PH300 Modern Physics SP11

"I will never believe that god plays dice with the universe."
- Albert Einstein

3/8 Day 15:
Questions?
Repeated spin measurements
Probability

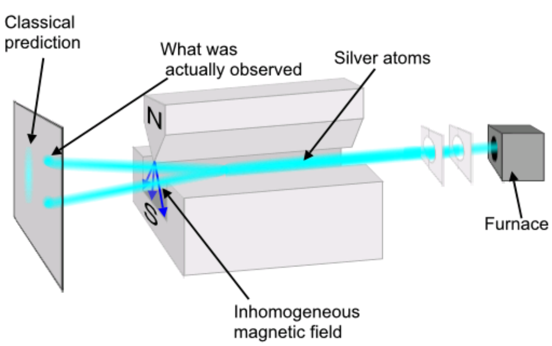
Thursday:
Entanglement
Quantum Physics & Reality
Quantum Cryptography

Last Week:
- Balmer formula, atomic spectra
- Bohr model, de Broglie waves
- Stern-Gerlach experiments

Today:
- Repeated spin measurements
- Probability

Thursday:
- Reading on Blackboard before class
- Entanglement/EPR "paradox"
- Quantum physics and reality

Stern-Gerlach Experiment with Silver Atoms (1922)*



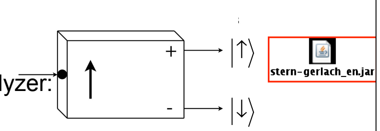
*Nobel Prize for Stern in 1943

In the future…

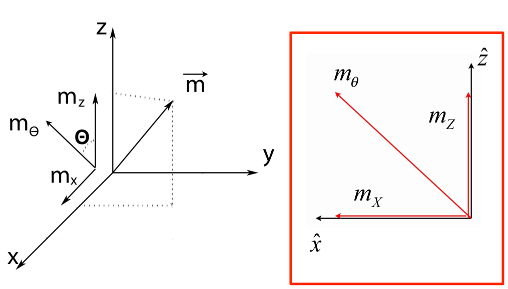
Some atoms have more than two possible projections of the magnetic moment vector along a given axis, but we will only deal with "two-state" systems when talking about atomic spin.

When we say a particle is measured as "***spin up***" (or ***down***) along a given axis, we are saying that we measured the projection of the magnetic moment along that axis (+m_B or -m_B), and observed it exiting the ***plus-channel*** (***minus-channel***) of a Stern-Gerlach type apparatus.

A simplified Stern-Gerlach analyzer:

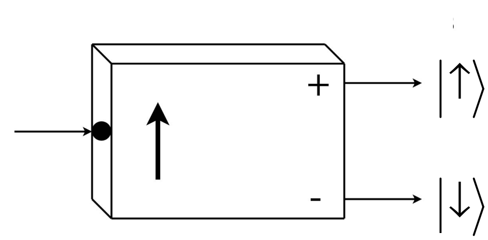


Notation for projection values:



Head-on view

A simplified Stern-Gerlach analyzer:

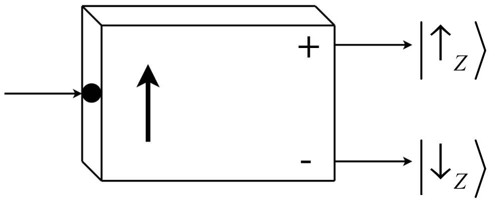


However the analyzer is oriented, we always measure one of two values: ***spin-up*** (+m_B) or ***spin-down*** (-m_B).

