

Separation of Variables.

we just solved Poisson's eq'n for some very special cases (oo grounded conducting sheet, e.g.). But it's not general enough, we want to tackle $\nabla^2 V = 0$, given boundary conditions, for more circumstances, i.e. more generally....

The idea is this: $V(x, y, z)$ could be complicated! It might

e.g. be $\sim \frac{1}{\sqrt{x^2+y^2+z^2}}$. But in some (many!) circumstances, we

will find that $V(x, y, z) = \Phi(f_n \text{ of } x) * (\text{another fn of } y) * (\text{another fn of } z)$

Like, say $e^x \cdot \cos(y) \cdot \sin(z)$ or something.

And if it ~~isn't~~, it might still be some combination of such fns,

Like say $e^x \cos y \sin z + e^{2x} \cos y \sin z + e^x \cos 2y \sin 3z + \dots$
 ↗ In fact, we can pretty much build any function V up like this!

So, we'll see that if $V(x, y, z) = \Phi(x) \Psi(y) \Omega(z)$,

(or any sum of such fns) then we can solve $\nabla^2 V = 0$

in many many situations!

- This method will only help us if we can then "match our boundary conditions", but is quite general + powerful.

Bottom line for "sep. of variables"

we have $\nabla^2 V = 0$ and given B.C.'s

we try $V(x, y, z) = X(x) Y(y) Z(z)$. Just try it!

If it works, and if it satisfies our B.C.'s, uniqueness \Rightarrow done.

(+ If it fails, but a sum of such fns "", "", \Rightarrow done)
use superposition.

By the way, it might work better to try, e.g.

$$V = R(r) \Theta(\theta) \Phi(\phi) \quad (\text{in spherical coords!})$$

we'll get there soon :)

$$\text{OK, so } \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\text{Try } V = X(x) Y(y) Z(z), \text{ and note e.g. } \frac{\partial^2 V}{\partial x^2} = \frac{d^2 X(x)}{dx^2} Y(y) Z(z)$$


Partial deriv \Rightarrow Total deriv * No x-dep,
(do you see why?) they're constants

$$\text{so we have } X''(x) Y(y) Z(z) + X(x) Y''(y) Z(z) + X(x) Y(y) Z''(z) = 0$$

where X'' means $\frac{d^2}{dx^2}$. Next, divide both sides by $X(x) Y(y) Z(z)$

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so we have a lot of cancellation (do it, check!)

$$\underbrace{\frac{\partial^2 \chi(x)}{\partial x}} + \underbrace{\frac{\partial^2 \gamma(y)}{\partial y}} + \underbrace{\frac{\partial^2 \zeta(z)}{\partial z}} = 0.$$

$$\text{pure function of } x + \text{pure fn of } y + \text{pure fn of } z = 0.$$

Now you must convince yourself that this is nuts!

How could $f(x) + g(y) = 0$ for all x and y ?

No way!! Vary x , with y fixed, + you'll change the sum...

unless $f(x)$ is just a constant, i.e. it really doesn't depend on x at all!

Conclusion : $\nabla^2 V = 0$ and $V = \chi(x)\gamma(y)\zeta(z)$ requires

$$\frac{\partial^2 \chi(x)}{\partial x} = c_1 \quad \frac{\partial^2 \gamma(y)}{\partial y} = c_2 \quad \frac{\partial^2 \zeta(z)}{\partial z} = c_3$$

$$\text{with } c_1 + c_2 + c_3 = 0.$$

we have three simple 2nd order ordinary diff eq's to solve.

Consider $\chi''(x) = c_1 \chi(x)$. Look familiar! It has quite simple solns, (depending on the sign of c_1 , only!)

~~$\chi(x) = A\cos(x) + B\sin(x)$~~

2 undetermined constants

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$\ddot{\chi}(x) = C_1 \chi(x)$. General sol'n is $\chi(x) = A e^{\sqrt{C_1} x} + B e^{-\sqrt{C_1} x}$

If $C_1 > 0$, $\ddot{\chi}(x) = A e^{\sqrt{C_1} x} + B e^{-\sqrt{C_1} x}$ check for yourself!

