Comment on aribitrary phase factor in ψ:
If Φ is sol'n of SE, then so is $(Ae^{i\theta})\Phi = \psi$
where $(Ae^{i\theta})$ is <u>any</u> complex nbr. A is determined
by normalization,

$$\underset{\substack{\uparrow \\ \text{normalized}}}{\psi} = A\, e^{i\theta} \cdot \underset{\substack{\nwarrow \\ \text{unnormalized}}}{\Phi} \Rightarrow \int |\psi|^2 dx = A^2 \int |\Phi|^2 dx = 1$$

$$\Rightarrow A = 1 \Big/ \sqrt{\int |\Phi|^2 dx}$$

The phase factor $e^{i\theta}$ is indeterminate, but it has no effect on any physical quantities (such as eigenvalues, expectation values, or $|\psi|^2$)

$\Rightarrow$ for <u>any</u> real const θ, ψ and $e^{i\theta}\psi$ represent the same physical state.

Note special case: $e^{i\pi} = -1 \Rightarrow \psi, -\psi$ are <u>same</u> st<u>ate</u>
this justifies throwing out negative values of n in sol'ns of ∞ square well.

The Free Particle ($V = 0$ everywhere)

TISE: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} = E \cdot \psi$ ← determines $\psi(x)$

(Remember: $\Psi(x,t) = \psi(x) \cdot e^{-iEt/\hbar}$)

$$\psi'' = \frac{-2mE}{\hbar^2} \cdot \psi = -k^2 \psi$$

$$k^2 \equiv \frac{2mE}{\hbar^2}, \quad E = \frac{\hbar^2 k^2}{2m} \overset{\text{(de Broglie)}}{=} \frac{p^2}{2m}$$

$$\Rightarrow \psi(x) = A e^{+ikx} + B e^{-ikx} \quad , \quad k = 2\pi/\lambda$$

$$\Psi(x,t) = A e^{+i(kx-\omega t)} + B e^{-i(kx+\omega t)}$$

$$(\omega \equiv E/\hbar)$$

$$\Psi(x,t) = \underbrace{A e^{ik(x-\frac{\omega}{k}t)}}_{\substack{f(x-v\cdot t)\\ \text{right-going wave}}} + \underbrace{B e^{-ik(x+\frac{\omega}{k}t)}}_{\substack{g(x+v\cdot t)\\ \text{left-going wave}}}$$

speed of wave $= v = \dfrac{\omega}{k} = \dfrac{E}{\hbar k} = \dfrac{E}{p} = \dfrac{p^2/2m}{p}$

$v = \dfrac{p}{2m}$ Problem: (but classically, should have $v = p/m$!)

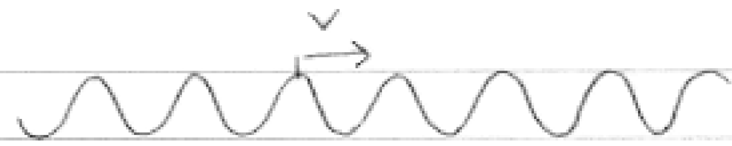
can include both rt- ~~&~~ or left-going waves in single expression by allowing k (+) or (−)

$$\Psi_k(x,t) = A e^{i(kx-\omega t)} \quad \begin{cases} k>0, & \text{rt-going} \\ k<0, & \text{left going} \end{cases}$$

any real value of k allowed. 2 values of k allowed for each value of E:

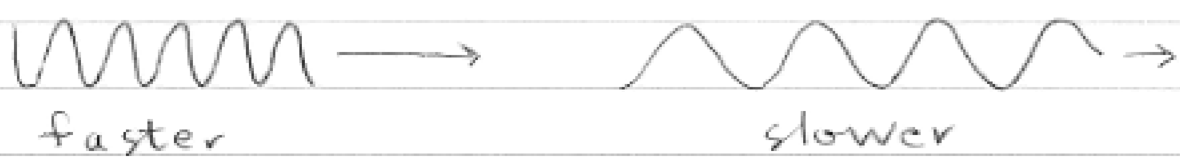
$$k = \frac{\pm\sqrt{2mE}}{\hbar} \quad , \quad \lambda = 2\pi/|k|$$

$$\omega = \omega(k) = \frac{E}{\hbar} = \frac{\hbar k^2}{2m} \quad \leftarrow \text{"dispersion relation"}$$



wave speed $v = \frac{\omega}{k}$ ("phase velocity") $= \frac{\hbar k}{2m}$

$v = \frac{h}{2m\lambda}$ different λ, waves same m travel w/ different speeds!



Another problem w/ "plane-wave" sol'ns

$$\Psi_k(x,t) = A\, e^{i(kx - \omega t)}$$

They can't be normalized! (because they extend to $x = \pm\infty$)

$$\int |\Psi_k|^2 dx = A^2 \int_{-\infty}^{+\infty} 1\, dx = A^2 \cdot \infty$$

so Ψ_k's cannot represent physical normalizable states

However, can construct linear combination of Ψ_k's which is normalizable. Recall that SE linear $\Rightarrow$ <u>any</u> linear combo of Ψ_k's is sol'n of SE

Most general linear combination of Ψ_k's:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, \underbrace{\Psi_k(x,t)}_{e^{ikx}\, e^{-i\omega t}}\, dk$$

$\omega = \omega(k)$

Note similarity to general sol'n of ∞ square-well:

$$\Psi(x,t) = \sum_n c_n \underbrace{\Psi_n(x,t)}_{\psi_n(x)\, e^{-i\omega t}}$$

(c_n's are like $\phi(k)\,dk$)

$\phi(k) = ?$ $\quad \Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{+ikx}\, dk$