$m_B = 9.27\times10^{-24}$ joule/tesla

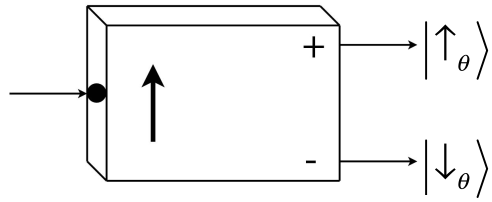
A simplified Stern-Gerlach analyzer:

If the analyzer is oriented along the +z-direction:

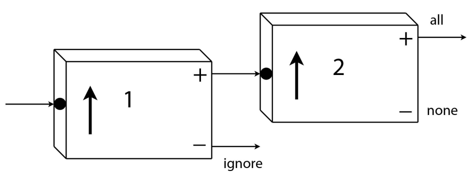


A simplified Stern-Gerlach analyzer:

If the analyzer is oriented along the θ-direction:



Repeated spin measurements:



Ignore the atoms exiting from the ***minus-channel*** of **Analyzer 1**, and feed the atoms exiting from the ***plus-channel*** into **Analyzer 2**.

All of the atoms also leave from the ***plus-channel*** of **Analyzer 2**.

State Preparation

The atoms leaving the plus-channel of vertical **Analyzer 1** were all in the state $|\uparrow_Z\rangle$

We confirmed this when all these atoms also leave the plus-channel of vertical **Analyzer 2**. We can repeat this as often as we like…

All of these atoms were in the ***definite state*** $|\uparrow_Z\rangle$

Bracket Notation

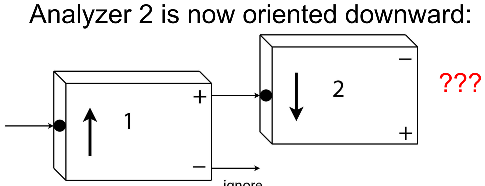
We'll place ***any information*** we have about a ***quantum state*** inside a bracket: $|\Psi\rangle$ Ψ = "Psi" is a generic symbol for a quantum state

For the atoms leaving the plus-channel of **Analyzer 1** we write:

$|\Psi\rangle = |\uparrow_Z\rangle$ **OR** $|m_Z = +m_B\rangle$

$m_B = 9.27\times10^{-24}$ joule/tesla

Analyzer 2 is now oriented downward:

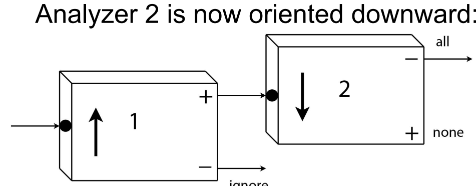


Ignore the atoms exiting from the ***minus-channel*** of **Analyzer 1**, and feed the atoms exiting from the ***plus-channel*** into **Analyzer 2**.

What happens when these atoms enter **Analyzer 2**?

A) They all exit from the plus-channel.
B) They all exit from the minus-channel.

Analyzer 2 is now oriented downward:

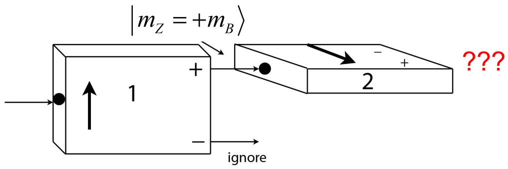


All the atoms leave from the ***minus-channel*** of **Analyzer 2**.

Remember, with **Analyzer 2** we are now measuring $m_{(-Z)}$

An atom in the state $|\uparrow_Z\rangle$ is also in the state $|\downarrow_{(-Z)}\rangle$

Analyzer 2 is now oriented horizontally (+x):

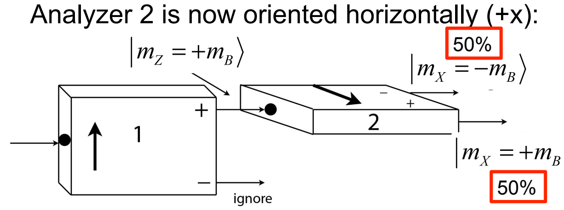


Ignore the atoms exiting from the ***minus-channel*** of **Analyzer 1**, and feed the atoms exiting from the ***plus-channel*** into **Analyzer 2**.

What happens when these atoms enter **Analyzer 2**?

