I. Some basics about operators:

1. An operator $\hat{Q}$ operates on functions, giving back other functions, $\hat{Q}\ f(x) = g(x)$

Fill in the blanks below!

$\langle\psi|\hat{Q}|\phi\rangle = \langle\psi|\hat{Q}\phi\rangle\ = \int d\,??$

$\langle\hat{Q}\psi|\phi\rangle = \int d\,??$

Make up a simple operator $\hat{Q}$ where (at least for some ψ(x) or φ(x)), $\langle\psi|\hat{Q}\phi\rangle \neq \langle\hat{Q}\psi|\phi\rangle$

2. Now suppose that $\hat{Q}$ is Hermitian, which means $\langle\psi|\hat{Q}\phi\rangle = \langle\hat{Q}\psi|\phi\rangle$ for all ψ(x) and φ(x).

Let's think about the eigenvalues of $\hat{Q}$:

$$\hat{Q}\ \underbrace{f(x)}_{\text{eigenfunction}} = \underbrace{q}_{\text{eigenvalue*}}\ \underbrace{f(x)}_{\text{eigenfuntion, back again!}}$$

We say "f(x) is an eigenfunction of operator $\hat{Q}$, with eigenvalue q." This q is a number - but in principle you would think it might be complex. Let's see!

$\langle\hat{Q}\rangle \equiv \langle f|\hat{Q}f\rangle Q = \langle f|\hat{Q}|f\rangle = \int d\,??$

If Q is hermitian this equals $\langle\hat{Q}f|f\rangle = \int d\,??$

Conclusion: Can q be imaginary? complex?

II. <u>The quantum mouse and operators</u>

SETUP: Consider a quantum object (a "quantum lab mouse") and some new properties we can measure. E.g., suppose "quantum weight", **W**, is a hermitian operator.
The corresponding physical measurement is "put the mouse on a quantum scale".
Interestingly, this scale reads either 1 (skinny mice) or 10 (heavy ones), but nothing else (!)

W|▬> = |▬> , W|⬭> = 10|⬭> .

(Because I'm no artist, I will simplify the icons for skinny and heavy mice to |▬ > and |⬭> respectively) Note! Being skinny or heavy is *normal.*
In fact, let us assume it is *ortho*normal (and complete)
Stare at these two eigen-equations and make sure you, and your group, understand the notation: which symbols are the eigenvectors here, what are the eigenvalues, in those equations?

What can you say about, e.g. <▬ |⬭> ? (Explain)

"Quantum happiness", **H**, is also Hermitian. The corresponding physical measurement is "look at the mouse's expression", yielding either a smile (happiness = +1), or frown (happiness = -1)

$\hat{\mathbf{H}}$ |☺> = |☺> , but

$\hat{\mathbf{H}}$ |☹> = -|☹>

Note! Being happy or sad is *normal*. (Again, orthonormal, and complete)
Again, make sure you follow this notation: what are the possible outcomes of a measurement of H? Which symbols are the eigenvectors here, what are the eigenvalues?
(Note that "H" in this Tutorial has nothing to do with a Hamiltonian, sorry for the common letter)

What is < ☺ | ☹> ?

Lastly, let us suppose that $|\text{▬}\rangle = \frac{1}{\sqrt{5}}|\text{☺}\rangle + \frac{2}{\sqrt{5}}|\text{☹}\rangle$
I guess this means skinny quantum-mice are a little *stressed*. (Do you see this from the equation?)
Let's see some consequences of the above assumptions!

A) In this quantum world, suppose I give you a mouse, you measure W and get eigenvalue 1.
i) What quantum state is the mouse in? Is there any ambiguity at this point?

ii) If you now remeasure W on this state, what results can you get, with what probabilities?
After this second measurement of W, what state will you be in? (Is there any ambiguity at this point?)

iii) Following the above, what is the probability that a subsequent measurement of H will yield a result of -1, i.e. “unhappy”? Explain.

B)i) Use orthonormality and completeness to expand the W=10 eigenstate in the "happiness basis":

|⬭> = a |☺> + b |☹>

i.e. find the numerical constants a and b.
Is your answer unique? (If not, does it *matter*?)

ii) Suppose I give you a quantum mouse, and you measure H and find eigenvalue -1.
What state is it in after your measurement? (Is there any ambiguity at this point?)

iii) What is the probability that a subsequent measurement of W will yield a result of 1, i.e. “light”?
(Don’t intuit answers at this point, work it out from the postulates of quantum mechanics!)

C) If I have an ensemble of happy mice, and measure W on all of them, do they each get the same result? Explain.

What is the *average result* of measuring weight of this ensemble of happy mice?

More on Operators and Eigenvalues

Let **P** be the "feed a quantum pizza" operator. This curious operator does the following:

$$\hat{\mathbf{P}}\,|\text{—}> \;=\; |\bigcirc> \quad \text{, but}$$

$$\hat{\mathbf{P}}\,|\bigcirc> \;=\; 0$$

This means pizza make a skinny mouse heavy, but *kills* the heavy mice. (it gives back 0, nothing)

D) Are skinny mice eigenstates of **P**?
Are heavy mice?

E) Show that **P** is NOT a Hermitian operator

Can you observe mice eating pizza in this quantum world? Explain.

F) What would $\mathbf{P}^{\dagger}$ do to a heavy mouse? What would it do to a skinny mouse?
This is tricky, work it out!

Based on the above, in the spirit of this Tutorial, make up a plausible name for the $\mathbf{P}^{\dagger}$ operator.

Simultaneous measurements:

G) i) Do **H** and **W** commute?
(Again, tricky. If they DO commute, then ***HW*** *gives the same result as* ***WH*** *for any state. A single counterexample will disprove it! Consider operating on, say, a skinny state…)*

ii) Based on the above, does measuring the quantum weight of a mouse affect the outcome of a future measurement of its quantum happiness? (Vice versa?)

H) Given a mouse in the +1 happiness state, if you then measure weight, what is the probability that you will measure 1? (Colloquially, we might phrase this "what is the probability that a happy mouse is skinny?")

Time evolution

Now suppose that the mouse Hamiltonian commutes with W (One way to think of this would be if the *energy* of a mouse is proportional to its measured weight)
Suppose I give you a mouse which I have carefully prepared. I know what state it is in, but I don't tell you. It might be in a superposition of skinny and heavy states!

I) You can measure weight at any time you like. Does the probability that your measurement will yield "10" (i.e. heavy) depend on the amount of time that you wait before measuring? Explain.

J) Suppose instead that you chose NOT to measure W but instead to measure H. Does the probability that your measurement will yield "+1" (i.e, that it's "happy) depend on the amount of time that you wait before measuring? Explain.
(*You might consider the simple case that I handed you a particle prepared in a happy state at t=0.)*