -(\vec{r}' - \vec{r})$

so $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{Vol}} \vec{P}(\vec{r}') \cdot + \left(\vec{\nabla}' \frac{1}{|R|} \right) d\tau'$

But since $\vec{\nabla}' \left(\frac{\vec{P}(\vec{r}')}{R} \right) = \underbrace{\vec{P}(\vec{r}') \cdot \vec{\nabla}' \frac{1}{R}} + \vec{\nabla}' \cdot \vec{P}(\vec{r}') \cdot \frac{1}{R}$
Get by itself and integrate

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{Vol}} -\vec{\nabla}' \cdot \frac{\vec{P}(\vec{r}')}{R} d\tau' + \frac{1}{4\pi\epsilon_0} \int_{\text{Vol}} \vec{\nabla}' \left(\frac{\vec{P}(\vec{r}')}{R} \right) d\tau'$$

↓ Div. Theorem

$$= \frac{1}{4\pi\epsilon_0} \int_{\text{Vol}} \frac{P_B(\vec{r}') d\tau'}{R} + \frac{1}{4\pi\epsilon_0} \oint_S \frac{\vec{P}(\vec{r}') \cdot d\vec{n}'}{R}$$

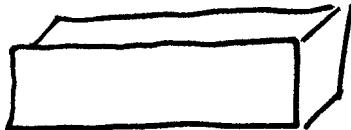
with $P_B \equiv -\vec{\nabla}' \cdot \vec{P}(\vec{r}')$

Define $\sigma_R = \vec{P}(\vec{r}') \cdot \hat{n}$

(As claimed)

so this is $\oint \sigma d\vec{n}' / R$

Ex 1:



Dielectric slab in "Uniform"

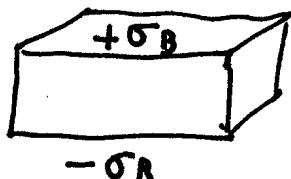
$$\vec{E}_{\text{em}} = E_{\text{ext}} \hat{z} \quad (\text{e.g., capacitor!})$$

we know the dielectric will polarize, with uniform

$$\vec{P} = N g d \hat{z} = P_0 \hat{z} \quad \left[N = \frac{\# \text{ atoms}}{\text{volume}} \quad g d = \text{dipole moment} \right. \\ \left. \text{of each atom.} \right]$$

(Note that d will itself depend on / be created by \vec{E})But in any case, P_0 will be a uniform constant value.so $\nabla \cdot \vec{P} = 0$, no bound volume charges

$$\sigma_B = |\vec{P} \cdot \hat{n}| = N g d \quad \text{on edges.} \quad (+N g d \text{ on top}) \\ = P_0 \quad (-N g d \text{ on bottom,})$$

This creates a new
field

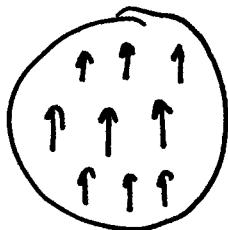
$$\downarrow \downarrow \downarrow \downarrow \quad \vec{E}_{\text{new}} = \frac{\sigma_B}{\epsilon_0} (-\hat{z}) = \frac{P_0}{\epsilon_0} (-\hat{z})$$

$$\vec{E}_{\text{total inside}} = E_{\text{ext}} - E_{\text{new}} \quad \text{is } \underline{\text{weaker}}, \cancel{\text{stronger}}$$

(we "polarized" the dielectric, which effectively
weakens total \vec{E} field inside)

(If it's large slab, \vec{E}_{new} is only inside the slab!)

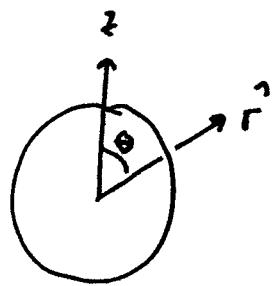
Ex. 2: (classic example!)