A) They all exit from the plus-channel.
B) They all exit from the minus-channel.
C) Half leave from the plus-channel, half from the minus-channel.
D) Nothing, the atoms' magnetic moments have zero projection along the +x-direction

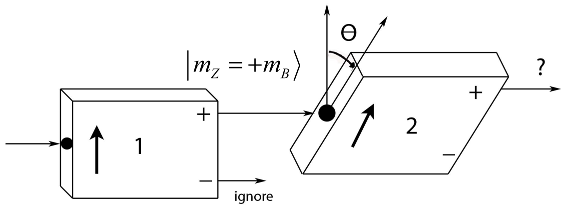
Analyzer 2 is now oriented horizontally (+x):



Half the atoms leave from the ***plus-channel*** of **Analyzer 2**, and ***half*** leave from the ***minus-channel***.

Remember, we ***always get one of two values*** ("plus" or "minus"), regardless of how the analyzer is oriented, and ***regardless of what state*** the atom is in when it enters **Analyzer 2**.

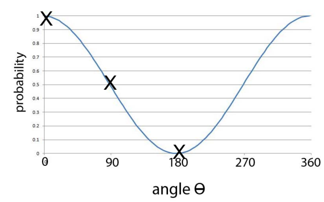
Analyzer 2 is oriented at an angle Θ:



Ignore the atoms exiting from the ***minus-channel*** of **Analyzer 1**, and feed the atoms exiting from the ***plus-channel*** into **Analyzer 2**.

What is the ***probability*** for an atom to exit from the ***plus-channel*** of **Analyzer 2**?

For atoms entering in the state $|\uparrow_z\rangle$



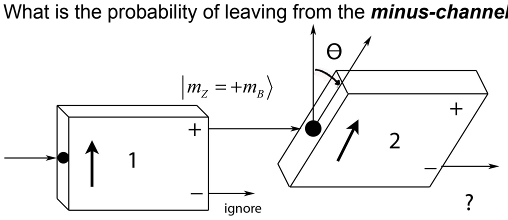
$$P\left[|\uparrow_\theta\rangle\right] = \cos^2\left(\frac{\theta}{2}\right)$$

For Θ = 0, 100% of the atoms exit from the plus-channel.

For Θ = 90^0, 50% of the atoms exit from the plus-channel.

For Θ = 180^0, 0% of the atoms exit from the plus-channel.

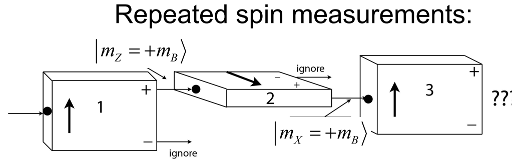
What is the probability of leaving from the ***minus-channel***?



Hint: $P\left[|\uparrow_\theta\rangle\right] + P\left[|\downarrow_\theta\rangle\right] = 1$

$$P\left[|\downarrow_\theta\rangle\right] = 1 - P\left[|\uparrow_\theta\rangle\right] = 1 - \cos^2\left(\frac{\theta}{2}\right) = \sin^2\left(\frac{\theta}{2}\right)$$

Repeated spin measurements:

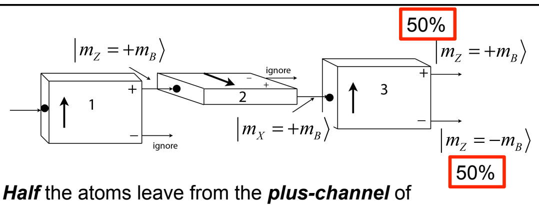


Ignore the atoms exiting from the ***minus-channel*** of **Analyzer 1**, and feed the atoms exiting from the ***plus-channel*** into **Analyzer 2**.

Ignore the atoms exiting from the ***minus-channel*** of **Analyzer 2**, and feed the atoms exiting from the ***plus-channel*** into **Analyzer 3**.

