University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#12
due Wed, Nov 18, 2PM at start of class

1. Vectors (Total: 20 pts)

   a) (Griffiths, Problem A.1) Consider the ordinary vectors in 3D ($a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$), with complex components. For each of the following three subsets find out whether or not it constitutes a vector space. If so, what is the dimension of the vector space? If not, why is it not a vector space?
      (i) The subset of all vectors with $a_z = 0$.
      (ii) The subset of all vectors whose z-component is 1.
      (iii) The subset of all vectors whose components are all equal.

   b) Does the subset of all 2×2 matrices form a vector space? Assume the usual rules for matrix addition and multiplication by a scalar, namely:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}, \quad \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

   If it does not form a vector space, why not? If it does form a vector space, state the dimensionality and give an example of a set of basis vectors.

   c) Does the set of all functions $f(x)$ defined on the range $0 < x < 1$ that vanish at $x = 0$ and $x = 1$ form a vector space? If so, state its dimensionality. If not, why not?

2. Properties of the harmonic oscillator (Total: 20 pts)
   The ladder operators are defined as

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) \quad \text{and} \quad \hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})$$

   a) Use the relations for the Hamiltonian $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + 1/2) = \hbar\omega(\hat{a}_-\hat{a}_+ - 1/2)$ and $E_n = (n + 1/2)\hbar\omega$ to show

$$\hat{a}_-\hat{a}_+\chi_n = (n+1)\chi_n \quad \text{and} \quad \hat{a}_+\hat{a}_-\chi_n = n\chi_n$$

   where χ_n is a normalized wave function.

   b) Show that for any square-integrable functions $f(x)$ and $g(x)$

$$\int_{-\infty}^{\infty} f^*(x)(\hat{a}_\pm g(x))dx = \int_{-\infty}^{\infty} (\hat{a}_\mp f(x))^* g(x))dx$$

   c) The ladder operators must take one stationary state to the next, times an overall constant: $\hat{a}_+\chi_n = c_n\chi_{n+1}$ and $\hat{a}_-\chi_n = d_n\chi_{n-1}$, where c_n and d_n are constants to be determined. Consider the expression $\int_{-\infty}^{\infty} (\hat{a}_+\chi_n)^*(\hat{a}_+\chi_n)dx$. Evaluate it using the results from parts a) and b), to solve for c_n. Now consider $\int\limits_{-\infty}^{\infty} (\hat{a}_-\chi_n)^*(\hat{a}_-\chi_n)dx$ and do something similar to solve for d_n. You should get $\hat{a}_+\chi_n = \sqrt{n+1}\chi_{n+1}$ and $\hat{a}_-\chi_n = \sqrt{n}\chi_{n-1}$.

3. Expectation values in the harmonic oscillator (Total: 20 pts)

   a) Find an expression for the operators $\hat{x}$ and $\hat{p}$ in terms of the ladder operators $\hat{a}_+$ and $\hat{a}_-$ as well as constants.

   b) Calculate $< \hat{x} >$, $< \hat{p} >$, $< \hat{x}^2 >$ and $< \hat{p}^2 >$ in the nth stationary state using the expressions from part a). (Hint: You don't ever need to write out the functional form of the χ_n if you use results from the previous problem.)

   c) How must $< \hat{H} >$ be related to the expectation values you calculated in the previous part? Check that this relationship works given what you know $< \hat{H} >$ must be for a stationary state. How much do the kinetic and potential energies each contribute to the total expectation value $< \hat{H} >$?