Statistics and the Wavefunction

Let's review some elementary statistics about random variables that can assume *discrete* values. Suppose we make many repeated measurements of a random discrete variable called x. An example of x is the mass, rounded to nearest kg, or the height, rounded to the nearest cm, of a randomly-chosen adult male.

We label the possible results of the measurements with an index i. For instance, for heights of adults males, we might have x_1=25 cm, x_2 = 26 cm, etc (nobody is shorter than 25 cm). The list $\{x_1, x_2, \ldots x_i, \ldots\}$ is the called the *spectrum* of possible measurement results. Notice that x_i is **not** the result of the i^{th} trial (the common notation in statistics books). Rather, x_i is the i^{th} possible result of a measurement in the list of all possible results.

N = total # of measurements.
n_i = # times that the result x_i was found among the N measurements.

Note that $N = \sum_i n_i$ where the sum is over the *spectrum* of possible results, *not* over the N different trials.

In the limit of large N (which we will almost always assume), then the *probability* of a particular result x_i is $P_i = \frac{n_i}{N}$ = (fraction of the trials that resulted in x_i).

The average of many repeated measurements of x = expectation value of x =

$$\langle x \rangle = \frac{\text{sum of results of all trails}}{\text{nbr of trials}} = \frac{\sum_i n_i\, x_i}{N} = \sum_i \left(\frac{n_i}{N}\right) x_i = \sum_i P_i\, x_i$$

The average value of x is the weighted sum of all possible values of x: $\langle x \rangle = \sum_i P_i\, x_i$

We can generalize this result to any function of x:

$$\langle x^2 \rangle = \sum_i P_i\, x_i^{\,2}\;, \qquad \langle f(x) \rangle = \sum_i P_i \cdot f(x_i)$$

The brackets $\langle \ldots \rangle$ means "average over many trials".

A measure of the expected spread in measurements of x is the standard deviation σ, defined as "the rms average of the deviation from the mean". "rms" = root-mean-square = take the square, average that, then square-root that.

$$\sigma = \sqrt{\sigma^2} = \sqrt{\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle}$$ σ^2 is called the *variance*.

Let us disassemble and reassemble: The *deviation* from the mean of any particular result x is $\Delta x = x - \langle x \rangle$. The deviation from the mean is just as likely to be positive as negative, so if we average the deviation from the mean, we get zero: $\langle \Delta x \rangle = \langle (x - \langle x \rangle) \rangle = 0$. To get the average *size* of Δx, we will square it first, before taking the average, and then later, square-root it: $\sigma = \sqrt{\langle (\Delta x)^2 \rangle} = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$

It is not hard to show that another way to write this is $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

Proof:

$$\sigma^2 = \left\langle (x - \langle x \rangle)^2 \right\rangle = \sum_i (x_i - \langle x \rangle)^2 \cdot P_i = \sum_i \left(x_i{}^2 - 2x_i \langle x \rangle + \langle x \rangle^2 \right) \cdot P_i$$

$$\sigma^2 = \underbrace{\sum_i x_i{}^2 \cdot P_i}_{\langle x^2 \rangle} - 2\langle x \rangle \underbrace{\sum_i x_i \cdot P_i}_{\langle x \rangle} + \langle x \rangle^2 \underbrace{\sum_i P_i}_{1} = \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Now we make the transition from thinking about *discrete* values of x (say x = 1, 2, 3, ...) to a *continuous* distribution (say, x any real number).

We define a *probability density* $\rho(x)$ as

$\rho(x)\,dx$ = Prob(randomly chosen x lies in the range $x \rightarrow x+dx$)

In switching from discrete x to continuous x, we make the following transitions:

$$P_i \rightarrow \rho(x_i)\,dx$$

$$\sum_i P_i = 1 \rightarrow \int_{-\infty}^{+\infty} \rho(x)\,dx = 1$$

$$\langle x \rangle = \sum_i x_i P_i \rightarrow \langle x \rangle = \int_{-\infty}^{+\infty} x \cdot \rho(x)\,dx$$

From Postulate 3, we make the identification $\left|\Psi(x,t)\right|^2 dx \;=\; \rho(x)\,dx$ and so we have
$\langle x \rangle = \int_{-\infty}^{+\infty} x \cdot \left|\Psi(x)\right|^2 dx$ or more generally , for any function f = f(x), we have

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x) \cdot \left|\Psi(x)\right|^2 dx$$