If $C_1 < 0$, get complex exponentials, which you ~~never~~ can think of as combos of $\sin + \cos$ by Euler's theorem

$$\hookrightarrow \chi(x) = A' \sin(\sqrt{-C_1} x) + B' \cos(\sqrt{-C_1} x)$$

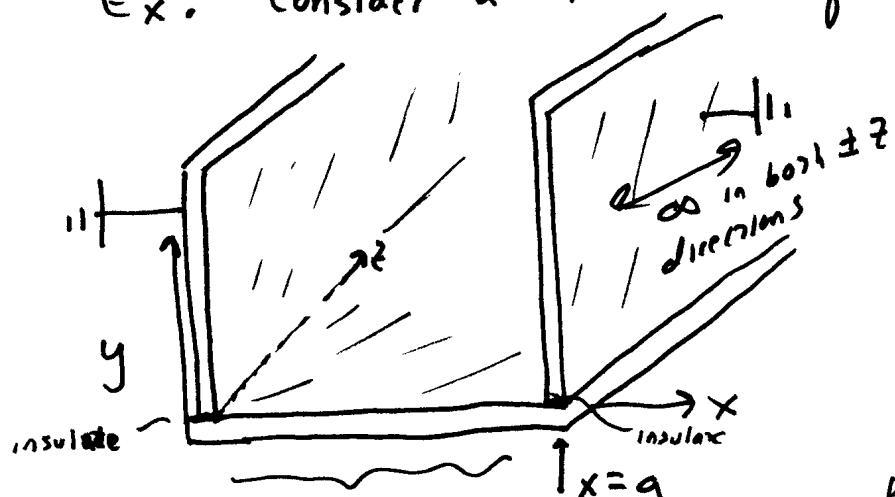
If $C_1 = 0$, $\ddot{\chi}(x) = A'' + B'' x$.

So we have traded our Partial Diff Eq for 3 (easy) ODE's, at a cost of lots of undetermined constants popping up.

Our boundary conditions will have to determine all those!

At this point we just need to get concrete! Let's start with a 2-D problem to warm up. This just means $V = V(x, y)$ it has no z -dependence. Physically, we just need a setup which is uniform in z , so nothing will vary in that direction.

Ex: Consider a metallic (square) ~~plate~~^{gutter} which extends in $\pm z$ directions, forever.



"hor" part: Voltage here = $V_0(x)$

of course, if this base is a conductor, $V_0(x) = V_0 = \underline{\text{constant}}$
 (But maybe the base is not a conductor, or e.g. long strips of conductor separated by thin insulation?)

Inside the pipe (i.e. $0 \leq x \leq a$) is empty so $\nabla^2 V = 0$ there.

Boundary conditions $V(x=0, y>0) = 0$ grounded!

$V(x=a, y>0) = 0$ grounded

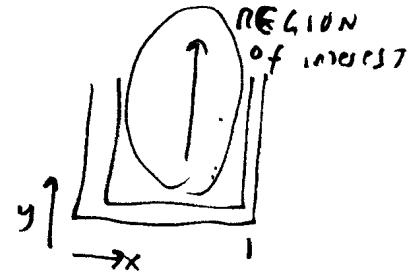
$V(\text{any } x, y=0) = V_0(x)$ fixed voltage, somehow.

$V(\text{any } x, y=\infty) = 0$ ← physically reasonable,

at $y=\infty$ we're far from the "hor" sheet, $V \rightarrow 0$

Note: No B.C. on z , but V can't depend on z by symmetry.
 So it's really a 2-D problem!

- So to go: we have region with $\nabla^2 V = 0$ and we know V at all 4 boundaries!
(zero on left, right, and top ($\rightarrow \infty$), finite but given at bottom.)



Try $V(x, y) = X(x)Y(y)$. Separate variables

$$\text{so } X''(x) = C_1 \cdot X(x)$$

$$AY''(y) = C_2 Y(y) \quad \text{but } C_1 + C_2 = 0, \text{ so } C_2 = -C_1.$$

which is positive, C_1 or C_2 ? Physics of Boundary tells us!!

If $C_1 > 0$, then $X(x) = A e^{+\sqrt{C_1}x} + B e^{-\sqrt{C_1}x}$.

