Back to angular eq'n: $\hat{L}^2 Y_\ell^m = \hbar^2 \ell(\ell+1) Y_\ell^m$

Want to solve for the Y_ℓ^m's "spherical harmonics"

Before, started w/ commutation relations

$$[\hat{L}_i, \hat{L}_j] = i\hbar \hat{L}_k \quad , \quad [L^2, L_i] = 0$$

and, using operator algebra, solved for the eigenvalues of L^2, L_z. We found

$$\left.\begin{aligned} L^2 Y_\ell^m &= \hbar^2 \ell(\ell+1) Y_\ell^m \\ L_z Y_\ell^m &= m\hbar Y_\ell^m. \end{aligned}\right\} \quad \begin{aligned} &\text{where } \ell = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \ldots \\ &m = -\ell, -\ell+1 \ldots +\ell \end{aligned}$$

In the process, we defined raising & lowering operators:

$$\hat{L}_\pm = L_x \pm i L_y$$

$$\hat{L}_+ Y_\ell^m = c_m Y_\ell^{m+1} \qquad (\text{for } m < m_{max} = \ell)$$

$$\hat{L}_- Y_\ell^m = c_m Y_\ell^{m-1} \qquad (\text{for } m > m_{Min} = \ell)$$

(c_m is some constant)

$$\hat{L}_+ f_{top} = \hat{L}_+ Y_\ell^\ell = 0 \quad \text{and} \quad \hat{L}_- Y_\ell^{-\ell} = 0$$

So, if we can find (for a given ℓ) a single eigenstate Y_ℓ^m, then we can generate all the others (other m's) by repeated application of $\hat{L}_+$ or $\hat{L}_-$

$Y_\ell^m = Y_\ell^m(\theta, \varphi)$.

It's easy to ~~~~ find the φ-dependence; don't need the $\hat{L}_\pm$ business yet.

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \quad \text{(showed in HW)}$$

$$\hat{L}_z Y = \frac{\hbar}{i}\frac{\partial Y}{\partial\varphi} = \hbar m Y$$

Assume $Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi) \Rightarrow$

$$\frac{d\Phi}{d\varphi} = im\Phi \Rightarrow \boxed{\Phi(\varphi) = e^{+im\varphi}}$$

If we assume (postulate) that Ψ is single-valued then

$$\Phi(\varphi + 2\pi) = \Phi(\varphi) \Rightarrow e^{im2\pi} = 1$$

$\Rightarrow m = 0, \pm 1, \pm 2, \ldots$ But $m = -\ell, \ldots +\ell$

So for orbital angular momentum, ℓ must be integer only ! $\ell = 0, 1, 2 \ldots$ (throw out $\frac{1}{2}$ int. values)

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$$L_+ = L_x + iL_y \overset{\text{(algebra!)}}{=} \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_- = L_x - iL_y = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$\left(L_- = L_+^{+} \leftarrow \text{adjoint} \quad \langle f | A^+ g\rangle = \langle Af | g\rangle\right)$

Can deduce Y_ℓ^ℓ from $\hat{L}_+ Y_\ell^\ell = 0$

$$\Rightarrow \quad \frac{\partial Y_\ell^\ell}{\partial \theta} + i \cot\theta \frac{\partial Y_\ell^\ell}{\partial \varphi} = 0$$

Solution: $\boxed{Y_\ell^\ell(\theta, \varphi) = (\sin\theta)^\ell e^{i\ell\varphi}}$ (un-normalized)

Check: Plug back in.

$$\ell (\sin\theta)^{\ell-1} \cos\theta \, \cancel{e^{i\ell\varphi}} + i \cot\theta (\sin\theta)^\ell (i\ell) \cancel{e^{i\ell\varphi}} = 0$$

$$\ell (\sin\theta)^\ell \underbrace{\frac{\cos\theta}{\sin\theta}}_{\cot\theta} - \cot\theta (\sin\theta)^\ell \cdot \ell = 0 \quad \checkmark$$

Now, can get other Y_ℓ^m's by repeated application of $\hat{L}_-$. Somewhat messy. Done in HW.

Normalization from $\int d\theta \int d\varphi \sin\theta \, |Y_\ell^m|^2$.

Notice case $\ell = 0$: $\boxed{Y_0^0 = \text{const} = 1/\sqrt{4\pi}}$

(since $\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta = \int d\Omega = 4\pi$)

"solid angle"

Example: $Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{+i\varphi}$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_1^{-1} = +\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

Convention: $Y_\ell^{-m} = (-1)^m (Y_\ell^m)^*$
on $\pm$ sign

The spherical harmonics form a complete, orthonormal set (since eigenfcns of hermitean operators)

$$\int d\Omega \, (Y_\ell^m)^* \, Y_{\ell'}^{m'} = \delta_{\ell\ell'} \, \delta_{mm'}$$

A<u>ny</u> function of angles $f = f(\theta, \varphi)$ can be written as linear combo of Y_ℓ^m's :

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} Y_\ell^m$$

Likewise: $\int_0^\infty dr \, r^2 \, (R_{n\ell})^* \, R_{n'\ell'} = \delta_{nn'} \, \delta_{\ell\ell'}$

$\Rightarrow$ H-atom <u>energy</u> <u>eigenstates</u> are

$$\psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r) \, Y_\ell^m(\theta, \varphi) = R_{n\ell} \, \Theta_{\ell m} \, e^{im\varphi}$$

$n = 1, 2, \ldots \; ; \; \ell = 0, 1 \ldots (n-1) \; ; \; m = -\ell \ldots +\ell$

Arbitrary (bound) state is

$$\psi = \sum_{n,\ell,m} c_{n\ell m} \cdot \psi_{n\ell m}$$ (c's are any complex constants)

energy of state (n, ℓ, m) depends only on n

$E_n = -\text{const}/n^2$ (states ℓ, m w/ same n are <u>degenerate</u>)

	$\ell =$ 0	1	2	3
$n =$ 4	4s (1)	4p (3)	4d (5)	4f (7)
3	3s	3p	3d	
2	2s	2p		
1	1s			

Degeneracy of nth level is n^2

($2 \cdot n^2$ if you include spin)