What happens when these atoms enter **Analyzer 3**?

A) They all exit from the plus-channel.
B) They all exit from the minus-channel.
C) Half leave from the plus-channel, half from the minus-channel.
D) Nothing, the atoms' magnetic moments have zero projection along the +z-direction



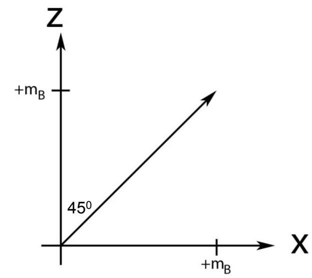
Half the atoms leave from the ***plus-channel*** of **Analyzer 3**, and ***half*** leave from the ***minus-channel***.

Atoms leaving the ***plus-channel*** of **Analyzer 1** are in the state $|\uparrow_Z\rangle$

Atoms leaving the ***plus-channel*** of **Analyzer 2** are in the state $|\uparrow_X\rangle$

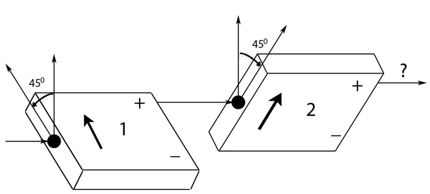
The atoms that were in a definite state $|\uparrow_Z\rangle$ are ***no longer*** in a definite state of m_Z ***after measuring*** m_X.

Measurement along the x-direction ***erases*** the information about the projection of the atom's magnetic moment along the z-direction.



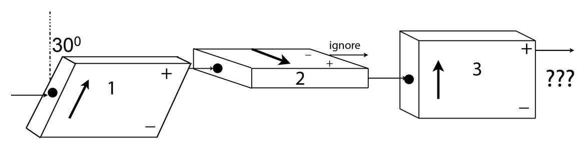
An atom can't have BOTH $|m_X = +m_B\rangle$ AND $|m_Z = +m_B\rangle$ since a measurement along m_{45^0} would yield $m_{45^0} = +\sqrt{2}m_B$.

Measurement along any axis forces the ***projection*** of the atom's magnetic arrow to assume one of two values ($+m_B$ or $-m_B$) along that axis.



Analyzer 1 is tilted 45^0 to the ***left*** of vertical, while **Analyzer 2** is tilted 45^0 to the ***right*** of vertical. Atoms leaving the ***plus-channel*** of **Analyzer 1** are fed into the input of **Analyzer 2**. What is the ***probability*** that an atom entering **Analyzer 2** will leave through the ***plus-channel***?

A) 0% B) 50% C) 100% D) Something else.

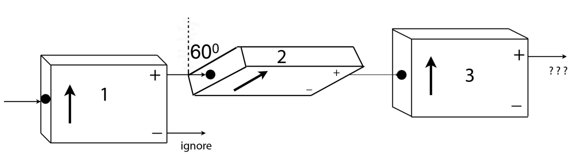


Instead of vertical, suppose **Analyzer 1** makes an angle of 30^0 from the vertical. **Analyzers 2** & **3** are left unchanged.

What is the ***probability*** for an atom leaving the ***plus-channel*** of **Analyzer 2** to exit from the ***plus-channel*** of **Analyzer 3**?

A) 0% B) 25% C) 50% D) 75% E) 100%

Hint: Remember that $P\left[|\uparrow_\theta\rangle\right] = \cos^2\left(\frac{\theta}{2}\right)$



Instead of horizontal, suppose **Analyzer 2** makes an angle of 60^0 from the vertical. **Analyzers 1** & **3** are unchanged.

What is the ***probability*** for an atom leaving the ***plus-channel*** of **Analyzer 2** to exit from the ***plus-channel*** of **Analyzer 3**?

A) 0% B) 25% C) 50% D) 75% E) 100%

Hint: Remember that $P\left[|\uparrow_\theta\rangle\right] = \cos^2\left(\frac{\theta}{2}\right)$

Probability

In general, the probability for a particular outcome is the ***ratio*** of the number of desired outcomes to the total number of possible outcomes.