That's no good! Can't make that function vanish at $x=0$
(convince yourself!) and $x=a$!!

~~so~~ $C_1 = 0$ also no good, can't make $A + BX$ vanish at $x=0$
(convince yourself!) and $x=a$

so $C_1 < 0$, call it $-k^2$ to make it obvious it's negative.

+ $C_2 = -C_1 = +k^2$ is clearly positive.

$$\begin{aligned} \text{so } X(x) &= A \sin(kx) + B \cos(kx) \\ Y(y) &= C e^{+ky} + D e^{-ky} \end{aligned}$$

~~Bottom~~ Four undetermined coefficients, plus K !

To go further, we need to use more boundary conditions

① $V(x=0)$ has to vanish (left wall is grounded)

so $\Phi(0) = 0$ which tells me $B = 0$.

$$\text{so } \Phi(x) = A \sin(Kx)$$

② $V(y \rightarrow \infty)$ has to vanish (far away, Voltage must $\rightarrow 0$)

this tells me $C e^{ky} \rightarrow 0$ for large y , so $C = 0$.

So we've got

$$V(x,y) = \Phi(x) \Psi(y) = \underline{AD} \sin(Kx) e^{-Ky}$$

Looks like who cares what you call it, it's just some const'n!
need 2 constants, but really \longrightarrow call it C'

Third boundary condition:

③ $V(x=a) = 0$ has to vanish, right wall is grounded.

$$\text{so } C' \sin(\frac{Ka}{a}) e^{-Ky} = 0 \quad \text{for any } y.$$

But I cannot now set $C'=0$, 'cause if I did, $V=0$,
+ that's not right at the bottom!

so we need $\sin(Ka) = 0$. This tells us K (the
"separation constant" cannot be any old shing!!

$$K = \frac{n\pi}{a} . \quad \text{so certain } K \text{'s will work!}$$

(n can be an integer, > 0 . But $n=0$ is bad
 $n<0$ isn't different!)

So far,

$$V(x,y) = C' e^{-ky} \sin(kx) \quad \text{with } k = \frac{n\pi}{a} \\ (n=1, 2, 3, \dots)$$

$[n=0$ gives you $V=0$, which is no good
 n negative is the same sol'n, just changes sign of C']

This satisfies $\nabla^2 V = 0$ (by construction, but check if you like)
 and all boundary conditions except $V(x, y=0) = V_0(x)$.

So can we use sep. of variables?

well, if $V(x,0) = C' \sin(kx) = V_0(x)$

then we're golden. So if we start with

$V_0(x) = C' \sin(kx)$, with $k = n\pi/a$, we're done!

we have V everywhere.

But gosh, we should be able to set $V_0(x)$ to be whatever we want, doesn't have to be $\sin(\frac{n\pi x}{a})$ physically! Are we

stuck? No. Because if we have $V_1(x,y)$ which "works"

and another $V_2(x,y)$ which "works", then $aV_1 + bV_2$ also

works. ($\nabla^2 aV_1 + bV_2 = a\nabla^2 V_1 + b\nabla^2 V_2 = 0 + 0 = 0$)

3310 Notes 3-21

We have many solns. For any + integer n

$$V_n(x, y) = C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

satisfies $\nabla^2 V_n = 0$ (and 3 b.c.'s).

$$\text{so } V(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}.$$

also satisfies $\nabla^2 V = 0$ (and all 3 b.c.'s Check!)

this sum is still a sum of zeros at $x=0, x=a$, and $y = +\infty$).

so Now the question is, can you pick your C_n 's such that

$$V(x, 0) = V_0(x). \text{ If so, we've got it!!}$$

this means
$$\boxed{\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) = V_0(x)}$$

(because $e^0 = 1$ when $y=0$)

remember, $V_0(x)$ is a given function, it's a b.c.

C_n is as yet undetermined. If we can pick C_n 's, then we win.

But we can! This is Fourier's Trick, any function $V_0(x)$ (reasonable)

(for $0 < x < a$) can always be written, uniquely, like this!

Fourier's Trick (Math interlude).

