University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#11
due Wed, Nov 4, 2PM at start of class

1. Analytic solution of the harmonic oscillator (Total: 20 pts)
In this problem we go through the analytic solution of the time-independent Schrödinger equation for the harmonic oscillator

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \chi(x) = E\chi(x) \tag{1}$$

a) It is convenient to simplify the problem by introducing the two dimensionless variables $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ and $K = \frac{2E}{\hbar\omega}$ Show that the time-independent Schrödinger equation can be written as

$$\frac{d^2\chi(\xi)}{d\xi^2} = \left(\xi^2 - K\right)\chi(\xi)$$

b) Show that in the limit $|x| \to \infty$ the above equation can be approximated as

$$\frac{d^2\chi(\xi)}{d\xi^2} \approx \xi^2\chi(\xi)$$

and an approximate solution is given by

$$\chi(\xi) \approx A\exp(-\xi^2/2) + B\exp(+\xi^2/2)$$

What is the constraint needed to be sure that $\chi(\xi)$ can be normalized? (Don't actually try to normalize the function.)

c) The original equation turns out to simplify if we extract the asymptotic (large $|x|$) behavior of χ and solve for what's left. Accordingly, we define $H(\xi)$ by means of

$$\chi(\xi) = H(\xi)\exp(-\xi^2/2)\,.$$

Substitute in the time-independent Schrödinger equation (1) and show that it is equivalent to the so-called Hermite equation

$$\frac{d^2H(\xi)}{d\xi^2} - 2\xi\frac{dH(\xi)}{d\xi} + (K-1)H(\xi) = 0$$

d) The solutions are Hermite polynomials which can be written as series:

$$H(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$$

where the a_j are constants. Show that the result of part c) implies the following recursion relation for the constants:

$$a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)}a_j$$

e) Normalizability of the wave function implies that the series can't go on forever. Thus to ensure we can normalize our wave function, the series must stop at some point. Assume that there exists a value n such that an $a_n \neq 0$ but $a_{n+2} = 0$, and find the resulting constraint on the energy E_n associated with this wave function.

2. Properties of the harmonic oscillator (Total: 20 pts)
The ladder operators are defined as

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) \quad \text{and} \quad \hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})$$

a) Use the relations for the Hamiltonian $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + 1/2) = \hbar\omega(\hat{a}_-\hat{a}_+ - 1/2)$ and $E_n = (n+1/2)\hbar\omega$ to show

$$\hat{a}_-\hat{a}_+\chi_n = (n+1)\chi_n \quad \text{and} \quad \hat{a}_+\hat{a}_-\chi_n = n\chi_n$$

where χ_n is a normalized wave function.

b) Show that for any square-integrable functions $f(x)$ and $g(x)$

$$\int_{-\infty}^{\infty} f^*(x)(\hat{a}_\pm g(x))dx = \int_{-\infty}^{\infty} (\hat{a}_\mp f(x))^* g(x))dx$$

c) The ladder operators must take one stationary state to the next, times an overall constant: $\hat{a}_+\chi_n = c_n\chi_{n+1}$ and $\hat{a}_-\chi_n = d_n\chi_{n-1}$, where c_n and d_n are constants to be determined. Consider the expression $\int_{-\infty}^{\infty} (\hat{a}_+\chi_n)^*(\hat{a}_+\chi_n)dx$. Evaluate it using the results from parts a) and b), to solve for c_n. Now consider $\int_{-\infty}^{\infty} (\hat{a}_-\chi_n)^*(\hat{a}_-\chi_n)dx$ and do something similar to solve for d_n. You should get $\hat{a}_+\chi_n = \sqrt{n+1}\chi_{n+1}$ and $\hat{a}_-\chi_n = \sqrt{n}\chi_{n-1}$.

3. Expectation values in the harmonic oscillator (Total: 20 pts)

a) Find an expression for the operators $\hat{x}$ and $\hat{p}$ in terms of the ladder operators $\hat{a}_+$ and $\hat{a}_-$ as well as constants.

b) Calculate $< \hat{x} >$, $< \hat{p} >$, $< \hat{x}^2 >$ and $< \hat{p}^2 >$ in the nth stationary state using the expressions from part a). (Hint: You don't ever need to write out the functional form of the χ_n if you use results from the previous problem.)

c) How must $< \hat{H} >$ be related to the expectation values you calculated in the previous part? Check that this relationship works given what you know $< \hat{H} >$ must be for a stationary state. How much do the kinetic and potential energies each contribute to the total expectation value $< \hat{H} >$?