If we roll a six-sided die, what is the probability that we obtain a 1?

There are six possible outcomes, and each one is equally likely, so the probability is 1/6.

Probability

When tossing a die, what is the probability of rolling either a 1 ***or*** a 3?

There are six possible outcomes and two desired outcomes, meaning the probability of success is:

1/6 + 1/6 = 2/6 = 1/3.

So there is a one in three chance of rolling a 1 or a 3.

In general*, the word "**OR**" is a signal to add probabilities:

$$P_{12} = P_1 + P_2$$

*Only if P1 & P2 are independent

Probability

Simultaneously toss a die and flip a coin. What is the probability of getting 2 ***and*** Tails?

There is a 1/6 chance of getting a two and a 1/2 probability of getting Tails:

1/6 x 1/2 = 1/12

We could also find this by counting up the number of possible outcomes: 12.

In general*, the word "**AND**" is a signal to multiply:

$$P_{12} = P_1 \cdot P_2$$

*Only if P1 & P2 are independent

Probability

Flip a coin three times. What is the probability of obtaining Heads twice?

A) 1/4
$\boxed{\text{B) 3/8}}$
C) 1/2
D) 5/8
E) 3/4

Probability

Flip a coin three times. What is the probability of obtaining Heads twice?

OUTCOME	PROBABILITY	# OF HEADS
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

- The probability of obtaining three Heads is 1/8, of two Heads 3/8, of one Head 3/8, and no Heads 1/8.
- Note that these probabilities must add up to ***one*** – we must get one of the four possibilities with 100% certainty.

Probability

Toss two dice simultaneously.

What is the probability that the sum of the results is four?

The probability for any single outcome is 1/36 = 1/6 x 1/6.

The probability that the sum result is four is:

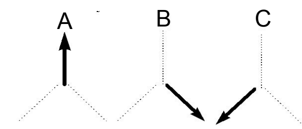
P[2,2] + P[1,3] + P[3,1].

The probability for each of these outcomes is 1/36, so the probability that the outcome adds to four is:

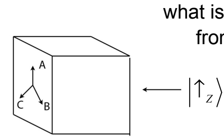
1/36 + 1/36 + 1/36 = 3/36 = 1/12

Probability

Imagine a Stern-Gerlach analyzer with three settings, each oriented 120^0 from the other:



If we send in an atom in the definite state $|\uparrow_z\rangle$, what is the ***probability*** the atom will leave from the plus-channel if the setting on the analyzer is ***random***?



Probability

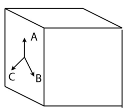
If the setting is in the +z-direction (**A**), then the probability to leave through the ***plus-channel*** is 1.

For either of the two other settings (**B** & **C**), the probability of leaving through the ***plus-channel*** is 1/4. [$\cos^2(120^0/2)$]

If the settings are ***random***, then there is 1/3 probability that the analyzer is set to a specific orientation.

The probability of leaving through the plus-channel is:

[1/3 x 1] + [1/3 x 1/4] + [1/3 x 1/4]
= 4/12 + 1/12 + 1/12 = 6/12 = 1/2



$\longleftarrow |\uparrow_z\rangle$

Probability Distributions

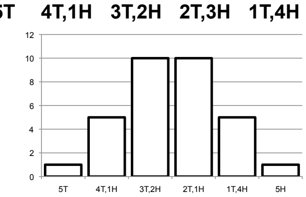
There are two types of probability distributions:

- Discrete
- Continuous

Discrete Probability Distributions

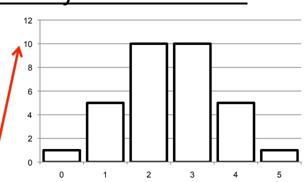
Flip a coin five times. There are 32 possible outcomes:

5T	4T,1H	3T,2H	2T,3H	1T,4H	5H
TTTTT	HTTTT	HTTTH	THHHT	THHHH	HHHHH
	THTTT	HTTHT	THHTH	HTHHH	
	TTHTT	THTTH	HTHHT	HHTHH	
	TTTHT	TTHTH	THTHH	HHHTH	
	TTTTH	THTHT	HTHTH	HHHHT	
		HTHTT	HHTHT		
		TTTHH	TTHHH		
		TTHHT	HTTHH		
		THHTT	HHTTH		
		HHTTT	HHHTT		



Discrete Probability Distributions

Can also represent this in terms of the number of Heads obtained:

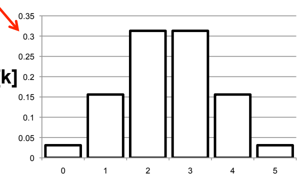


The probability P[k] for obtaining a specific number of Heads (k) is found by dividing through by the total number of possible outcomes [32]:

$$\sum_{k=0}^{5} \mathrm{P}[k] = 1^{*}$$

*P[k] is ***normalized***

P[k]



k = # of Heads

Discrete Probability Distributions

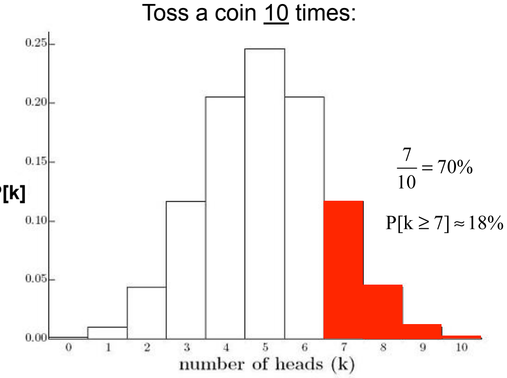
- Flip a coin ten times. (Yes, actually do this!)
- Record the number of times the outcome is heads.

K=# of heads	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
# of students	0	1	2	0	1	1	0	2	0	0

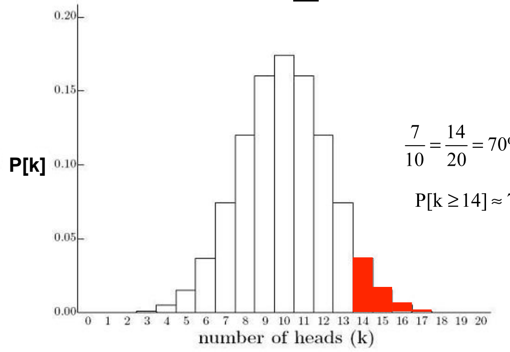
How can we decide if a coin is fair (not biased)?

Toss a coin <u>10</u> times:

P[k]



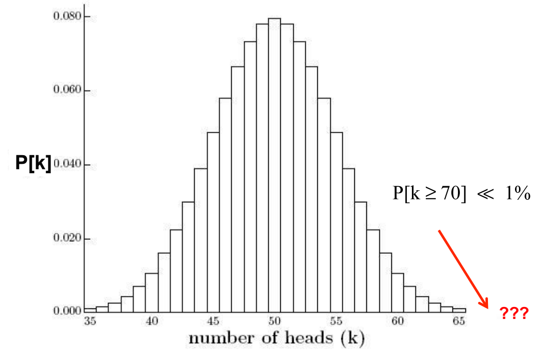
Toss a coin <u>20</u> times:



$\frac{7}{10} = \frac{14}{20} = 70\%$

P[k $\geq$ 14] $\approx$ 7%

Toss a coin <u>100</u> times:



P[k $\geq$ 70] $\ll$ 1%

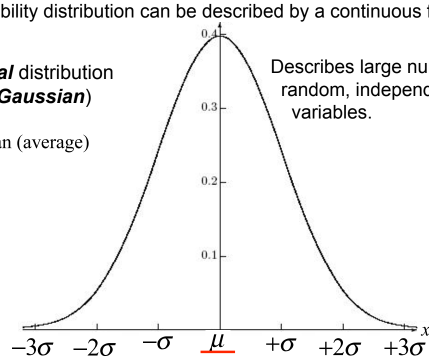
Continuous Probability Distributions

Central Limit Theorem says that, with more and more coin tosses, the probability distribution can be described by a continuous function:

Normal distribution
(or ***Gaussian***)

μ = mean (average)

Describes large numbers of random, independent variables.

