

3310 (1-2) - 27'

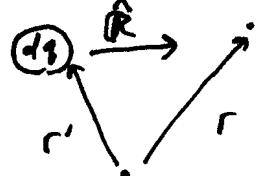
Maxwell's Eq'n #2 (Faraday's Law in Electrostatics)

$$\vec{\nabla} \times \vec{E} = ?$$

We know $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ now.

What about $\vec{\nabla} \times \vec{E}$? This adds more info about \vec{E} !

I know $\frac{d\vec{E}}{dq}$ from a pt charge = $\frac{dq}{4\pi\epsilon_0} \frac{\hat{R}}{R^2}$ with $\vec{R} = \vec{r} - \vec{r}'$



or, as we've been writing it:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{vol}} p(r') dV' \frac{\hat{R}}{R^2}$$

I could just take $\vec{\nabla} \times \vec{E}(\vec{r})$ and "grind":

$$\text{It's a calc III problem, } \vec{\nabla} \times \frac{(\vec{r} - \vec{r}')}{(\vec{r} - \vec{r}')^2}.$$

~~It's a Griffiths ch. problem 200-13~~

(Though we'll see
there are other,
more useful
ways to
see the result!)
(See next page!)

And the result is... zero! (No matter what \vec{r}' is, this
function is not "curly")

$$\boxed{\vec{\nabla} \times \vec{E} = 0}$$

- \vec{E} fields (in electrostatics!) are not curly. Certainly true for a pt charg

but, $\vec{\nabla} \times (\vec{E}_1 + \vec{E}_2 + \dots) = \vec{\nabla} \times \vec{E}_1 + \vec{\nabla} \times \vec{E}_2 + \dots = 0 + 0 + 0 + \dots = 0$ //

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MATH INTERLUDE

Th #1: For any f , $\boxed{\vec{\nabla} \times (\vec{\nabla} f) = 0}$ reminds me of $\vec{r} \times \vec{r} = 0$!

$\vec{\nabla} f$ points "up the hill", this $\vec{\nabla} f$ is a radial-like field, no curl!

[the proof: just do it! E.g. $(\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$]
 here, $(\vec{\nabla} \times (\vec{\nabla} f))_z = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 0$. etc.

Th #2: $\vec{\nabla} \frac{1}{R} = -\frac{1}{R^2} \hat{R}$ reminds me of $\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$.

[the proof: just do it! $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$, write out $\vec{R} = \vec{r} - \vec{r}'$]
 this is Griffiths HW prob. 1.13 b ...

Now look: If $\vec{E} = \frac{1}{4\pi\epsilon_0} \int \rho(r') d\tau' \frac{\hat{R}}{R^2}$,

then Th #2 above says $\vec{E} = \frac{1}{4\pi\epsilon_0} \int \rho(r') d\tau' - \vec{\nabla} \left(\frac{1}{R} \right)$

pull $\vec{\nabla}$ out of $\int d\tau'$ ($\vec{\nabla}$ has nothing to do with $d\tau'$!)

$$\vec{E} = -\vec{\nabla} \left(\frac{1}{4\pi\epsilon_0} \int \frac{\rho(r') d\tau'}{R} \right) \equiv -\vec{\nabla} V(r) \text{ with } V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r') d\tau'}{R}$$

and since $\vec{E} = -\vec{\nabla} V$

then $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} V) = 0$ from Th. #1

$\boxed{\vec{\nabla} \times \vec{E} = 0}$, always, in electrostatics

\vec{E} is "radial-like", has no curl to it.

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Another Math include: Stokes' Theorem (Griff 1.3.5)

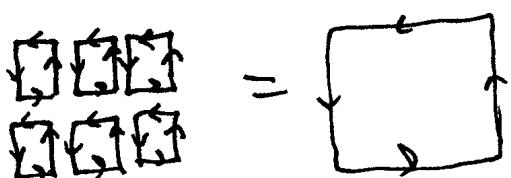
$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_{\text{Line around } S} \vec{F} \cdot d\vec{L} \quad (\text{for any } \vec{F})$$

Any open

again, like $\int \frac{df}{dx} dx = f(b) - f(a)$, ("integral of deriv = fn at boundary")

In words, $\vec{\nabla} \times \vec{F}$ is circulation or swirl at a point.

