University of Colorado, Department of Physics
PHYS3220, Spring 11, HW#11
due Fri, Apr 15, in class
- Please note: Late Homeworks will not be accepted -

1. Angular momentum operators
   The x, y, and z components of the orbital angular momentum operator expressed in spherical coordinates are:

$$\begin{aligned}\hat{L}_x &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\\ \hat{L}_y &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)\\ \hat{L}_z &= -i\hbar\frac{\partial}{\partial\phi}\end{aligned}$$

   a) Prove the above expression for $\hat{L}_z$ by showing it is equivalent to the expression for $\hat{L}_z$ in Cartesian coordinates.
   (*Hint: Work backwards from the desired result and use the chain rule.*)
   b) Find and simplify the generalized uncertainty relation between $\hat{L}_z$ and the angle ϕ. What does the result remind you of?
   c) Show that

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \pm\hbar\exp(\pm i\phi)\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

   and

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

2. Commutators with angular momentum operators
   a) Work out the following commutators:

$$[\hat{L}_z, \hat{p}_x] = i\hbar\hat{p}_y, \qquad [\hat{L}_z, \hat{p}_y] = -i\hbar\hat{p}_x, \qquad [\hat{L}_z, \hat{p}_z] = 0,$$

   b) Use the results obtained in part a) to evaluate the commutator $[\hat{L}_z, \hat{p}^2]$.
   c) Use that $[\hat{L}_x, \hat{p}^2] = [\hat{L}_y, \hat{p}^2] = [\hat{L}_z, \hat{p}^2]$ (you do *not* have to show these relations) and the result obtained in part b) to show that the Hamiltonian $H = (p^2/2m) + V$ commutes with $\hat{L}^2$ and $\hat{L}_z$, provided that $V = V(r)$ depends only on r.

- Please note: There is another problem on the next page -

3. Superposition of orbital angular momentum eigenstates
   Consider a system which is initially in the state

$$\Psi(\theta,\phi) = \frac{1}{\sqrt{5}} Y_{1,-1}(\theta,\phi) + \sqrt{\frac{3}{5}} Y_{1,0}(\theta,\phi) + A Y_{1,1}(\theta,\phi)$$

   where A is a *real* number.
   ($Y_{l,m}$ are the spherical harmonics and as we have shown / will show in class the simultaneous eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$.)

   a) Find A such that the state is normalized. Is your answer unique? Explain.
   b) Find $< \Psi|\hat{L}_+|\Psi >$.
      (Hint: We have shown / will show in class, that $\hat{L}_\pm Y_{l,m} \propto Y_{l,m\pm 1}$. For the problem you need the proportionality factor. It is $\hat{L}_\pm Y_{l,m} = C^\pm_{l,m} Y_{l,m\pm 1}$ with $C^\pm_{l,m} = \hbar\sqrt{l(l+1) - m(m\pm 1)}$.)
   c) If $\hat{L}_z$ is measured what values will one obtain and with what probabilities? What is the expectation value of $\hat{L}_z$?
   d) Assume that we measure $\hat{L}_z$ and the result of the measurement is $-\hbar$. What is the state of the system *after* the measurement? Determine the product $\Delta L_x \Delta L_y$ *after* the measurement.