University of Colorado, Department of Physics
PHYS3220, Spring 11, HW#3
due Wed, Jan 26, in class
- Please note: Late Homeworks will not be accepted -

1. A wavefunction

   a) Consider the following wavefunction for a particle of mass m

   $$\Psi(x,t) = A\left[\exp(-i\alpha\hbar t/m) + \beta x \exp(-3i\alpha\hbar t/m)\right]\exp(-\alpha x^2) \tag{1}$$

   where α and β are real constants. Determine A such that the wave function is normalized. Is the answer unique? Explain.
   (*Hint: You'll find the solutions to problem 3b) of the last HW set helpful here. Also, think about whether you can argue some terms are zero without explicitly calculating them.*)

   b) What is the probability density for this wavefunction? Is the probability density independent of time?

   c) What is the expectation value $\langle x\rangle$? What kind of motion is the average value of the particle's position executing?

2. Thinking about the Schrödinger equation

   a) Show that if $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are solutions of the Schrödinger equation

   $$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi(x,t) \tag{2}$$

   then the superposition $\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$ is a solution too.

   b) The composite Schrödinger equation

   $$i\hbar\frac{\partial}{\partial t}\Psi(x_1,x_2,t) = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}\right) + V_1(x_1,t) + V_2(x_2,t)\right]\Psi(x_1,x_2,t) \tag{3}$$

   describes a system of two independent particles. Use the interpretation that $|\Psi(x_1,x_2,t)|^2$ is a probability density to show that the state of such a system must be given in the product form $\Psi(x_1,x_2,t) = \Psi_1(x_1,t)\Psi_2(x_2,t)$.

- This problem continues on the back -

c) Consider now two Schrödinger equations with different potential terms $V_1(x_1, t)$ and $V_2(x_2, t)$

$$i\hbar\frac{\partial}{\partial t}\Psi_1(x_1,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + V_1(x_1,t)\right)\Psi_1(x_1,t) \tag{4}$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2(x_2,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V_2(x_2,t)\right)\Psi_2(x_2,t) \tag{5}$$

Show that if $\Psi_1(x_1, t)$ is a solution of equation (4) and $\Psi_2(x_2, t)$ is a solution of equation (5), then the product form $\Psi(x_1, x_2, t) = \Psi_1(x_1, t)\Psi_2(x_2, t)$ is a solution of the Schrödinger equation composed of the two equations above, namely

$$i\hbar\frac{\partial}{\partial t}\Psi(x_1,x_2,t) = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}\right) + V_1(x_1,t) + V_2(x_2,t)\right]\Psi(x_1,x_2,t) \tag{6}$$

(*Note: This is the same equation as in part b).*)

d) Replace in part c) all time derivatives $\partial/\partial t$ by $\partial^2/\partial t^2$ and show that in this case the product form is *not* solution of the composite equation.

e) Review your answers and conclusions from parts b) to d) and argue why the time derivative in the Schrödinger equation should be $\partial/\partial t$ and *not* $\partial^2/\partial t^2$.

3. Phase factors
Let $\Psi(x, t)$ be a solution of the (one-dimensional) Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t). \tag{7}$$

Suppose you add a constant V_0, which is independent of x and t, to the potential energy term $V(x, t)$.

a) Show that the solution of the new Schrödinger equation is given by $\Phi(x, t) = \Psi(x, t)\exp(-iV_0t/\hbar)$.

b) What effect does this time-dependent phase factor have on the probability density $|\Phi(x, t)|^2$ and the expectation value of the position $\langle x\rangle$? Explain.

4. Current density
In class we have shown that the continuity equation for probability is given by

$$\frac{d}{dt}|\Psi(\mathbf{r},t)|^2 + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0. \tag{8}$$

In deriving the equation we have assumed that the potential $V(\mathbf{r}, t)$ is a real quantity.

a) Prove that if $V(\mathbf{r}, t)$ would be complex the continuity equation would become

$$\frac{d}{dt}|\Psi(\mathbf{r},t)|^2 + \nabla\cdot\mathbf{J}(\mathbf{r},t) = \frac{2}{\hbar}Im(V(\mathbf{r},t))|\Psi(\mathbf{r},t)|^2 \tag{9}$$

b) How do you interpret the term on the right hand side of equation (9)? Discuss in particular the cases $Im(V(\mathbf{r}, t)) > 0$ and $Im(V(\mathbf{r}, t)) < 0$.