If add up all the swirls \Rightarrow "total swirl", this is just what the circulation $\oint v \cdot dL$ around outside gives



If fish swims around edge of pond and has a current with \rightarrow the whole way, there must be "whirlpools" somewhere in the middle! (Maybe it's in a toilet?)

So if $\vec{\nabla} \times \vec{E} = 0$, then $\oint \vec{E} \cdot d\vec{L} = 0$ by Stokes th!

Any loop

\Rightarrow Maxwell's Eq'n in electrostatics (Faraday's Law)

$$\oint \vec{E} \cdot d\vec{L} = 0$$

or

$$\vec{\nabla} \times \vec{E} = 0$$

Both say
Same thing, by
STOKE'S.

Griff worked from here, showing $\int \left(\frac{1}{4\pi\epsilon_0 r^2} \hat{r} \right) \cdot d\vec{r} = 0$ for any path.

One more math interlude: fund. theorem of calc:

$$\boxed{\int_A^B (\vec{\nabla} F) \cdot d\vec{L} = F(B) - F(A)}$$

$$(1; \text{like: } \int_A^B \frac{df}{dx} dx = f(b) - f(a))$$

But, since $\vec{E} = -\vec{\nabla} V$ we have

$$-\int_A^B \vec{E} \cdot d\vec{L} = + \int_A^B \vec{\nabla} V \cdot d\vec{L} = + (V(B) - V(A))$$

- V is a scalar fn, the "potential".

It's a number at every point in space.

(Not pos. Energy, we'll
get to that!)

- For a point charge,

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R}}$$

(Because $p \rightarrow q \delta^{(3)}$)

Except, (ambiguity!) you could always add constant ($\propto 1/r$),

Because $\vec{E} = -\vec{\nabla} V$ would not be changed.

- Usually choose $V(\vec{r} \rightarrow \infty) = 0$ to set the value of this constant

so $\vec{\nabla} V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

at origin

By superposition, $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r') dr'}{|r-r'|}$, (as we had before)
with $\rho(r')$

- $\oint \vec{E} \cdot d\vec{L} = \iint (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = 0$. $\left(\vec{E} \text{ is "conservative".} \right)$

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Summary so far: If define $V(A) = 0$

$$V(r) = - \int_A^r \vec{E} \cdot d\vec{L} \quad \leftarrow \text{Find } V \text{ from } \vec{E}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(r') dr'}{|r-r'|} \quad \leftarrow \text{or, Find } V \text{ from } \rho.$$

$$\vec{E} = -\vec{\nabla}V \quad \leftarrow \text{Find } \vec{E} \text{ from } V$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(r') dr'}{(r-r')^3} (\vec{r}-\vec{r}') \quad \leftarrow \text{or, Find } \vec{E} \text{ from } \rho$$

- The integral for V is easier, so we may want (prefer!) to find V ,
then $\vec{E} = -\vec{\nabla}V$ is a "1-stepper".

Finally, $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla}V) = -\nabla^2 V$

But $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

so $\boxed{\nabla^2 V = -\rho/\epsilon_0}$ \leftarrow Poisson's eq'n.
It's really Gauss + $\vec{\nabla} \times \vec{E} = 0$
combined!!

This is all we really need. Given ρ , solve for V , then get \vec{E} , yay!

If empty space

$\boxed{\nabla^2 V = 0}$ \leftarrow Laplace's Eq'n.
we'll spend a lot of time learning tricks
to solve this, often with no integrals needed.

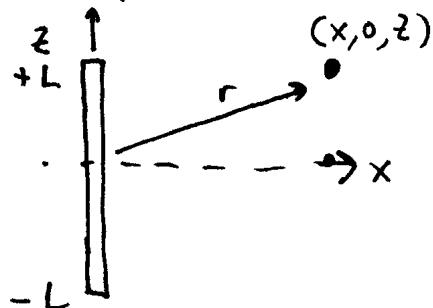
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Examples: let's find $V(r)$

Case 1: Point charges: $V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

superposition, if several.

Case 2:



Uniform Line of charge, density λ ,

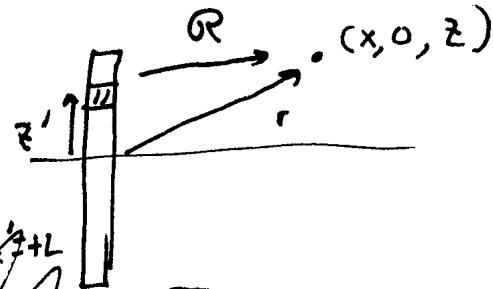
finite length, $-L \leq z \leq +L$.

