Physics 3220 – Quantum Mechanics 1 – Fall 2008
Problem Set #13

Due Wednesday, December 10 at 2pm

Problem 13.1: Surveys! (20 points)

Please take the following surveys. **You will not be graded for accuracy for these surveys, you get credit just for participating.**

a) http://www.colorado.edu/sei/surveys/Fall08/Clicker_Phys3220_fa08-post.html

b) http://www.colorado.edu/physics/EducationIssues/baily/SurveyFa08/MPASFall08Post_3220.htm

Problem 13.2: Projection operators. (20 points)

Consider first a Hilbert space spanned by a basis of orthonormal states $\{|n\rangle\}$ labeled by one discrete quantum number, which we will call n. The states $|n\rangle$ are the eigenstates of some operator $\hat{Q}$, and n labels the various eigenvalues, $\hat{Q}|n\rangle = q_n|n\rangle$. (For example, $\hat{Q}$ could be the Hamiltonian, and n could label the allowed energies.)

For each n, we define the *projection operator onto the state* $|n\rangle$ as

$$\hat{P}_n \equiv |n\rangle\langle n| \,. \tag{1}$$

Thus there is a different $\hat{P}_n$ for each state $|n\rangle$.

a) Demonstrate that $\hat{P}_n$ is Hermitian, and that $\hat{P}_n^2 = \hat{P}_n$.

b) What is the result of acting $\hat{P}_n$ on an arbitrary state $|\psi\rangle = \sum_m c_m|m\rangle$? Explain why the name "projection operator" is justified. If there are N distinct values of n, all operators will be $N \times N$ matrices; what does $\hat{P}_n$ look like as such a matrix?

c) In general $\hat{P}_n|\psi\rangle$ is not normalized; show that the state $\hat{P}_n|\psi\rangle/\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}$ is properly normalized.

d) How are the number $\langle\psi|\hat{P}_n|\psi\rangle$ and the state $\hat{P}_n|\psi\rangle/\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}$ related to the result of making a measurement of Q? Relate them to the postulates of quantum mechanics.

Now consider a system where the Hilbert space is labeled by more than one quantum number: the hydrogen atom, with states $|n\,\ell\,m\rangle$ labeled by n, ℓ and m. The projection operator associated to a given value of n now has a sum over all values of the *other* quantum numbers:

$$\hat{P}_n = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} |n\,\ell\,m\rangle\langle n\,\ell\,m| \,. \tag{2}$$

In the following consider the hydrogen atom wavefunction

$$|\psi\rangle = \frac{1}{2}\left(|2\,1\,0\rangle + \sqrt{2}|2\,0\,0\rangle + |1\,0\,0\rangle\right) . \tag{3}$$

e) Consider a measurement of energy. Which values of n might be observed? Write down the projection operators associated with each possible result, and use them to calculate the probabilities of each outcome, and the result of the collapse of the wavefunction. Do these results agree with what you would have expected?

f) If you were to make a measurement of L_z instead, how would you define the projection operator(s) you need? Repeat the calculation of e) for this case.

Problem 13.3: Spin angular momentum operators. (20 points)

a) Demonstrate that the 2×2 spin matrices $\vec{\hat{S}} = (\hbar/2)\vec{\sigma}$ with $\vec{\sigma}$ the three Pauli matrices, which operate on spin-1/2 particles, satisfy the correct angular momentum commutation relations,

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z\,, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x\,, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y\,. \tag{4}$$

b) Consider the spin-1 case. There are three eigenvectors; let them be called

$$|s=1, m_s=1\rangle \equiv (1,0,0)\,, \quad |s=1, m_s=0\rangle \equiv (0,1,0)\,, \quad |s=1, m_s=-1\rangle \equiv (0,0,1)\,, \tag{5}$$

and construct the matrices for $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ for this spin. Hint: as with the spin-1/2 case, $\hat{S}_z$ can be deduced from the eigenvectors, and $\hat{S}_x$ and $\hat{S}_y$ can be deduced from the action of $\hat{S}_+$ and $\hat{S}_-$,

$$\hat{S}_\pm|s\ s_m\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}\,|s\ s_m \pm 1\rangle\,. \tag{6}$$

c) Verify that the matrices you constructed in part b) also satisfy the correct angular momentum commutation relations.

Problem 13.4: Spin of an electron. (20 points)

An electron is in the spin state

$$|\chi\rangle = A\begin{pmatrix} 3 \\ 4i \end{pmatrix} . \tag{7}$$

a) Determine the normalization constant A.

b) Calculate the expectation values of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$.

c) An observation of $\hat{S}_z$ is made. What are the possible values that may be observed, and with what probabilities?

d) Imagine we never made an $\hat{S}_z$ measurement and make an observation of $\hat{S}_y$ instead. What are the possible values that may be observed, and with what probabilities? To determine this, find the eigenvalues and eigenvectors of $\vec{S}_y$ in general, and then use them in this case.

