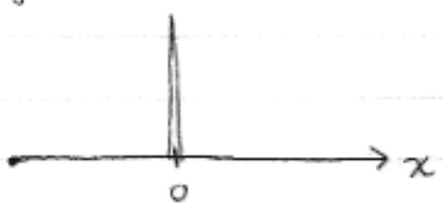
Dirac Delta Function, $\delta(x)$

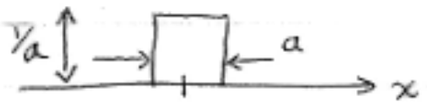
Very tall, very narrow spike at $x=0$ w/ area $=1$



$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

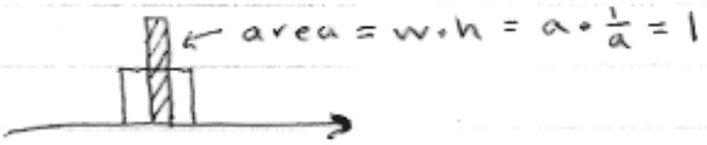
$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

Many ways to construct $\delta(x)$



$$f(x,a) = \begin{cases} \frac{1}{a} & , |x| < a/2 \\ 0 & , |x| > a/2 \end{cases}$$

$$\delta(x) = \lim_{a \to 0} f(x,a)$$



Most important property of $\delta(x)$:

$$\int dx\ f(x)\,\delta(x) = f(0)$$ (assuming $f(x)$ continuous at $x=0$)

Similarly, 

, $\int dx\ \delta(x-x_0) = 1$

$$\int dx\ f(x)\,\delta(x-x_0) = f(x_0)$$

Proof: $\int_{-\infty}^{+\infty} dx\ f(x)\,\delta(x-x_0) = \int_{x_0-\varepsilon}^{x_0+\varepsilon} dx\ f(x)\,\delta(x-x_0)$

$$= \lim_{\varepsilon \to 0} f(x_0) \underbrace{\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx\ \delta(x-x_0)}_{1} = f(x_0) \quad \checkmark$$

Other important properties of $\delta(x)$ (homework!)

- $\delta(-x) = \delta(x)$
- $\delta(c \cdot x) = \dfrac{\delta(x)}{|c|}$, c = real constant

units of $\delta(x)$? $\int \delta(x)\,dx = 1 \Rightarrow [\delta] = [x^{-1}]$

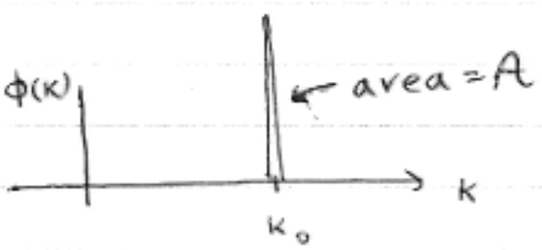
Example: Recall wavepacket

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk\, \phi(k)\, e^{i(kx - \omega t)}$$

$\omega(k) = \dfrac{\hbar k^2}{2m}$

Spse $\phi(k) = A \cdot \delta(k - k_0)$

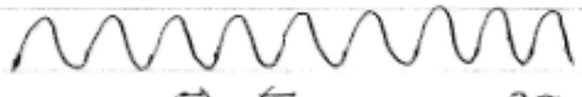
$\left(\int dk\, \phi(k) = A \int dk\, \delta(k-k_0) = A \right)$



$$\Psi(x,t) = \frac{A}{\sqrt{2\pi}} \int dk\, \delta(k-k_0)\, e^{i(kx - \omega t)}$$

$$= \frac{A}{\sqrt{2\pi}}\, e^{i(k_0 x - \omega_0 t)}$$

$\omega(k_0) = \dfrac{\hbar k_0^2}{2m}$

Re(Ψ):  $\rightarrow$ $v_{phase} = \omega_0 / k_0 = \dfrac{\hbar k_0}{2m}$

λ_0, $k_0 = 2\pi/\lambda_0$