The functions $\sin \frac{n\pi x}{a}$ are a complete, orthonormal set of fns.

$$\int_0^a \sin \frac{n\pi x}{a} \cdot \sin \frac{n'\pi x}{a} dx = \begin{cases} 0 & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

check for yourself!

$$\text{so if } V_0(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a}$$

$$\text{then } \sin \frac{n'\pi x}{a} \cdot V_0(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sin \frac{n'\pi x}{a} \quad \leftarrow \begin{array}{l} \text{multiply} \\ \text{both sides} \\ \text{by } \sin \frac{n'\pi x}{a} \end{array}$$

$$\begin{aligned} \int_0^a V_0(x) \sin \frac{n'\pi x}{a} dx &= \sum_{n=1}^{\infty} C_n \int_0^a \sin \frac{n\pi x}{a} \sin \frac{n'\pi x}{a} dx \quad \leftarrow \text{integrate} \\ &= C_{n'} \cdot \frac{a}{2} \quad \underset{\text{all}}{\equiv} \quad \begin{array}{l} \uparrow \\ \text{vanish} \end{array} \text{ except for } n=n' \end{aligned}$$

$$\text{so } C_{n'} = \frac{2}{a} \int_0^a V_0(x) \sin \frac{n'\pi x}{a} dx \quad \boxed{\text{A formula for all the } C_n\text{'s.}}$$

Once you have all the C_n 's... you're done!

So this gives us $V(x, y)$ everywhere, no matter what $V_0(x)$ on the base we started with.

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Example: $V_0(x) = V_0$, the base is also metal, just at high voltage.

$$C_{n'} = \frac{2}{a} \int_0^a V_0 \sin \frac{n' \pi x}{a} dx = \frac{2V_0}{a} \cdot \frac{-a}{n' \pi} \left(\cos \frac{n' \pi x}{a} \right)_0^a \\ = -\frac{2V_0}{\pi n'} (\cos n' \pi - 1)$$

Pick any / every n' from 1 - ∞ .

$$\text{e.g. } C_1 = -\frac{2V_0}{\pi} (-2) = \frac{4V_0}{\pi}$$

$$\text{e.g. } C_2 = -\frac{2V_0}{\pi \cdot 2} (1-1) = 0.$$

In general $C_{n \text{ odd}} = \frac{4V_0}{n\pi}$, $C_{n \text{ even}} = 0$.

thus $V(x,y) = \sum_{n=1,3,5,\dots}^{\infty} \underbrace{\frac{4V_0}{\pi} \cdot \frac{1}{n}} \cdot \sin \left(\frac{n\pi x}{a} \right) e^{-\frac{n\pi y}{a}}$
 this was C_n .

valy? But, analytic, calculable, ... we did it.

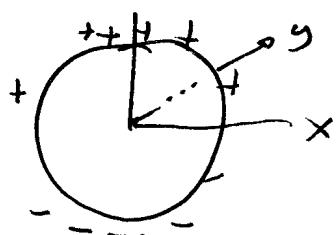
(Any function $V_0(x)$ can be used as the top, you'll get a different set of constants C_n 's...)

- See Griffiths for more examples! Work through a couple...

Let's instead consider problems that have "boundaries" that look spherical instead of planar. So, $\mathcal{E}(x)\mathcal{A}(y)\mathcal{B}(z)$ will work, but it'll be a terrible mess. It's much more natural to try $R(r)\Theta(\theta)\Phi(\phi)$.

In this course, we will only look at problems with $\frac{\partial V}{\partial \phi} = 0$

dependence (e.g. a sphere of charge)



ϕ could depend on θ (polar)

but not ϕ .

$$\nabla^2 V = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + 0$$

Again, we try $V(r, \theta) = R(r)\Theta(\theta)$, plug it in, and

then divide both sides by $R(r)\Theta(\theta)$, giving

$$\frac{\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right)}{R(r)} + \frac{\frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta(\theta)}{d\theta} \right)}{\Theta(\theta)} = 0$$

Multiply through by r^2 , to get

$$\frac{1}{R(r)} \frac{d}{dr} (r^2 R'(r)) + \frac{1}{\Theta(\theta)} \frac{d}{d\theta} (\sin \theta \Theta'(\theta)) = 0$$

|| ||

$$C_1 + C_2 = 0$$

as before, no function of r alone can add to a fn of θ to give 0 unless each is a constant.

