University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#12
due Wed, Nov 18, 2PM at start of class

1. Vectors (Total: 20 pts)

   a) (Griffiths, Problem A.1) Consider the ordinary vectors in 3D $(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k})$, with complex components. For each of the following three subsets find out whether or not it constitutes a vector space. If so, what is the dimension of the vector space? If not, why is it not a vector space?
      (i) The subset of all vectors with $a_z = 0$.
      (ii) The subset of all vectors whose z-component is 1
      (iii) The subset of all vectors whose components are all equal.

   b) Does the subset of all 2×2 matrices form a vector space? Assume the usual rules for matrix addition and multiplication by a scalar, namely:
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}, \qquad \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$
   If it does not form a vector space, why not? If it does form a vector space, state the dimensionality and give an example of a set of basis vectors.

   c) Does the set of all functions $f(x)$ defined on the range $0 < x < 1$ that vanish at $x = 0$ and $x = 1$ form a vector space? If so, state its dimensionality. If not, why not?

2. Hermitian adjoint (or Hermitian conjugate) of an operator (Total: 20 pts)
   The Hermitian conjugate or Hermitian adjoint of an operator $\hat{A}$ is denoted as $\hat{A}^\dagger$, which in class we have defined as
   $$< \Psi|\hat{A}^\dagger|\Phi > \equiv < \Psi|\hat{A}^\dagger\Phi > \equiv < \hat{A}\Psi|\Phi >$$

   a) Griffiths defines $\hat{A}^\dagger$ by (Griffiths, Equation 3.20, with $f = \Phi$ and $g = \Psi$)
   $$< \hat{A}^\dagger\Phi|\Psi > \equiv < \Phi|\hat{A}\Psi >$$
   Show that this definition is equivalent to that one used in class.

   b) Show that a Hermitian operator is self-adjoint, i.e. $\hat{A}^\dagger = \hat{A}$

   c) Show the following two properties of Hermitian adjoints of operators:
   (i) $\left(\hat{A}^\dagger\right)^\dagger = A$ and (ii) $(AB)^\dagger = B^\dagger A^\dagger$

   d) Find the Hermitian conjugates of i, $\partial/\partial x$ and the raising operator $\hat{a}_+$.

   e) An operator $\hat{A}$ can be represented by a matrix $(A)_{ij}$, i.e.
   $$(A)_{ij} = \begin{pmatrix} A_{11} & A_{22} & \dots & A_{1j} & \dots \\ A_{21} & A_{22} & \dots & A_{2j} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_{i1} & A_{i2} & \dots & A_{ij} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$
   Show that the Hermitian adjoint operator $\hat{A}^\dagger$ is represented by the transpose conjugate $(A^*)_{ji}$, that is it consists of the complex conjugates of each element, but with the rows and columns interchanged.

3. Schwarz inequality (Total: 10 pts)
   Prove the Schwarz inequality

$$< A|A >< B|B > \geq |< A|B >|^2$$

(Hint: Let $|C >= |B > - \left(\frac{<A|B>}{<A|A>}\right) |A >$ and use the fact that for any vector $< C|C > \geq 0$

4. Commutator and Anti-Commutator (Total: 10pts)

   a) Show that the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is anti-Hermitian, that means

$$[\hat{A}, \hat{B}] = -[\hat{A}, \hat{B}]^\dagger$$

   and that the anti-commutator $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is Hermitian.

   b) Show that $[\hat{x}^n, \hat{p}] = i\hbar n \hat{x}^{n-1}$.