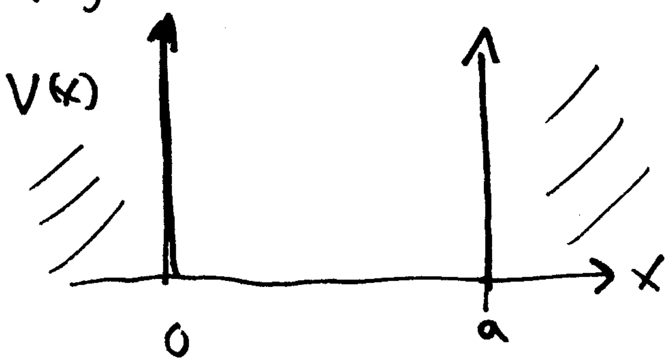
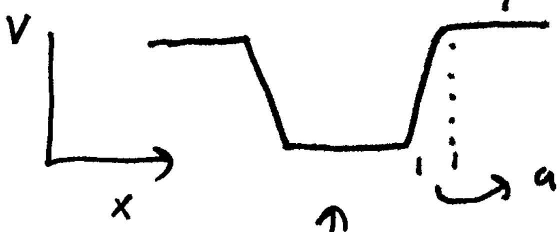
SJP QM 3220 ch. 2.8.

The Infinite Square Well : Let's make a choice for V(x) which is solvable + has some (approximate) physical relevance



$$V(x) = \begin{cases} 0 & \text{for } 0 < x < a \\ \infty & \text{elsewhere} \end{cases}$$

It's like the limit of



as this gets big (height)

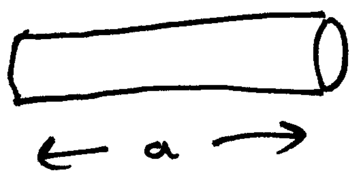
and this gets small. (distance where it rises)

classically,

$\vec{F} = -\dfrac{dV}{dx}$ is 0 in the middle (free)

but big at the edges. Like walls at the edges.

(Like an electron in a small length of wire :



free to move inside, but stuck, large force at ends prevents it from leaving.)

Given V(x), we want to find the special! stationary states

(+ then we can construct any physical state by some linear combination of those!)

Recall, we're looking for $u_n(x)$ such that

$\hat{H} u_n(x) = E_n u_n(x)$ (and then $\Psi_n(x,t) = u_n(x) e^{-i E_n t/\hbar}$.)

(or, dropping "n" for a sec) (and then at last → $\Psi_{general} = \sum_n c_n \Psi_n(x,t)$)

$$\boxed{-\frac{\hbar^2}{2m} u''(x) + V(x)\, u(x) = E\, u(x)}$$

Inside the well, $0<x<a$, $V(x)=0$, so in that region

$$\frac{d^2u}{dx^2} = -\frac{2mE}{\hbar^2} u(x) \equiv -k^2 u(x)$$

↳ Here, I simply defined $k \equiv \sqrt{\frac{2mE}{\hbar}}$

It's just shorthand, $k \equiv \sqrt{2mE}/\hbar$ or $E = \frac{\hbar^2}{2m} k^2$

(However, I have used the fact that $E>0$, you can't ever get $E < V_{min} = 0$! ← convince yourself!)

I know this 2^{nd} order ODE, + its general sol'n is

$$u(x) = A \sin kx + B \cos kx \quad \text{or} \quad \alpha e^{ikx} + \beta e^{-ikx}$$

But Postulate I says $u(x)$ should be continuous.

Now, outside $0<x<a$, $V(x) \to \infty$. This is unphysical, the particle can't be there! So $u(x) \equiv 0$ at $x=0$ & $x=a$.

This is a BOUNDARY CONDITION.

$u(x=0) = A \cdot 0 + B \cdot 1 = 0$ so $B = 0$. (required!)

$u(x=a) = A \sin Ka = 0$. But Now, I can't set $A=0$

'cause then $u(x) = 0$ and that's not normalized!

So $\sin Ka = 0$, i.e. $K = \frac{n\pi}{a}$ with $n = 1, 2, 3, \ldots$

Ah ha! The Boundary condition forced us to allow only certain K's. Call them $K_n = \frac{n\pi}{a}$.

Then since $E = \frac{\hbar^2 K^2}{2m}$, we get $E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$

(Note: negative n's just re-define "A", it's not really a different state, $A \sin(Kx)$ is the same function as $A\sin(-Kx)$)

(Note: $n=0$ no good, because $\sin(0x) = 0$ not normalized!)

Thus, our sol'ns, the "energy eigenstates" are

$$u_n(x) = A \sin(K_n x) = A \sin\left(\frac{n\pi x}{a}\right)$$

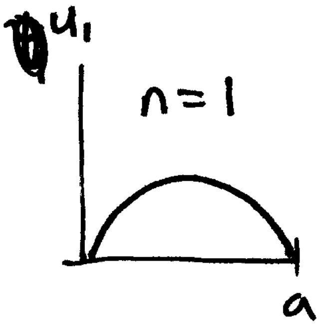
$n = 1, 2, 3, \ldots$
$(0 < x < a)$
(and 0 elsewhere)

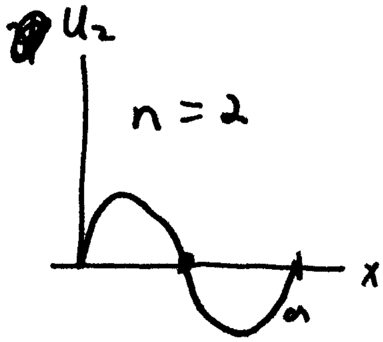
$$\text{and } E_n = n^2 \left(\frac{\pi^2 \hbar^2}{2ma^2}\right)$$