Once again, our PDE \Rightarrow 2 simple ODE's.

" " , which needs to be positive, C_1 or C_2 ? This time we have a "built in" B.C., namely $\Theta(\theta)$ must not blow up (we can't allow infinite Voltages at finite r 's! At least, not if we have smooth charge distributions) It turns out that this forces C_2 to have special features: it must be negative, and it must be writable as $-l(l+1)$ with l an integer (I won't prove this!)

$$\text{so } C_1 = +l(l+1) \quad \text{for some integer } l \geq 0$$

$$C_2 = -l(l+1) \quad \text{(Here, } l=0 \text{ is not trivial!)}$$

This ODE's are solvable. (Remember, you can just check!)

The general sol'n to $\frac{d}{dr} (r^2 R') = l(l+1) R$

is $R(r) = Ar^l + B/r^{l+1}$

2 constants, to be determined by b.c.'s!

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The angular eq'n is uglier. But the sol'n's are not so bad: For instance

$$l=0: \frac{d}{d\theta} (\sin\theta \Theta'(\theta)) = 0 \text{ is solved by } \Theta_0(\theta) = 1$$

($\Theta_0(\theta) = \text{constant}$ works, + there is another function which, alas, blows up at $\theta=0$) I will choose $\Theta_0(\theta) = 1$ and slide any constant coefficient over into my A and B of $R(r)$

$$l=1, \text{ the sol'n (which is finite) is } \Theta_1(\theta) = \cos\theta$$

$$\left(\text{check: } \frac{d}{d\theta} \sin\theta \Theta'_1 = -1)(2) \{\Theta'_1\} \cdot \sin\theta \quad \text{it works!} \right)$$

$$\text{In general, } \Theta_l(\theta) = P_l(\cos\theta)$$

with $P_l(\cos\theta)$ = "Legendre Polynomial"

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{3}{2} \cos^2\theta - \frac{1}{2}$$

$$\text{thus, } V_l(r, \theta) = R_l(r) \Theta_l(\theta) = \left(A r^l + \frac{B}{r^{l+1}} \right) P_l(\cos\theta)$$

solves $\nabla^2 V_l = 0$. It's true for any l , and so, like before, we can combine ~~as many~~ V_l 's

3310 Notes 3-27

$$\text{so } V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

satisfies $\nabla^2 V = 0$, (and $V(r, \theta)$ is finite for any theta.)

So here's the game: If I give you boundary conditions on a sphere, like e.g. $V(a) = \text{constant}$, or $V(a, \theta) = C \times f_n(\theta)$, or whatever, we can build a $V(r, \theta)$ which satisfies this b.c., and $\nabla^2 V = 0$, by using the form above. You just need to find A_ℓ and B_ℓ 's, and uniqueness says if you find one such sol'n, that's it!

In the case $V(a, \theta) = V_0(\theta)$ is given, and suppose we want $V(r, \theta)$ outside this boundary



→ We need $V(\infty, \theta) \rightarrow 0$, so A_ℓ better vanish $\forall \ell$!

$$\text{so } V(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta) \quad \xrightarrow{\text{unknown constants!}}$$

$$\text{and } V(a, \theta) = V_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos \theta)$$

\uparrow \uparrow \searrow
Given given constns known functions

This is again Fourier's trick!

Fourier's trick for Legendre polynomials:

$$\text{I claim } \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \cdot \sin\theta d\theta = \begin{cases} 0 & \text{if } \ell \neq \ell' \\ \frac{2}{2\ell+1} & \text{if } \ell = \ell' \end{cases}$$

without proof.

$$\text{or, } \begin{aligned} x &= \cos\theta \\ dx &= -\sin\theta d\theta \end{aligned}$$

$$\int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2\ell+1} & \ell = \ell' \end{cases}$$

so to find B_ℓ , multiply both sides by $P_{\ell'}(\cos\theta)$ and integrate.

All terms in the sum $\sum_{\ell=0}^{\infty} \dots P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$ will vanish but one

$$\text{so } \frac{B_{\ell'}}{\alpha^{\ell'+1}} \cdot \frac{2}{2\ell'+1} = \int_0^\pi P_{\ell'}(\cos\theta) \cdot V_0(\theta) \sin\theta d\theta$$

this gives you all your B_ℓ 's!