Problem 13.5: Addition of two spin-1/2s. (Extra credit)

This extra credit problem cancels out previously missed HW points, up to 3% of the HW total grade.

Consider a composite particle composed of a bound state of a spin-1/2 constituent, particle A, and another spin-1/2 constituent, particle B. (For example, particle A could be a proton and particle B an electron, and the composite would be a hydrogen atom.) For this problem we ignore all properties of the particles except their spin. (In the real world you have to deal with spin and position-space wavefunctions at the same time!)

a) Each of the particles has its own spin operators, $\hat{S}_x^A$, $\hat{S}_y^A$ and $\hat{S}_z^A$ for particle A, and $\hat{S}_x^B$, $\hat{S}_y^B$ and $\hat{S}_z^B$ for particle B, as well as separate spin quantum numbers, which we will label s_A, m_A for particle A and s_B, m_B for particle B. Including only spin, the basis vectors are of the form

$$|s_A\, m_A\ ;\ s_B\, m_B\rangle\,. \tag{8}$$

(Note that a single state must give a spin configuration for *both* particles.) The spin operators for particle A don't care about the quantum number for particle B, and vice versa. Given this, write down the action of $(\hat{S}^A)^2$, $(\hat{S}^B)^2$, $\hat{S}_z^A$ and $\hat{S}_z^B$ on a state of the form $|s_A\, m_A\ ;\ s_B\, m_B\rangle$. Note: for now, don't assume that the particles are spin-1/2 yet, so s_A and s_B are still arbitrary.

b) Now let the particles both be spin-1/2: $s_A = s_B = 1/2$. Because these numbers will never change, we get bored with writing them over and over, and we'll just write the states of the system as

$$|m_A\, m_B\rangle\,. \tag{9}$$

How many different such states are there for two spin-1/2 particles? List them all. If you like, you can follow a standard notation and use an up-arrow $\uparrow$ to denote $m = 1/2$ and a down-arrow $\downarrow$ for $m = -1/2$, so one state would be

$$|\frac{1}{2}\,\frac{1}{2}\rangle \qquad \text{or equivalently} \qquad |\uparrow\uparrow\rangle\,. \tag{10}$$

We will take these states to be a basis for our spin-only Hilbert space.

c) When particles A and B are combined into a composite particle, it is often useful to ask about the *total* spin of the composite. In particular, interactions with other particles and fields will often depend on the total spin, so we would like to determine that.

Define the total spin vector operator,

$$\vec{\hat{S}} = \vec{\hat{S^A}} + \vec{\hat{S^B}}\,. \tag{11}$$

Are the states you listed in part b) eigenstates of $\hat{S}_z$? If yes, give the value of the quantum number associated to $\hat{S}_z$ for each, which we will just call M, and give a general relation between M, m_A and m_B. If no, explain why not and indicate what states would be eigenstates.

d) Once we know the M quantum number of each state, we would also like to know the total spin quantum number, S (the spin analog of the orbital angular momentum quantum number ℓ). To do this, we will need the operator $\hat{S}^2$, which squaring out equation (**??**) becomes

$$\hat{S}^2 = (\hat{S}^A)^2 + (\hat{S}^B)^2 + 2\vec{\hat{S^A}} \cdot \vec{\hat{S^B}}\,. \tag{12}$$

First, prove that $\vec{\hat{S^A}} \cdot \vec{\hat{S^B}}$ can be expressed as

$$\vec{\hat{S^A}} \cdot \vec{\hat{S^B}} = \frac{1}{2}(\hat{S}^A_+\hat{S}^B_- + \hat{S}^A_-\hat{S}^B_+) + \hat{S}^A_z\hat{S}^B_z\,, \tag{13}$$

where the raising and lowering operators for each particle are defined in the usual way.

Now, take the state from part b) with the highest M and act on it with $\hat{S}^2$. What is the value of S for this state? Could you have predicted this from the fact that this was the largest value of M?

e) How many states are there for a particle of the spin S you found in part d)? Act on the state with highest M with the lowering operator $\hat{S}_- = \hat{S}^A_- + \hat{S}^B_-$, and use equation (**??**), substituting for s and m_s either S and M, s_A and m_A, or s_B and m_B as the case may be. What are the S and M values of the state you get, and what is this state in terms of the original states from part b)?

Repeat the process until you have run out of states for that value of spin S. (Hint: there aren't too many more.)

f) Do the states you found in part e) span the whole Hilbert space? If not, how many states were left out? Explain what these state(s) are in terms of the original states from part b). What values of M do they have? Assume that all of these remaining state(s) fit into a single other spin, S'. What value must S' take?

In summary, what values of spin S and S' have you found in combining two spin-1/2 states?

You have now written two different bases for your Hilbert space: in terms of the spins of the individual particles, and in terms of the spins of the composite. Other particles and fields can interact with either the constituents or the whole bound state, so both are important!