Uniformly polarized sphere.

$$\vec{P} = P_0 \hat{z}$$

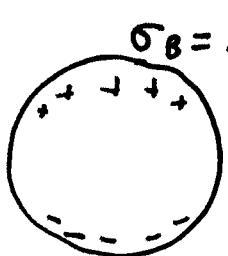
$$\text{So } \vec{\nabla} \cdot \vec{P} = 0 \text{ again.}$$



$$\text{But we again have } \sigma_B = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{r} = \underbrace{P_0 \cos \theta}_{\hat{z} \cdot \hat{r} = \cos \theta}$$

This was a homework problem, + is solved exactly in Griffiths

using spherical (Legendre) approach. [In ~~our~~ our, B.C. will force us to pure $P_1(\cos \theta) = \cos \theta$ term]



creates

$$V(r, \theta)_{\text{outside}} = \underbrace{\frac{P_0}{3\epsilon_0} \frac{R}{r^2} \cos \theta}_{\text{Pure dipole !!}}$$

No derivation, to check!

$$\text{outside: } \frac{C_{\text{out}} P_1(\cos \theta)}{r^2}$$

$$\text{inside: } C_{\text{in}} \cdot r \cdot P_1(\cos \theta)$$

$V(R)$ continuous \Rightarrow

$$C_{\text{out}}/R^2 = C_{\text{in}} R$$

$$\frac{\partial V}{\partial r} = -2 C_{\text{out}} / R^3 P_1 \text{ out}$$

$$= + C_{\text{in}} P_1 \text{ in}$$

$$\left. \frac{\partial V}{\partial r} \right|_{\text{out}=\text{in}} = \cancel{C_{\text{out}} \rightarrow \sigma_B / \epsilon_0} = 0$$

$$V(r, \theta)_{\text{inside}} = \underbrace{\frac{P_0}{3\epsilon_0} \frac{R}{r} \cos \theta}_{\text{Pure dipole !!}}$$

$$= \underbrace{\frac{P_0}{3\epsilon_0} Q}_{Z \theta}$$

~~Like in a capacitor!~~

$$\int_{\text{out}}^{\text{in}} -2 \frac{C_{\text{out}}}{R^3} - C_{\text{in}} = -\frac{P_0}{\epsilon_0}, \text{ and } C_{\text{out}} = R^3 C_{\text{in}}$$

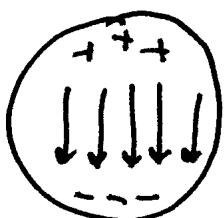
$$\text{so } C_{\text{in}} = P_0 / 3\epsilon_0, Q = 0$$

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Finding \vec{E} makes this result more intuitive.

$$\text{Inside } V = \frac{P_0}{3\epsilon_0} z \Rightarrow \vec{E} = -\vec{\nabla}V = -\frac{P_0}{3\epsilon_0} \hat{z}$$

Pure downward field, like a capacitor, cool!



(It is "fighting" whatever E_{ext} made our dipole in the first place)

$$\text{Outside: } V = \frac{P_0 R^3}{3\epsilon_0} \frac{\cos\theta}{r^2} = \frac{\vec{P} \cdot \hat{r}}{r^2} \cdot \frac{R^3}{3\epsilon_0}$$

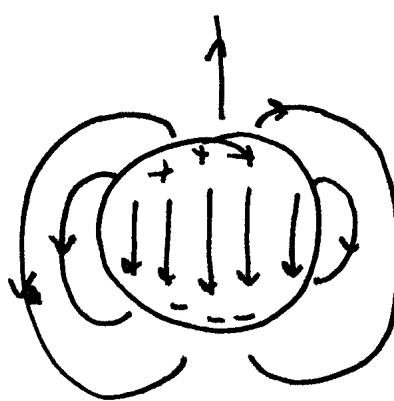
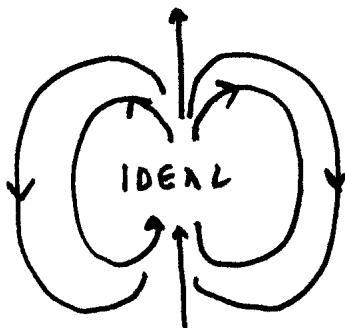
That's V of an ideal dipole ~~with~~: ~~\vec{P}~~ $V_{\text{ideal}} = \frac{\vec{P} \cdot \hat{r}}{4\pi\epsilon_0 r^2}$

$$\text{If } \frac{\vec{P}}{4\pi} = \frac{\vec{P}}{3} \frac{R^3}{r^2} \Rightarrow \vec{P} = \vec{P} \left(\frac{4\pi R^3}{3} \right)$$

↑
 total ideal
 dipole moment

←
 dipole moment
 unit volume

←
 volume
 of sphere!



Discontinuity in \vec{E} at surface - as you would expect!
There's σ there