University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#13
due Wed, Dec 2, 2PM at start of class

1. Matrix representation of the eigenvalue problem (Total: 20 pts)
Suppose there are two observables A and B with corresponding Hermitian operators $\hat{A}$ and $\hat{B}$. The eigenfunctions to $\hat{A}$ as well as to $\hat{B}$ form a complete set of basis functions, and satisfy the eigenvalue equations:

$$\hat{A}|a_n >= a_n|a_n > \qquad \text{and} \qquad \hat{B}|b_n >= b_n|b_n > \ .$$

(Here we are adopting a common practice of labeling an eigenstate by its eigenvalues, so $|a_n >$ is the eigenvector of $\hat{A}$ with eigenvalue a_n. Do not confuse the eigenvalues, which are numbers, with their eigenvectors, which are vectors in Hilbert space).

   a) Since the eigenfunctions form a complete set, any wavefunction $|\Psi(t) >$ can be written as an expansion in either basis (we assume for simplicity a discrete basis):

   $$|\Psi(t) >= \sum_n c_n(t)|a_n >= \sum_n d_n(t)|b_n >$$

   Derive a formula that expresses any particular d_n in terms of the set of c_n's.

   b) Using the identity operator $\hat{I} = \sum_i |b_i >< b_i|$ show that $\hat{A}$ can be written as:

   $$\hat{A} = \sum_j \sum_i A_{ji}|b_j >< b_i|$$

   with $A_{ji} =< b_j|\hat{A}|b_i >$ is the ji-th element of the matrix representing $\hat{A}$.

   c) Show that the eigenvalue equation $\hat{A}|a_n >= a_n|a_n >$ can be written as

   $$\sum_i [A_{ji} - a_n\delta_{ij}] < b_i|a_n >= 0\,. \qquad (1)$$

   d) Equation (1) has nonzero solutions only if the determinant

   $$\det(A - a_n I) = 0\,,$$

   where A is the matrix representing the operator $\hat{A}$ and I is the identity matrix. The solutions of this so-called secular or characteristic equation yield the eigenvalues a_n. Use this result to find the eigenvalues a_n of

   $$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix}$$

   Then, use the matrix representation of the eigenvalue equation to find the eigenvectors $|a_n >$ corresponding to the eigenvalues a_n. Make sure that the eigenvectors are normalized.

2. Uncertainty relation and time dependence of expectation values (Total: 20 pts)

   a) Show that the generalized uncertainty principle for the operator $\hat{x}$ and $\hat{H}$ yields

   $$\sigma_x \sigma_H \geq \frac{\hbar}{2m} | < p > |$$

   What can you deduce about the value of $< p >$ in a stationary state?

   b) Use

   $$\frac{d}{dt} < \hat{A} >= \frac{1}{i\hbar} < [\hat{A}, \hat{H}] > + < \frac{\partial \hat{A}}{\partial t} >$$

   with $\hat{A} = \hat{I}$ (identity operator), $\hat{A} = \hat{H}$ and $\hat{A} = \hat{p}$ to prove the conservation of probability, the conservation of energy, and Ehrenfest's theorem, respectively.

3. Angular momentum operators (Total: 20 pts)
   The x, y, and z components of the orbital angular momentum operator expressed in spherical coordinates are:

   $$\begin{aligned}
   \hat{L}_x &= -i\hbar \left(-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right) \\
   \hat{L}_y &= -i\hbar \left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right) \\
   \hat{L}_z &= -i\hbar \frac{\partial}{\partial\phi}
   \end{aligned}$$

   a) Prove the above expression for $\hat{L}_z$ by showing it is equivalent to the expression for $\hat{L}_z$ in Cartesian coordinates. (Hint: Work backwards from the desired result and use the chain rule.)

   b) Find and simplify the generalized uncertainty relation between $\hat{L}_z$ and the angle ϕ. What does the result remind you of?

   c) Show that

   $$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \pm\hbar \exp(\pm i\phi) \left(\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi} \right)$$

   and

   $$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] .$$