PH300 Spring 2011

Homework 04

Total Points: 30

1. (1 Point) Each week you should review both your answers and the solutions for the previous week's homework to make sure that you understand all the questions and how to answer them correctly. Each week you will receive credit for reviewing your old homework, which will be returned to you every Tuesday. Please review your homework and the solutions from last week and let me know that it was graded correctly. If it was not, state here which problems were incorrectly graded, and then contact me (via email or before/after class).

2. (2 Points) Your first homework question this week is to submit one homework correction from the previous week's homework. Select one problem for which you had the wrong answer, and then:

1. Identify the question number you are correcting.

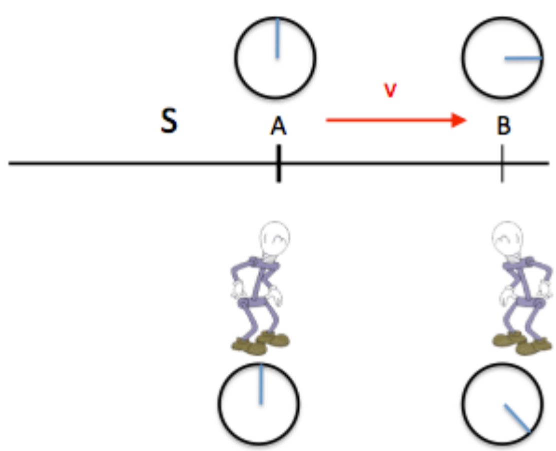
2. State (copy) your original wrong answer

3. Explain where your original reasoning was incorrect, the correct reasoning for the problem, and how it leads to the right answer.

If you got all the answers correct, Great!!! Then state which was your favorite or most useful homework problem and why.

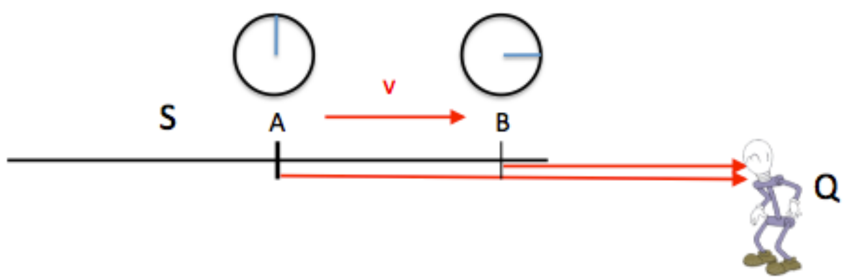
3. (19 Points Total, see individual parts for point values) It is important to understand that the effects of special relativity are in no way due to the fact that light travels at a finite speed, but rather due to the fact that the speed of light is measured as the same finite quantity in any inertial reference frame.

Time dilation implies that if a clock moves relative to frame S, careful measurements made by local observers in S with synchronized clocks will find that the clock runs slow:



This is not the same as saying that a single observer in S will ***see*** the clock running slow; there is a difference between what a ***single*** observer ***perceives*** (with her eyes), and what a system of ***local observations*** made with synchronized clocks ***will record***.

Consider a single Observer Q in frame S, and suppose that the clock in motion measures the passage of time (the proper time) as $\Delta\tau$ while it moves from Point A to Point B. As measured by local observers in S, the time between these two events ("Event A" and "Event B") is $\Delta t = \gamma\,\Delta\tau$.



3**A. (2 Points)** Explain in words why Observer Q will actually ***see*** (perceive with her eyes) the time elapsed between Event A and Event B (call this Δt_Q) as ***less than*** Δt (i.e., explain in words why $\Delta t_Q < \Delta t$).

3B. (4 Points) Show that in fact: $\Delta t_Q = \Delta t\left(1 - \frac{v}{c}\right)$.

How does Δt_Q compare with Δt?

3C. (4 Points) Show that the expression for Δt_Q may also be written as: $\Delta t_Q = \Delta\tau\sqrt{\frac{1 - v/c}{1 + v/c}}$.

How does Δt_Q compare with $\Delta\tau$?

3D. (2 Points) Now, explain in words why Observer Q ***perceives*** the clock as running ***fast***, even though the local observers ***measure*** the moving clock as running ***slow***.

3E. (2 Points) What would Observer Q perceive if the clock were moving away from her (say, from B to A, instead of from A to B).

3F. (5 Points) Look back at Problem #6 from Homework 03 (solutions available on Blackboard), and compare the conclusions from that problem with those from this problem. Now, explain in your own words the difference between events ***recorded*** by local observers, and events ***perceived*** by a single observer. We should conclude from this that the effects of special relativity (time dilation, length contraction) are not a result of our perceptions (regarding when and where events take place) being distorted by the finite speed of light. It is a result of the fact that local observers with synchronized clocks in any inertial reference frame will always ***measure*** the speed of light to be c.

4. (2 Points) As seen from earth (rest frame S) two Death Stars (A and B) are approaching in opposite directions, each with speed 0.9c relative to earth (frame S). Find the velocity of Death Star B, as measured by the pilot of Death Star A.

[Hint: Consider a coordinate system S' traveling along with Death Star A; your problem is then to find the velocity u' of Death Star B relative to S', knowing its velocity u relative to S.]