Find $\vec{E}(r)$?

Hard! Gauss' law: no good, no symmetry. What surface would you draw? (\vec{E} not constant on any obvious surface)

Could integrate, but let's try using/find $V(r)$ instead.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{\lambda dz'}{\sqrt{x^2 + (z-z')^2}}$$



~~$$\frac{1}{4\pi\epsilon_0} \log \left(\frac{z+z' + \sqrt{x^2 + (z+z')^2}}{z-z' + \sqrt{x^2 + (z-z')^2}} \right)$$~~

$$= \frac{\lambda}{4\pi\epsilon_0} \log \frac{L+z + \sqrt{x^2 + (L+z)^2}}{+L-z + \sqrt{x^2 + (+L-z)^2}}$$

(could always add +C, but no need, since $V(x \rightarrow \infty) \rightarrow 0$)

(Take $\frac{\partial}{\partial x} \rightarrow 0$ Find E_x , etc.)

Last Example Sometimes you know \vec{E} already.

Then V is even easier! $V(r) - V(a) = - \int_a^r \vec{E} \cdot d\vec{L}$.

Consider our "Shell" of σ : $E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$ outside

$E = 0$ inside.

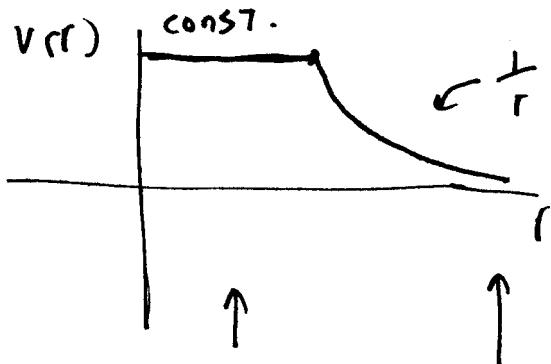
Define $V(\infty) = 0$

$$V(r > R) = - \int_{-\infty}^r \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} \cdot dr' = - \frac{q}{4\pi\epsilon_0} \left(-\frac{1}{r'} \right)_{\infty}^r = + \frac{q}{4\pi\epsilon_0 r}$$