If I give you $V_0(\theta)$ on the sphere and want V inside,

the argument is similar, except now $B_\ell = 0$ (so $V(0)$ is fine)

$$\text{so } V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta), \text{ with}$$

$$\text{and: } A_\ell \cdot \alpha^{\ell'} \cdot \frac{2}{2\ell'+1} = \int_0^\pi P_{\ell'}(\cos\theta) V_0(\theta) \sin\theta d\theta$$

Comments: What we have is a method to find $V(r, \theta)$ if we know $V_0(\theta) \stackrel{\equiv V_0(\theta)}{=}$. The sol'n will be a sum, we need only to find some numerical coefficients (A_l 's if our region includes the origin, or B_l 's if our region includes $r = \infty$, assuming no funny business)

The integrals look scary, $\int_0^\pi P_l(\cos\theta) V_0(\theta) \sin\theta d\theta$.

In general, if $V_0(\theta)$ is nasty, we may be in trouble. The trick

is to remember orthogonality, $\int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta = 0$ if $l \neq l'$

so this approach works best if $V_0(\theta)$ looks like a single, simple Legendre Polynomial (or perhaps, a sum of a couple)

e.g. if $V_0(\theta) = V_0$, we see that $V_0(\theta) = V_0 \underline{\underline{P_0(\cos\theta)}}$

and so we'll only need one integral, with $l=0$, all others vanish.

$$P_0 = 1$$

$$P_1 = \cos\theta$$

$$P_2 = \frac{3}{2} \cos^2\theta - \frac{1}{2}$$

so e.g. If $V_0(\theta) = \sin^2\theta$,
I notice this is $1 - \cos^2\theta$
 $= -\frac{2}{3} P_2 - \frac{1}{3} P_0$.

So only $l=0$ and $l=2$ terms will live!

classic Example Let's put a metal sphere (radius a) into an existing, external, uniform field $\vec{E} = E_0 \hat{z}$. Now, this field will polarize the sphere, which in turn superposes, making a complicated field.

what is $V(r, \theta)$ everywhere?

• $V(r < a), \theta) = V_0$ conductors are equipotentials!

And I can pick one spot to be $V=0$, so

let's let $V(0)=0$, which makes $V_0=0$.

Outside, well, we have new b.c.'s! $V(r=\infty, \theta)$ is not 0!

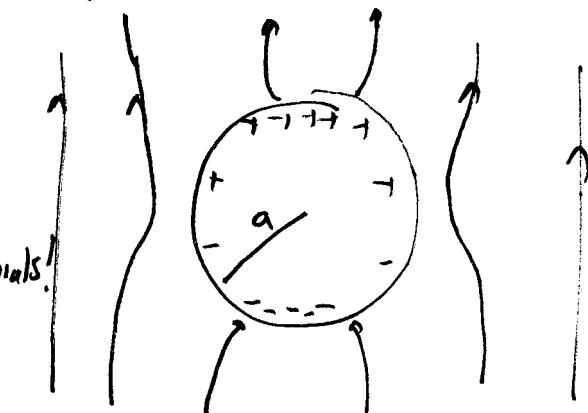
With $\vec{E}_{ext} = E_0 \hat{z}$, far away you must have $\vec{E}_{ext} \rightarrow E_0 \hat{z}$
 $= -\vec{\nabla}V$

thus $V(r \rightarrow \infty, \theta) \rightarrow -E_0 z$ (no constant, because need
 $V(x \rightarrow \infty, z=0) = 0$ too!)

$$\text{So we have } V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Consider first $r \rightarrow \infty$, wish $V(\infty, \theta) \rightarrow -E_0 z = -E_0 r \cos \theta$
 $= -E_0 r P_1(\cos \theta)$

$$\text{so } -E_0 r P_1(\cos \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) \Big|_{r \rightarrow \infty} P_l(\cos \theta)$$



In, The right side, the B.C. terms don't contribute,

$$\text{so } -E_0 r P_1(\cos\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

L.H.S. is pure P_1 . So, only $l=1$ can contribute!!

so $A_1 r P_1 = -E_0 r P_1$ which means $A_1 = -E_0$ is fixed by the B.C. at ∞

and so are all other A_l 's, all rest vanish!

