

INSTRUCTORS MANUAL: TUTORIAL 5

Note:

There are two versions of this Tutorial. This version should be used when the pace of the course only allows for a single tutorial session on Separation of Variables. If it is possible to dedicate more time to this topic, use the extended version of this Tutorial, 5a and 5b, over two consecutive periods.

Goals:

1. Understand how Laplace's equation produces a separation of variables problem
2. Describe a physical situation mathematically, and see the connection between physics and math (course scale learning goal 1)
3. Learn how to solve a separation of variables problem (which arises in many other fields of physics in addition to E&M) (course scale learning goals 5, 6)
4. Check the solution (course scale learning goal 7)
5. Learn how to apply Fourier's trick (course scale learning goal 5)

This tutorial is based on:

- “Griffiths by Inquiry” Lab 3 – Bruce Patton and Catherine Crouch
- Written by Darren Tarshis, Steven Pollock, and Stephanie Chasteen, with modifications by Ed Kinney, Michael Dubson, Markus Atkinson and Bethany Wilcox.

Tutorial Summary:

Laplace's equation is applied to a general separable potential function in order to derive ordinary differential equations (ODEs). The ODEs are solved generally at first, then boundary conditions are applied to solve for the voltage specifically. Students check that their answer agrees with the boundary conditions. Finally, Fourier's trick is used.

Reflections on this tutorial

Students' diverse mathematical backgrounds result in different degrees of comfort with this tutorial. Those with good math backgrounds complete the tutorial with minimal difficulty, though the problems that they do have are fundamental. Other students can spend 2 hours just finishing separation of variables and never reaching Fourier's Trick, despite having seen separation of variables in 2 or 3 lectures. For these students in particular, the tutorial is extremely valuable.

In part I. In **question (i)** a few students were unsure if a "separable" equation was of the form: $X(x)+Y(y)+Z(z)$ or $X(x)Y(y)Z(z)$. When writing the ODE in **question (ii)**, many students can write down $X''(x) = -cX(x)$ from memory, but only some can explain where this comes from. Some students have been seen to struggle with plugging their separable equation (from question (i)) into Laplace's Equation. When solving the ODE, some students are unclear why $B \bullet \cos(kx)$ is required since $A \bullet \sin(kx)$ works fine (and similarly for Be^{-ky}). Some do not use different coefficients for these solutions (i.e., A and B), and don't understand the role of this constant.

When applying the boundary conditions in **question (iv)** many students get lost and thus we prompted them to write down $V(x,y,z)$ after each step to help them organize their problem-solving strategy, as well as remind them what they are doing (finding the potential!). Almost every student missed the following step: Boundary condition #2 tells us that c must be $n\pi/b$. At this point, V becomes an infinite sum over n , and the constants are functions of n . Not one student acknowledged this. Also, one of the students in the slower group thought that $n\pi/b$ was only substituted for the constant in the $Y(y)$ equation, and didn't realize that the constant in the $X(x)$ equation must be the same.

Some other common difficulties in this step include:

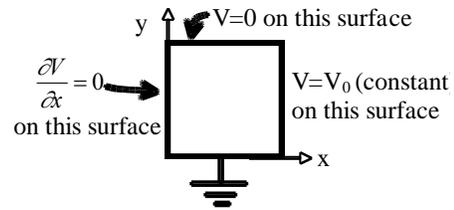
- Skipping boundary conditions 1, 2, and 3 because they're equal to zero, which "doesn't give us any information."
- Only solving for the constants in front of the functions (and not the constant in the argument of the sin, cos, or exponential)
- Discomfort or unfamiliarity with hyperbolic sines and cosines
- Forgetting what $X(x)$ and $Y(y)$ represent
- Failing to recognize that $Z(z)$ is constant and the second derivative of Z disappears.

Part II. Not all students will make it this far. Of all the students that did make it this far, most of them did not remember the specifics of Fourier's trick without prompting. They knew that they needed to integrate something, but they were not sure what the limits of integration needed to be. None of the students recognized that all of the odd eigenfunctions would have to contribute zero in order to build a constant even function.