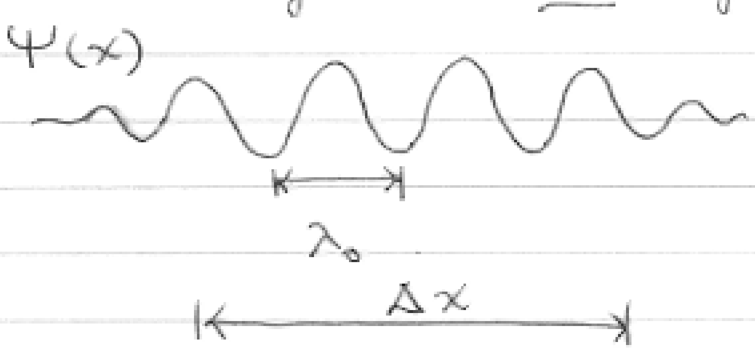
Recall Fourier Transforms:

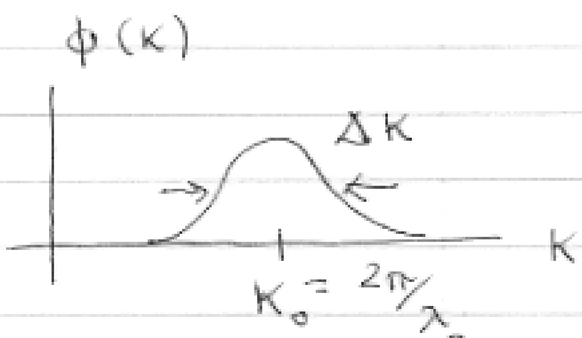
$$f(x) = \frac{1}{\sqrt{2\pi}} \int F(k)\, e^{+ikx}\, dk$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{-ikx}\, dx$$

$$\Rightarrow \phi(k) = \frac{1}{\sqrt{2\pi}} \int \Psi(x,0)\, e^{-ikx}\, dx$$

Everything OK so long as $\Psi(x,0)$ is normalizable wave packet

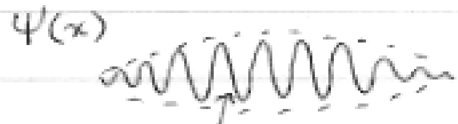


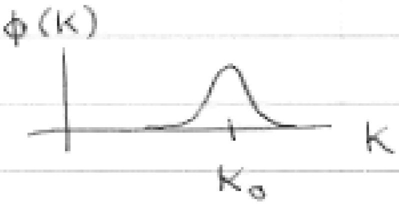


small $\Delta k \Leftrightarrow$ big Δx $\qquad \Delta x \cdot \Delta k \simeq 1$

$\Delta p = \hbar\, \Delta k$ (de Broglie) $\quad \Rightarrow \quad \Delta x \cdot \Delta p \simeq \hbar$ $\quad$ Uncertainty Principle

Back to problem of velocity of free particle

Wave Packet : 



Will show that ripples inside packet move w/ phase velocity ω/k but envelope  moves w/ group velocity $d\omega/dk$

ripples × envelope = packet

packet $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{i(kx - \omega(k)\cdot t)}\, dk$

$\omega = \omega(k) = \frac{E}{\hbar} = \frac{\hbar k^2}{2m}$

$\phi(k)$ is assumed ≈ 0 except near $k_0 = 2\pi/\lambda_0$

$\Rightarrow$ only contributions to integral from k's near k_0

$\Rightarrow$ can expand $\omega(k)$ about k_0

Taylor Series : $\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k_0} \cdot (k - k_0)$

$\omega(k) = \omega_0 + \omega_0' \cdot \Delta k$

$\Delta k = k - k_0 , \quad k = k_0 + \Delta k ,$

$$e^{i(kx - \omega t)} = e^{i[(k_0 + \Delta k)x - (\omega_0 + \omega_0' \cdot \Delta k)\cdot t]}$$

$$= e^{i(k_0 x - \omega_0 t)} \cdot e^{i(\Delta k \cdot x - \omega_0' \Delta k t)}$$

$$= e^{i(k_0 x - \omega_0 t)} \cdot e^{i \Delta k (x - \omega_0' t)}$$

So, can rewrite $\Psi(x,t)$ as

$$\Psi(x,t) = \underbrace{\frac{e^{i(k_0 x - \omega_0 t)}}{\sqrt{2\pi}}}_{\substack{g\left(x - \frac{\omega_0}{k_0}t\right) \\ \text{(ripples)}}} \underbrace{\int_{-\infty}^{+\infty} d(\Delta k)\, \phi(k_0 + \Delta k)\, e^{i \Delta k (x - \omega_0' t)}}_{\substack{f(x - \omega_0' \cdot t) \\ \text{(envelope)}}}$$

phase velocity $v_{phase} = \dfrac{\omega}{k} = \dfrac{\hbar k}{2m} = \dfrac{p}{2m}$

(recall $\omega = \omega(k) = \hbar k^2 / 2m$)

group velocity $v_{group} = \dfrac{d\omega}{dk} = \dfrac{\cancel{2} \hbar k}{\cancel{2} m} = \dfrac{\hbar k}{m} = \dfrac{p}{m}$

$\Rightarrow$ envelope of wavepacket moves w/ $v_{group} = p/m$
agrees w/ classical mechanics

Because different k's move w/ different $v_{phase} = \dfrac{\hbar k}{2m}$
the envelope tends to spread out as it
moves w/ higher k's moving to front of packet

Δx grows. OK w/ Uncertainty Principle which
only places lower limit $\Delta x \Delta p \geq \hbar/2$