Physics 3220 – Quantum Mechanics 1 – Fall 2008
Problem Set #2

Due Wednesday, September 3 at 2pm

Problem 2.1: Fun with statistics. (20 points)

To do this problem, while you may use a calculator, spreadsheet or mathematical software to do simple algebraic operations like adding and dividing, don't use any built-in statistics functions.

A box contains 18 small items, of various lengths. The distribution of lengths in cm of the set of objects is

$$3, 3, 3, 3, 4, 6, 6, 6, 8, 8, 8, 8, 8, 9, 9, 9, 11, 11, 11. \quad (1)$$

a) What is the probability that an object chosen at random from the box will have length 8 cm, assuming there is an equal probability of selecting any one object?

b) What is the average length $\langle L \rangle$ (also called the *expectation value of length*) of an object in the box?

c) What is the probability that an object chosen at random from the box will have length $\langle L \rangle$, assuming there is an equal probability of selecting any one object?

d) What is the average of the square of the length $\langle L^2 \rangle$ of an object in the box?

e) Use your previous results to compute the standard deviation σ for the length of an item drawn from the box.

f) What is the probability that an object chosen at random from the box will have length in the range $\langle L \rangle \pm \sigma$, assuming there is an equal probability of selecting any one object? Based on what you know about standard deviations, does this seem reasonable?

This problem is to give you practice and intuition with probability distributions, and to think about the differences between probabilities on the one hand, and averages/expectation values on the other. Quantum mechanics forces us to think in probabilistic terms, so we'd better get used to it!

Problem 2.2: Probability density. (20 points)

Consider a mass m on a spring with spring constant k, oscillating in simple harmonic fashion about $x = 0$ between the extremes $x = -A$ and $x = A$. The motion is entirely classical.

a) What is the probability density $\rho(x)$ for the mass's position? In other words, find the function $\rho(x)$ such that $\rho(x)dx$ is the probability of observing the mass between positions x and $x + dx$ when observed at a random moment in time. (Griffiths examines a similar problem in Example 1.1.)

b) Verify that your result integrates to a total probability of 1.

c) Calculate $\langle x \rangle$ and $\langle x^2 \rangle$, the average values (expectation values) of x and x^2, respectively, and write down σ_x, the standard deviation in x. (If you encounter integrals you don't know how to evaluate, look them up in a table or in a mathematical software package.) How could you have predicted the result for $\langle x \rangle$ without having to calculate?

d) Sketch $\rho(x)$ between $x = -A$ and $x = A$. Where is it smallest, and where is it largest? Explain why these results are sensible.

When we are interested in probabilities for a continuous variable (like the position x) instead of a discrete one, we must use probability densities. This problem gets practice with those densities in the context of a classical system. Note that in classical physics, we needed to average over time to get a probability distribution; in quantum mechanics, we will have probability distributions at a single moment in time!

Problem 2.3: The Bohr model. (20 points)

The Bohr model of the hydrogen atom, formulated in 1911, was a stepping stone in between classical and quantum physics. Although not totally correct, it incorporated some features of the atom that would prove to exist in the full quantum-mechanical treatment.

In this review problem, look at your notes from Phys2130 or Phys2170, or some other source on the Bohr model. This material was not covered in class and is not in Griffiths. Do your best to understand the model and write the answers in your own words.

a) The Bohr model can be thought of as a set of *classical physics* assumptions about the hydrogen atom, combined with additional assumptions that are fundamentally *non-classical*, meaning they do not follow in any way from classical physics. What are the classical assumptions of the model, and what are the non-classical assumptions? These should just be one or two sentences each.

b) Starting with these assumptions, show how to derive the Bohr expressions for both the allowed radii and allowed energies of the hydrogen atom. For simplicity refer to the constant that appears in Coulomb's law as k instead of $1/4\pi\epsilon_0$.

c) Calculate the *classical* frequency of the motion of the Bohr model. Show that this is the same result as the frequency of a photon emitted when the atom transitions from the state with quantum number n to the state with quantum number $n - 1$, in the limit where n is large. This is called the "Correspondence principle" — the quantum result agrees with a classical result, but only in the "classical limit", which here is the limit of large quantum number.

It may seem unnecessary to study an early quantum model that is not entirely correct, but we do it for a couple reasons. It's historically interesting because it paints a picture of a stepping stone that helped people manage to make the difficult mental journey from classical physics to quantum mechanics. And it also captures some essential features of quantum mechanics in a considerably simpler context, and hence functions as a nice warm-up for the real thing — a kind of "baby quantum".

Problem 2.4: A wavefunction. (20 points)

Consider the following example of a quantum mechanical wavefunction for a particle of mass m moving in one dimension,

$$\Psi(x,t) = Ae^{-\alpha(x^2+i\hbar t/m)}, \tag{2}$$

where α and A are constants.

a) Normalize the wavefunction — that is, find a value of A for which $\int_{-\infty}^{\infty} |\Psi|^2\, dx = 1$. A useful integral is

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}\,. \tag{3}$$

b) At what value of x is there equal probability for the particle to be either to the left or to the right of that x? Explain.

c) For the special value $\alpha = 1/2$, numerically estimate to two significant digits the probability of observing the particle between $x = -1$ and $x = +1$. You will need mathematical software (or a good table of integrals) to do this part.

d) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)\,. \tag{4}$$

For the wavefunction given, what must the potential $V(x,t)$ be in order for $\Psi(x,t)$ to satisfy the Schrödinger equation? What classical system has this kind of potential?

Here we get practice normalizing wavefunctions and using them to predict the probability of making certain observations — doing real quantum mechanics! Although you are just given this wavefunction without having solved for it, the last part helps you find a clue what kind of system this wavefunction might occur for — bear in mind that any given wavefunction only works (where "works" means solves the Schrödinger equation) for certain potentials, and different physical systems will call for different wavefunctions.