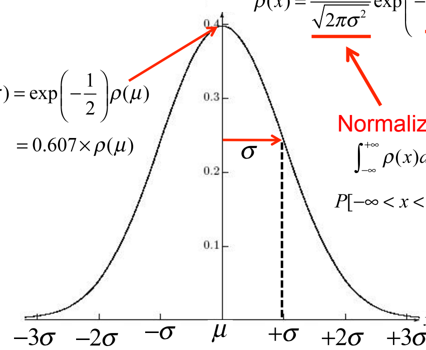


Continuous Probability Distributions

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\rho(\mu+\sigma) = \exp\left(-\frac{1}{2}\right)\rho(\mu)$$

$$= 0.607 \times \rho(\mu)$$



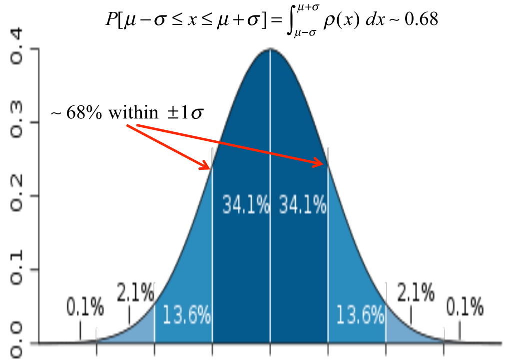
Normalization!!

$$\int_{-\infty}^{+\infty} \rho(x)dx = 1$$

$$P[-\infty < x < +\infty] = 1$$

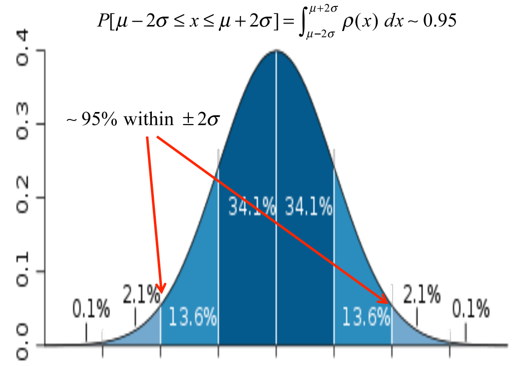
Continuous Probability Distributions

$$P[\mu-\sigma \leq x \leq \mu+\sigma] = \int_{\mu-\sigma}^{\mu+\sigma} \rho(x)\, dx \sim 0.68$$



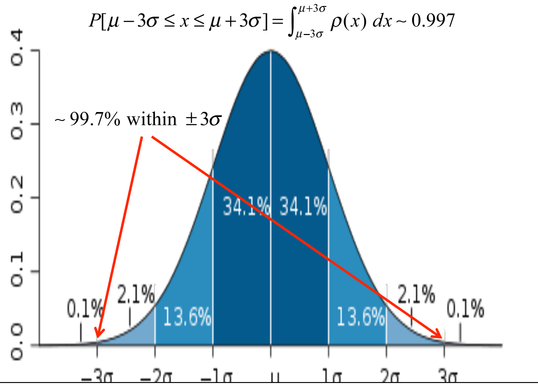
Continuous Probability Distributions

$$P[\mu-2\sigma \leq x \leq \mu+2\sigma] = \int_{\mu-2\sigma}^{\mu+2\sigma} \rho(x)\, dx \sim 0.95$$