~~cancel~~

$$V(r < R) - V(R) = - \int_R^r 0 \cdot dr' = 0.$$

$$\text{so } V(r < R) = V(R) = q/4\pi\epsilon_0 R$$



$$-\nabla V = -\nabla \frac{1}{r} = +\frac{1}{r^2} \text{ as it should be.}$$

as it should

be

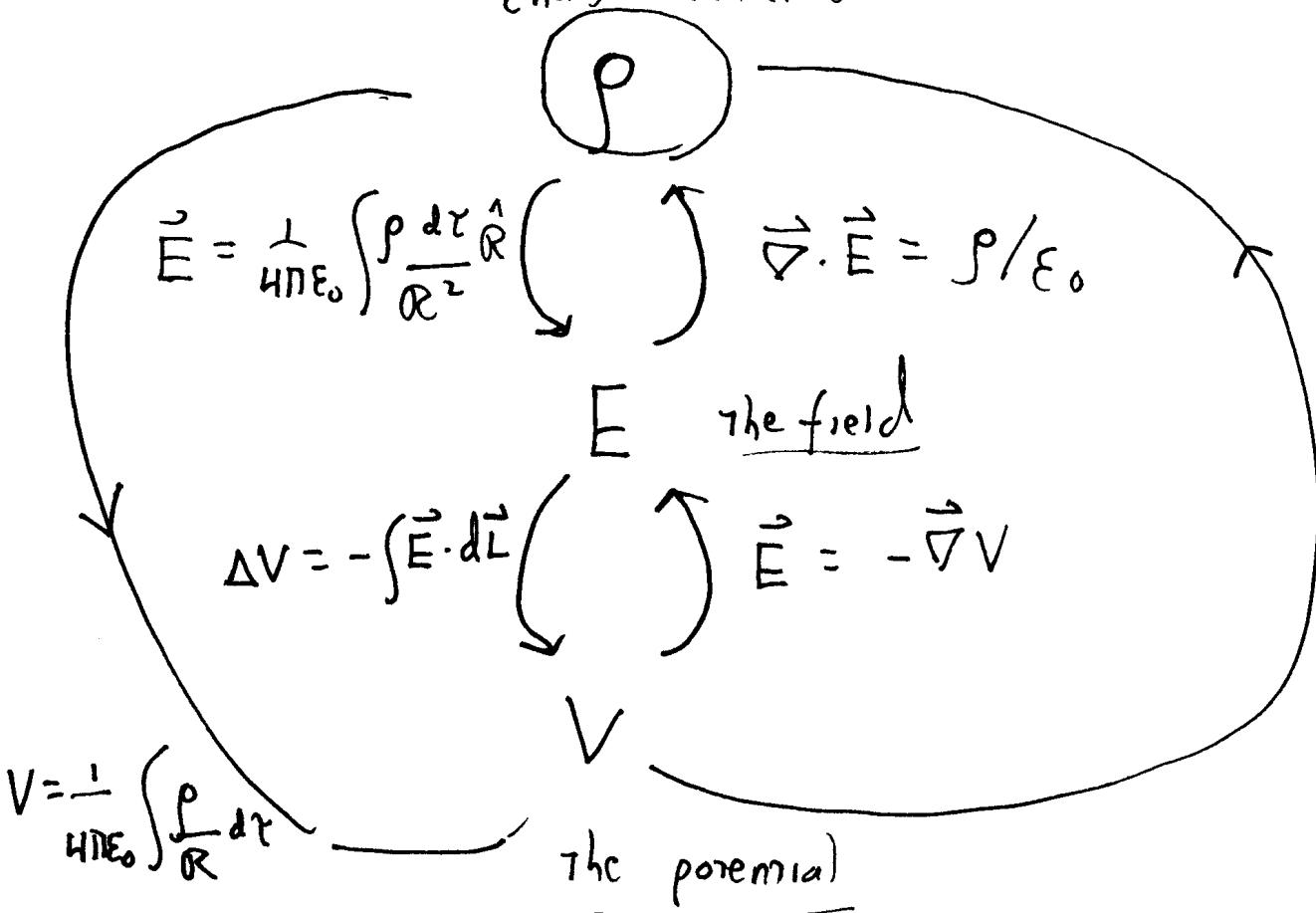
Boundary Conditions

Electrostatics is about figuring out V (or E) from charges, or "setups" ← e.g. location + shapes of conductors + insulators.

The field is determined by what is at the boundaries and we often talk about "Boundary conditions" which

establish \vec{E} 

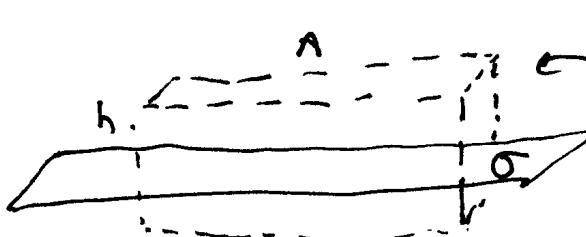
charges + materials



At physical boundaries: (e.g. sheet w. charges)

- \vec{E} can "jump" (because charges create field)
- but V is always continuous! ← Nice!!

How does \vec{E} jump? By Gauss' Law.



Small "pillbox" A

$$\oint \vec{E} \cdot d\vec{\lambda} = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma \cdot A}{\epsilon_0}$$

Let $A \rightarrow 0$

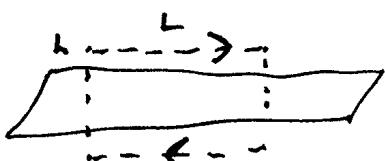
$$h \rightarrow 0$$

$$\vec{E} \cdot \hat{n}_{\text{above } A} = \frac{\sigma A}{\epsilon_0}$$

$$-\vec{E} \cdot \hat{n}_{\text{below } A}$$

so change in $\vec{E} \cdot \hat{n}$ is σ/ϵ_0

How about \vec{E} parallel to sheet? Use $\vec{\nabla} \times \vec{E} = 0$



On Loop, $E_{||}^{\text{above}} \cdot L - E_{||}^{\text{below}} \cdot L = 0$

If $h \rightarrow 0$

so $E_{||}$ to sheet is continuous

Since $\vec{E} = -\vec{\nabla} V$ can rewrite these expressions in terms of V . We'll return to this!