SETUP: Consider a quantum particle with some *new properties* that we can measure.
We won't talk much about the physics of these measurements yet, but the formalism of quantum mechanics will teach us a great deal, just from operator methods!

Consider a hermitian operator S_x which yields only 2 possible measurement outcomes, +1 or -1.
(It will represent the measurement of "the component of spin in the x direction". For now think of it as a measurement of some property of a particle which can be "rightwards" or "leftwards"…)
With only two possible measurement outcomes, S_x has *only* two (orthonormal) eigenvectors.

People get tired of writing out complicated "ket names", so a simple, common notation is

$\hat{S}_x|+\rangle = |+\rangle$,

$\hat{S}_x|-\rangle = -1\cdot|-\rangle$

I suppose you might refer to the |+> state as a "spin right" state, and |-> as "spin left", does that seem reasonable to you?

Stare at these two eigen-equations and make sure you, and your group, understand the notation: which symbols are the eigenvectors here, what are the eigenvalues, in those equations?
What can you say about, e.g. $\langle -|+\rangle$? (Explain)

Now consider a second observable, with corresponding Hermitian operator, $S_{z.}$
This operator is NOT the same as $S_{x,}$ although it does have the same spectrum.
Since the eigenvectors of S_z are different from those of $S_{x,}$ we need to give them a different name.
Here, people conventionally name the ket with a *symbol*:

$\hat{S}_z\,|\uparrow\rangle = |\uparrow\rangle$,

$\hat{S}_z\,|\downarrow\rangle = -1\cdot|\downarrow\rangle$

(Since S_z measures the vertical, or z, component of spin, I suppose you might refer to the |↑> state as a "spin up", and |↓> as "spin down" particle, does that seem reasonable to you?)
Again, make sure you follow this notation: what are the possible outcomes of a measurement of S_z?
Which symbols are the eigenvectors here, what are the eigenvalues? What is $\langle\uparrow|\downarrow\rangle$? (Explain)

Lastly, let us suppose that

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

This tells us how the eigenvectors of S_z and those of S_x are related.
(As I said earlier, they are NOT identical)

Let's look at some consequences of this setup:

A) In this quantum world, suppose I give you a particle and you measure S_z and find eigenvalue +1, i.e. it is "spin up".
i) What state is the particle in now? (Is there any ambiguity at this point?)

ii) If you remeasure S_z on this state, what results can you get, with what probabilities?
After this second measurement of S_z, what state will you be in? (Is there any ambiguity at this point?)

iii) Following the above, what is the probability that a subsequent measurement of S_x will yield a result of -1, i.e. "spin left"? Explain.

B)i) Use orthonormality and completeness to expand the "S_z=-1 eigenstate" in the "S_x basis":

$$\left|\downarrow\right\rangle = a\left|+\right\rangle + b\left|-\right\rangle$$

i.e. find the numerical constants a and b.
Is your answer unique? (If not, does it *matter*?)

ii) Suppose I give you a particle and you measure S_x and find eigenvalue -1.
What state is the particle in? (Is there any ambiguity at this point?)

iii) What is the probability that a subsequent measurement of S_z will yield a result of -1, i.e. "spin down"? *(Don't intuit answers at this point, work it out from the postulates of quantum mechanics!)*

C) If I have an ensemble of spin "right" particles, and we measure S_z on all of them, do they each get the same result? Explain.

What is the *average* result of measuring S_z on this ensemble of "spin right" particles?

More on Operators and Eigenvalues

Let σ_- be another operator which can act on all these kets (|+>, |->, |↑>, and/or |↓>).
I'm only going to tell you what this curious operator does to two of these states:

$\hat{\sigma}_- |\uparrow\rangle = |\downarrow\rangle$

$\hat{\sigma}_- |\downarrow\rangle = 0$

So it operates on |↑> and yields |↓>, but it "kills" the state |↓> (it gives you back 0, nothing)
If you want some intuition, it's a little bit like a "lowering operator". (If you act it on something which "points up", it then "points down", but it kills something which is already pointing down)

D) Are "spin up" states (i.e. |↑>) eigenstates of σ_-?
Are "spin down" states (i.e. |↓>)?

E) Show that σ_- is NOT a Hermitian operator!

I said σ_- is an operator. Does it correspond to something you can measure or observe? Explain.

F) What would $\sigma_-^\dagger$ do to a "spin up" state. How about to a "spin down" state?
(This is tricky, work it out!)

Based on the above, make up a plausible name for the $\sigma_-^\dagger$ operator.

Simultaneous measurements:

G) i) Do S_x and S_z commute?
(Again, a little tricky. If they DO commute, then S_xS_z *gives the same result as* $S_z S_x$ *for any state. A single counterexample will disprove it! Consider operating on, say, a spin "down" state)*

ii) Based on the above, does measuring the value of S_z on some particle affect the outcome of a measurement of S_x? (Vice versa?)

H) Given a particle in the state |+>, if you then measure S_z, what is the probability that you will measure -1? (Colloquially, we might phrase this "what is the probability that a "spin right" particle yields a "spin down" measurement?)

Time evolution
Now suppose that the Hamiltonian commutes with S_z *(*One way to think of this would be a system whose *energy* is proportional to S_z) Suppose I give you a particle which I have carefully prepared. I know what state it is in, but I don't tell you. It might be in a superposition of "spin up" and "spin down" states!

I) You can measure S_z at any time you like. Does the probability that your measurement will yield "+1" (i.e, that it's "spin up") depend on the amount of time that you wait before measuring? Explain

J) Suppose instead that you chose NOT to measure S_z, but instead to measure S_x. Does the probability that your measurement will yield "+1" (i.e, that it's "spin right") depend on the amount of time that you wait before measuring? Explain.
(*You might consider the simple case that I handed you a particle prepared in the state* $|+\rangle$ *at t=0.)*