5. (6 Points Total, see individual parts for point values) In this problem you will prove that if the relativistic definitions of momentum and energy are conserved in one inertial frame, then they are conserved in any inertial frame. We'll then find that the quantity $E^2 - (pc)^2 = (E')^2 - (p'c)^2$ is the same in any inertial reference frame, in exactly the same way that $(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2 = (\Delta s)^2$ is the same in all reference frames. Again, as physicists, we are interested in calculating quantities that are the same in any frame of reference ("invariants"), ones that we'll all agree on regardless of our relative speeds.

We'll start by understanding how momentum and energy transform between frames. The relativistic definitions of momentum and energy for a particle in motion relative to frame S are:

$$\vec{p} \equiv m\frac{d\vec{r}}{d\tau} = \gamma_u m\frac{d\vec{r}}{dt}$$

$E = \gamma_u mc^2 = mc^2 \dfrac{dt}{d\tau}$ (since the local time is related to the proper time via $dt = \gamma_u d\tau$ - time dilation), and $\gamma_u = \dfrac{1}{\sqrt{1-(u/c)^2}}$, where u is the velocity of that particle in frame S.

We'll now transform these to a new frame S', moving to the right along the +x-direction with speed v [Note that v is not equal to u; v is the speed of the S' frame relative to S, u is the velocity of the particle in S; its velocity in S' would be written as u'. Anytime you do a transformation to a new frame, you need to worry about how all the different parts transform. Of course, $d\vec{r}$ will become some new $d\vec{r}\,'$, and dt will become some new dt'.

What about m, c, and $d\tau$? Well, that's what's nice about these definitions – none of these quantities change! They are what they are! Therefore, momentum changes between frames, but only because $d\vec{r}$ becomes some new $d\vec{r}\,'$; similarly, energy transforms due to the change in dt.

What's left is to actually write down all the changes. Let's start with the momentum, and write it in the new frame using the Lorentz transformations. First, let's write the momentum out the long way in terms of the different vector components:

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{1}{d\tau}\left(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}\right)$$

So, in the new frame moving with speed v relative to S along the +x-direction, we have:

$$\vec{p}\,' = m\frac{1}{d\tau}\left(\Delta x'\,\hat{i} + \Delta y'\,\hat{j} + \Delta z'\,\hat{k}\right)$$

$$= m\frac{1}{d\tau}\left(\gamma\left(\Delta x - v\Delta t\right)\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}\right)$$

$$= \gamma\left(m\frac{\Delta x}{d\tau} - mv\frac{\Delta t}{d\tau}\right)\hat{i} + m\frac{\Delta y}{d\tau}\hat{j} + m\frac{\Delta z}{d\tau}\hat{k}$$

Here, $\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}}$ refers to the speed v of the frame S' relative to S, and not to the particle's velocity u in frame S.

Take the delta (Δ) above to very small infinitesimal values and you can immediately identify the vector components of the momentum and write this result as:

$$\vec{p}\,' = \gamma\left(p_x - mv\frac{dt}{d\tau}\right)\hat{i} + p_y\,\hat{j} + p_z\,\hat{k}$$

Since we have $E = \gamma_u mc^2 = \frac{dt}{d\tau} mc^2 \quad \rightarrow \quad m\frac{dt}{d\tau} = \frac{E}{c^2}$

$$\vec{p}\,' = \gamma\left(p_x - \frac{v}{c^2}E\right)\hat{i} + p_y\,\hat{j} + p_z\,\hat{k}$$

Next, look at the transformation for the energy in a new frame:

$$E' = \gamma_{u'}\, mc^2 = mc^2 \frac{dt'}{d\tau}$$

$$= mc^2 \frac{1}{d\tau}\gamma\left(\Delta t - \frac{v\cdot\Delta x}{c^2}\right) = \gamma\left(mc^2\frac{\Delta t}{d\tau} - v\cdot m\frac{\Delta x}{d\tau}\right)$$

$$= \gamma\left(\gamma_u mc^2 - v\cdot p_x\right)$$

$$E' = \gamma\left(E - v\cdot p_x\right).$$

These are the transformation rules for momentum and energy. You may notice a similarity between them and the transformation rules for space and time:

$$p_x' = \gamma\left(p_x - \frac{v}{c^2}E\right) \quad \leftrightarrow \quad \mathrm{t}' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$E' = \gamma\left(E - v\cdot p_x\right) \quad \leftrightarrow \quad x' = \gamma\left(x - vt\right)$$

5A. (2 Points) Now, use these results to prove that if the total momentum and energy of a system are conserved (as measured in one inertial frame S), the same is true in any other inertial frame S'. Note that you are not being asked to prove that momentum and energy are conserved in some frame, but rather ***assume*** that momentum and energy are conserved in some frame, then show that these transformed quantities are also conserved in some other frame S'.

5B. (2 Points) The relativistic expression for the total energy of a particle is: $E = \gamma_u mc^2$, and the relativistic momentum is defined as $p = \gamma_u mu$. Use these relations to show that: $(E)^2 - (pc)^2 = (mc^2)^2$.

What would this energy-momentum relation be for massless particles? (e.g., photons, about which we will learn much, much more!).

Notice that the quantity mc^2 is the same in all reference frames, which implies that the quantity $(E)^2 - (pc)^2 = (E')^2 - (p'c)^2$ in any inertial frame of reference (it is invariant).

5C. (2 Points) Use the energy-momentum transformations from the introduction to this problem to show that $\left(E'\right)^2 - \left(p'c\right)^2 = \left(E\right)^2 - \left(pc\right)^2$ [For this problem, assume the momentum is entirely along the x-direction].