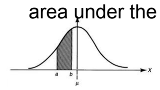
Continuous Probability Distributions

$$P[\mu - 3\sigma \le x \le \mu + 3\sigma] = \int_{\mu-3\sigma}^{\mu+3\sigma} \rho(x)\, dx \sim 0.997$$



Continuous Probability Distributions

- ρ(x) is a probability ***density*** (not a probability). We approximate the probability to obtain x_i within a range Δx with: $P[x_i - \frac{\Delta x}{2} \le x \le x_i + \frac{\Delta x}{2}] \approx \rho(x_i) \cdot \Delta x$
- The probability of obtaining a range of values is equal to the area under the probability distribution curve in that range:

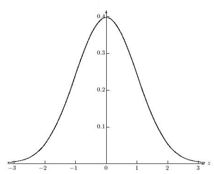


$$P[a \le x \le b] = \int_a^b \rho(x)\, dx$$

- For $x_i = x_1, x_2, x_3 \dots x_n$ (discrete values): $\langle x \rangle = \sum_{i=1}^{n} x_i P(x_i)$

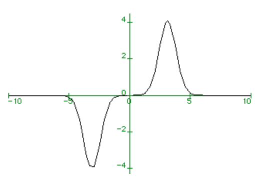
  $\langle x \rangle =$ average value of x
- For continuous x: $\langle x \rangle = \int_{-\infty}^{+\infty} x\, \rho(x)\, dx$

Continuous Probability Distributions



$\rho(x) \sim \exp(-x^2)$

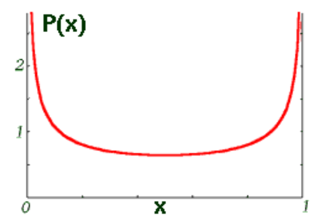
$x \cdot \rho(x) \sim x \cdot \exp(-x^2)$



$$\langle x \rangle = \int_{-\infty}^{+\infty} x\, \rho(x)\, dx = 0$$

- $\langle x \rangle = 0$ since $\rho(x)$ is symmetric about x = 0
- $\langle x \rangle = 0$ since $x \cdot \rho(x)$ is anti-symmetric about x = 0

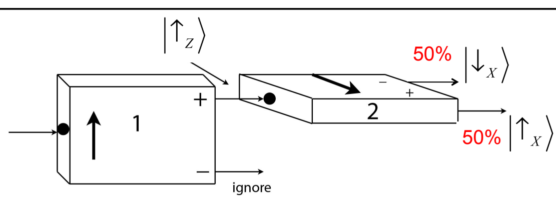
What if the probability curve is not normal?



What kind of system might this probability distribution describe?

$\langle x \rangle = ?$

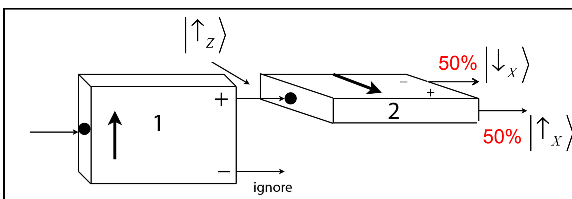
A) 0
$\boxed{\text{B) 1/2}}$
C) 1
D) Not defined, since there are two places where x is most likely.



We always get one of two possible results: $|\uparrow_X\rangle$ or $|\downarrow_X\rangle$

With **Analyzer 2** oriented at 90^0 to **Analyzer 1**, either result $|\uparrow_X\rangle$ or $|\downarrow_X\rangle$ is equally likely.

We can't predict ahead of time whether an atom will exit through the plus-channel or the minus-channel of **Analyzer 2**, only that there is a 50/50 chance for either to occur.



What would be the expectation (average) value for m_X? $\langle m_X \rangle = ?$

$$\langle m_X \rangle = P\left[|\uparrow_X\rangle\right](+m_B) + P\left[|\downarrow_X\rangle\right](-m_B)$$

$$= (0.50)(+m_B) + (0.50)(-m_B) = 0$$