I: Hydrogen atoms, angular momenta, and probabilities

Ignoring spin (for now), an electron is known to be in a hydrogen atom state given by

$$\psi(t=0) = \sqrt{\frac{1}{6}} R_{10} Y_0^0 + \sqrt{\frac{1}{6}} R_{21} Y_1^1 + c R_{32} Y_2^1$$

A. Pick a value of c which normalizes the wavefunction. Is the value unique? (Why/why not?)

B. What possible outcomes (with what associated probabilities) are there for a measurement of energy?

C. Does your answer above depend on the time you wait before measuring energy?

D. Consider the state given by $\hat{H}\psi$. Is it in any sense the "outcome" of a physical measurement of energy on our state ψ?

✓ Check your results with a tutorial instructor.

Continuing with the previous page, with (still)

$$\psi(t=0) = \sqrt{\frac{1}{6}} R_{10} Y_0^0 + \sqrt{\frac{1}{6}} R_{21} Y_1^1 + c R_{32} Y_2^1$$

E. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{L}^2$?

F. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{L}_z$?

G. Does your answer above depend on the time you wait before measuring $\hat{L}_z$?

H. Suppose you start in the state (from the previous page)

$$\psi(t=0) = \sqrt{\frac{1}{6}} R_{10} Y_0^0 + \sqrt{\frac{1}{6}} R_{21} Y_1^1 + c R_{32} Y_2^1$$

and measure $\hat{L}_z$, and find a value of $+\hbar$. Write a properly normalized wave function for this particle immediately after this measurement.

Is the total angular momentum well-defined? (Why/why not?)

Is the z-component of angular momentum well-defined? (Why/why not?)

Is the state of your particle well-defined? (Why/why not?)

I. Suppose you take the state you found in part H, and then you measured $\hat{L}^2$, and found a value of $6\hbar^2$. What is the (normalized) state of the electron now?

Has this final measurement affected the z-component of angular momentum? Why/why not?

At this moment, suppose you made a measurement of the x-component of the angular momentum. What are the possible values you might obtain? (You do *NOT* need to find their corresponding probabilities!)

II: Spin

Spin is an intrinsic form of angular momentum. In quantum mechanics, it has operators which mirror the angular momentum operators we developed in our study of central potentials. Instead of $\vec{\hat{L}}$ which we used for orbital angular momentum, we will use $\vec{\hat{S}}$ for spin angular momentum, however, there are a lot of similarities. We can call the components of the spin operator, $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. These operators obey commutation relations similar to the components of the $\hat{L}$ operator:

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z\,, \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x\,, \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$$

Just as with orbital angular momentum, we will choose to work in the basis vectors of the $\hat{S}_z$ and $\hat{S}^2$ operators. The eigenvalues and eigenvectors of these operators should also be familiar. The main difference is that while our orbital angular momentum quantum number, ℓ, could take many different values in general, the equivalent spin quantum number, s is an intrinsic property of the particle under study. In this tutorial, we will study electrons which all have a spin quantum number of $\frac{1}{2}$. Thus, if we call our basis vectors, χ_+ and χ_-, we have:

$$\hat{S}_z\chi_+ = \tfrac{1}{2}\hbar\chi_+ \qquad \hat{S}^2\chi_+ = \tfrac{1}{2}\tfrac{3}{2}\hbar^2\chi_+$$
$$\hat{S}_z\chi_- = -\tfrac{1}{2}\hbar\chi_- \qquad \hat{S}^2\chi_- = \tfrac{1}{2}\tfrac{3}{2}\hbar^2\chi_-$$

Now, let's try out a situation where an electron is known to be in the spin angular momentum state, χ_+ (we will neglect time dependence for now).

A. What is the value of $\hat{S}^2$ for this electron?

B. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{S}_z$?

What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{S}_x$?

Is the x-component of spin angular momentum well-defined for this electron? (How about the y-component?)

C. If you started with the state χ_+, and measured $\hat{S}_z$ several times in a row, what might you measure?

Now, suppose you measured $\hat{S}_x$ and got the result $-\hbar/2$. What is the state of the electron immediately after the measurement?

D. Following the previous part, if you now remeasured $\hat{S}_z$, what do you expect? Explain.

III: Combined Wave Functions

An electron in a hydrogen atom occupies the combined spin and position state

$$\psi(t=0) = R_{21}\left(\sqrt{\frac{1}{6}}Y_1^0\chi_+ + i\sqrt{\frac{5}{6}}Y_1^1\chi_-\right),$$

where $\chi_\pm$ are the usual "spin up" or "spin down" states of the electron.

A. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{L}^2$?

B. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{L}_z$?

C. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{S}^2$?

D. What possible outcomes (with what associated probabilities) are there for a measurement of $\hat{S}_z$?