University of Colorado, Department of Physics
PHYS3220, Spring 11, HW#10
due Fri, Apr 8, in class
- Please note: Late Homeworks will not be accepted -

1. Schwarz inequality
   Prove the Schwarz inequality

$$\langle A|A\rangle\langle B|B\rangle \geq |\langle A|B\rangle|^2$$

   (Hint: $Let|C >= |B > - \left(\frac{\langle A|B\rangle}{\langle A|A\rangle}\right)|A >$ and use the fact that for any vector, i.e. in particular also for the vector $|C >$ yields $< C|C > \geq 0$.)

2. Useful properties of commutators (and anti-commutators)

   a) Assume $\hat{A}$ and $\hat{B}$ are Hermitian operators. Show that the commutator $[\hat{A};\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is anti-Hermitian, that means $[\hat{A};\hat{B}] = -[\hat{A};\hat{B}]^\dagger$ and that the anti-commutator $\{\hat{A};\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is Hermitian.
   b) Show that $[\hat{x}^n;\hat{p}] = i\hbar n\hat{x}^{n-1}$.

3. Uncertainty relation and time dependence of expectation values

   a) Show that the generalized uncertainty principle for the operator $\hat{x}$ and $\hat{H}$ yields

$$\sigma_x\sigma_H \geq \frac{\hbar}{2m}| < p > |$$

   What can you deduce about the value of $< p >$ in a stationary state?
   b) Use

$$\frac{d}{dt} < \hat{A} >= \frac{1}{i\hbar} < [\hat{A},\hat{H}] > + < \frac{\partial\hat{A}}{\partial t} >$$

   with $\hat{A} = \hat{I}$ (identity operator), $\hat{A} = \hat{H}$ and $\hat{A} = \hat{p}$ to prove the conservation of probability, the conservation of energy, and Ehrenfest's theorem, respectively.
   (Hints: (i) In this class we did not and we will not consider problems, in which the potential is time-dependent. Thus, you can assume that $V(x,t) = V(x)$. (ii) In the last term of the right hand side $\partial\hat{A}/\partial t$ is the derivative of the *operator*, and only of the operator, with respect of time.)

- Please note: There is another problem on the next page -

4. Matrix representation of the eigenvalue problem

Suppose there are two observables A and B with corresponding Hermitian operators $\hat{A}$ and $\hat{B}$. The eigenfunctions to $\hat{A}$ as well as to $\hat{B}$ form a complete set of basis functions, and satisfy the eigenvalue equations:

$$\hat{A}|a_n >= a_n|a_n > \qquad \text{and} \qquad \hat{B}|b_n >= b_n|b_n > \ .$$

(Here we are adopting a common practice of labeling an eigenstate by its eigenvalues, so $|a_n >$ is the eigenvector of $\hat{A}$ with eigenvalue a_n. Do not confuse the eigenvalues, which are numbers, with their eigenvectors, which are vectors in Hilbert space).

a) Since the eigenfunctions form a complete set, any wavefunction $|\Psi(t) >$ can be written as an expansion in either basis (we assume for simplicity a discrete basis):

$$|\Psi(t) >= \sum_n c_n(t)|a_n >= \sum_n d_n(t)|b_n >$$

Derive a formula that expresses any particular d_n in terms of the set of c_n's.

b) Using the identity operator $\hat{I} = \sum_i |b_i >< b_i|$ show that $\hat{A}$ can be written as:

$$\hat{A} = \sum_j \sum_i A_{ji}|b_j >< b_i|$$

with $A_{ji} =< b_j|\hat{A}|b_i >$ is the ji-th element of the matrix representing $\hat{A}$.

c) Show that the eigenvalue equation $\hat{A}|a_n >= a_n|a_n >$ can be written as

$$\sum_i [A_{ji} - a_n\delta_{ij}] < b_i|a_n >= 0 \,. \tag{1}$$

d) Equation (1) has nonzero solutions only if the determinant

$$\det(A - a_n I) = 0 \,,$$

where A is the matrix representing the operator $\hat{A}$ and I is the identity matrix. The solutions of this so-called secular or characteristic equation yield the eigenvalues a_n. Use this result to find the eigenvalues a_n of

$$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix}$$

Then, use the matrix representation of the eigenvalue equation to find the eigenvectors $|a_n >$ corresponding to the eigenvalues a_n. Make sure that the eigenvectors are normalized.