Relevant Homework Problems

Separation of variables- Cartesian 2D

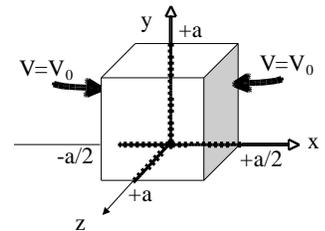
A square rectangular pipe (sides of length a) runs parallel to the z -axis (from $-\infty$ to $+\infty$)
 The 4 sides are maintained with boundary conditions given in the figure. (Each of the 4 sides is insulated from the others at the corners)



- i) Find the potential $V(x,y,z)$ at all points in this pipe.
- ii) Sketch the E-field lines and equipotential contours inside the pipe. (Also, state in words what the boundary condition on the left wall means - what does it tell you? Is the left wall a conductor?)
- iii) Find the charge density $\sigma(x,y=0,z)$ everywhere on the bottom conducting wall ($y=0$).

Separation of variables- Cartesian 3D

You have a cubical box (sides all of length a) made of 6 metal plates which are insulated from each other. The left wall is located at $x=-a/2$, the right wall is at $x=+a/2$. Both left and right walls are held at constant potential $V=V_0$. All four other walls are grounded.



(Note that I've set up the geometry so the cube runs from $y=0$ to $y=a$, and from $z=0$ to $z=a$, but from $x=-a/2$ to $x=+a/2$. This should actually make the math work out a little easier!)

Find the potential $V(x,y,z)$ everywhere inside the box.

(Also, is $V=0$ at the center of this cube? Is $E=0$ there? Why, or why not?)

Separation of Variables: Spherical

The potential on the surface of a sphere (radius R) is given by $V=V_0 \cos(2\theta)$.

(Assume $V(r=\infty)=0$, as usual. Also, assume there is no charge inside or outside, it's ALL on the surface!)

- i) Find the potential inside and outside this sphere.
 (Hint: Can you express $\cos(2\theta)$ as a simple linear combination of some Legendre polynomials?)
- ii) Find the charge density $\sigma(\theta)$ on the sphere.

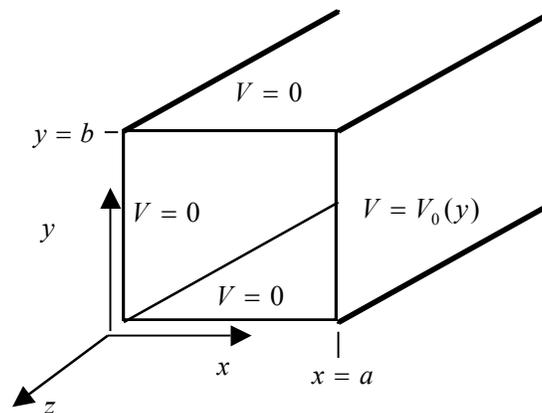
★TUTORIAL 5: BREAKING DOWN THE STEPS ★
SEPARATION OF VARIABLES

Part 1: Laplace's Equation and Separation of Variables

Within a very long, rectangular, hollow pipe, there are no electric charges. The walls of this pipe are kept at a known voltage (they are known because in a lab, you can control them).

Three of the walls are grounded: $V(x=0,y,z)=0$; $V(x,y=0,z)=0$; $V(x,y=b,z)=0$

The fourth wall maintains a potential that varies with y : $V(x=a,y,z)=V_0(y)$ which will be specified later.



In order to find out the voltage inside the pipe, you will need to solve Laplace's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- i. What does it mean to “separate variables” of $V(x,y,z)$. What advantage is there to using that approach here?

ii. Plug the separated form of V into Laplace's equation. After doing this, you should have several terms.

- Simplify as much as possible.
- Are any of the terms zero in this case?
- What must be true about the remaining terms in order to satisfy Laplace's equation?
- Write down the ordinary differential equations you need to solve to find V .

iii. The boundary conditions on the pipe are listed below. Which boundary condition(s) allow you to determine the direction (x or y) that must have sinusoidal behavior?

1. $V(x, y = 0, z) = 0$

3. $V(x = 0, y, z) = 0$

2. $V(x, y = b, z) = 0$

4. $V(x = a, y, z) = V_o$

Write down and modify your general expression for the voltage everywhere inside the pipe so that it satisfies the first three boundary conditions. **Do not apply the 4th boundary condition yet.**

3. Perform the integration. What happens to the infinite sum after you perform the integration?

4. Now find all the C_n 's and write down $V(x,y,z)$.