Next, at $r=a$, we have

the only surviving A_1 term

$$V(a, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) + \underbrace{A_1 a P_1(\cos\theta)}$$

○ so how can this be zero? All B_l 's better vanish, except $l=1$, which will have to kill that A_1 bit.

$$\text{so } \frac{B_1}{a^{1+1}} = -A_1 a . \quad \left[\text{so we have } \frac{B_1}{r^{1+1}} P_1(\cos\theta) \text{ term too} \right]$$

$$\text{thus } V(r, \theta) = A_1 r P_1(\cos\theta) + -\frac{A_1 a^3}{r^2} P_1(\cos\theta) \quad \text{with } A_1 = -E_0$$

$$\text{so } V(r, \theta) = E_0 P_1(\cos\theta) \left(\frac{a^3}{r^2} - r \right) = -\underbrace{E_0 r \cos\theta}_{\text{that was } E_0 r!} + \underbrace{\frac{a^3}{r^2} E_0 \cos\theta}_{\text{the "induced" } V!}$$

- If, instead of giving you V at some radius, I gave you $\sigma(a, \theta)$, you can still use this approach, because

$$E_{\text{out}}^\perp - E_{\text{in}}^\perp = \frac{\sigma}{\epsilon_0} \quad \text{with } \text{a sphere, and } \cancel{E^\perp = -\frac{\partial V}{\partial r}},$$

$$\left. \frac{\partial V}{\partial r} \right|_{r=a+\epsilon} - \left. \frac{\partial V}{\partial r} \right|_{r=a-\epsilon} = -\frac{\sigma}{\epsilon_0} \quad (\text{B.C.})$$

So, you can use separation of variables, and separately to get

$r > a$ case (where all A_ℓ 's vanish)

$r < a$ " " " B_ℓ 's vanish),

then use the (BC!) above to get a relation between A 's + B 's.

By the way, in previous Example, $V=0$ for $r \leq a$

$$\text{so } \left. \frac{\partial V}{\partial r} \right|_{\text{inside}} = 0$$

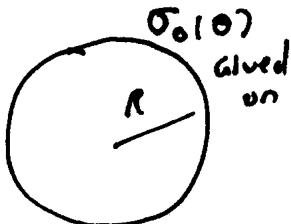
$$\left. \frac{\partial V}{\partial r} \right|_{\text{just outside}} = \frac{\partial}{\partial r} \left(-E_0 r \cos\theta + \frac{a^3}{r^2} E_0 \cos\theta \right) = -E_0 \cos\theta - \frac{2a^3}{r^3} E_0 \cos\theta$$

$$\text{so } \sigma = \epsilon_0 E_0 \cos\theta (1+2) = 3 \epsilon_0 E_0 \cos\theta$$

so our polarized sphere has a simple $\sigma(\theta)$ on the surface,
+ at north pole, - at south pole, like you might expect.

(+ By subtracting the uniform field everywhere, you now know the
 V (and field) just from this σ alone)

One last example:



$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V_{in}(r, \theta) = \sum_l A_l r^l P_l(\cos\theta)$$

continuity says ~~$A_l R^l = \frac{B_l}{R^{l+1}}$~~ (each term matches!)

$$\left. \frac{\partial V}{\partial r} \right|_{out, R} - \left. \frac{\partial V}{\partial r} \right|_{in, R} = -\frac{\sigma_0}{\epsilon_0} \Rightarrow \sum_l \left. -(l+1) \frac{B_l}{r^{l+2}} P_l \right|_{r=R} - \sum_l l A_l r^{l-1} \Big|_{r=R} = -\frac{\sigma_0}{\epsilon_0}$$

If know σ_0 , then 2nd eq'n tells you about

$$-\frac{(l+1) B_l}{R^{l+2}} - l A_l R^{l-1} = \dots \quad \begin{matrix} \text{(whichever the } l+1 \\ \text{coeff of } -\frac{\sigma_0}{\epsilon_0} \text{ is) } \end{matrix}$$

2 eq'n's in 2 unknowns A_l, B_l !

("Done")