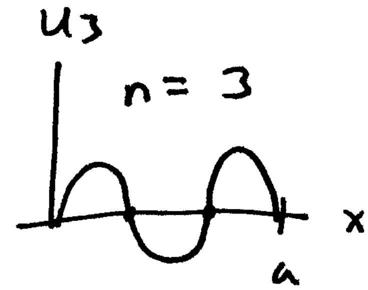
For normalization, $\int u_n(x)^2\, dx = 1 \Rightarrow |A|^2 \cdot \int_0^a \sin^2 \frac{n\pi x}{a}\, dx = 1$

convince yourself, $|A|^2 = \frac{2}{a}$ is required.

so $u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ (for $0<x<a$) ← [Note sign/phase out front is not physically important!] *





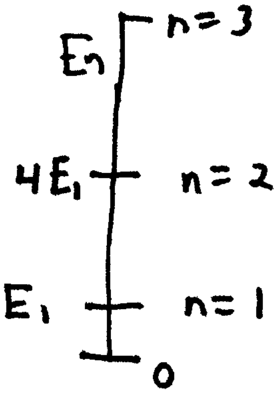


etc.

Energies are

$E_1, 4E_1, 9E_1, \ldots$

$\uparrow$ n=1, $\uparrow$ n=2, $\uparrow$ n=3



} these are excited states

← this is ground state

- Energy is quantized (due to boundary condition on u)
- Energies grow like n^2
- Lowest energy is NOT 0! (You cannot put an electron in a box and have it be at rest!)

---

* If you multiply $u_n(x)$ by $e^{i\theta}$, you have a wave fn with the same $|u(x)|^2$, it's physically indistinguishable.

So e.g. $u_n(x)$ and $-u_n(x)$ are not "different states".

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Key properties to note (many of which will be true for most potentials, not just this one!)

Energy eigenstates $u_n(x)$ are ...

- "even" or "odd" with respect to center of box (n=1 is even, n=2 is odd, this alternates)
- oscillatory, and higher energy $\Leftrightarrow$ more nodes or zero crossings. (Here, u_n has $(n-1)$ intermediate zeros)
- orthogonal: $\displaystyle\int_0^a u_n(x)\, u_m(x)\, dx = \delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$

  check this, if you don't know why $\sin\left(\frac{n\pi x}{a}\right)$ forms orthogonal states, work it out.

- complete: Dirichlet's theorem, the basis of "Fourier series" says ANY function $f(x)$ which is 0 at $x=0$ + $x=a$ can be written, uniquely,

$$f(x) = \sum_{n=1}^{\infty} c_n\, u_n(x)$$

← can always find the c_n's:

↳ recall, $\sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$

Fourier's trick finds those c_n's, given an $f(x)$!

If $f(x) = \sum_n c_n \psi_n(x)$ then do "the trick"...

$$f(x)\,\psi_m^*(x) = \sum_n c_n \psi_n(x)\,\psi_m^*(x)$$

so
$$\int_0^a f(x)\,\psi_m^*(x)\,dx = \sum_n c_n \underbrace{\int_0^a \psi_n(x)\,\psi_m^*(x)\,dx}_{\delta_{nm}}$$

δ_{nm}, (here ψ's are real so * isn't important in this example)

only one term, $n=m$, contributes!

$$c_m = \int_0^a f(x)\,\psi_m^*(x)\,dx$$

← this is how you figure out c_m's!

(The above features are quite general! Not just this problem!)

And last but not least, this was $u_n(x)$, but the time-dependent stationary states are $u_n(x)\,e^{-iE_nt/\hbar}$

$$\underbrace{\Psi_n(x,t)}_{} = \underbrace{\sqrt{\tfrac{2}{a}}\,\sin\left(\tfrac{n\pi x}{a}\right) e^{-i\,n^2\pi^2\hbar^2/2ma^2\left(\frac{t}{\hbar}\right)}}_{}$$

stationary state, full sol'n to "Schrod in a box" with definite energy E_n

this particular functional form is specific to "particle in box"

These are special, particular states., "eigenstates of $\hat{H}$"

The most general possible state of any "electron in a 1-D box", then, would be any linear combo of these:

$$\Psi(x,t) = \sum_n c_n \Psi_n(x,t) = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}$$

you can pick any c_n's you like (even complex!) and this will give you all possible physical states.

you might choose the c_n's to give a particular $\Psi(x, t=0)$ that you start with, using fourier's trick!

If $\Psi(x,0) = \sum_n c_n u_n(x)$ is given, find the c_n's,

& now the formula at top of page tells you $\Psi(x,t)$

So given initial conditions, we know the state at all times. That's the goal of physics!

For other potentials, game is same: find $u_n(x)$'s and corresponding E_n's, then form a linear combo at $t=0$ to match your starting state, + let it evolve, $\Psi(x,t) = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}$ ...

Last comment:

If $\Psi(x,t) = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}$

then <u>normalization</u> says $\int \Psi^* \Psi \, dx = 1$

But when you expand Ψ and Ψ, all cross terms vanish after integration, $\int u_n^*(x) u_m(x) \, dx = \delta_{nm}$

Leaving <u>only</u> the terms with $n=m$, which integrate simply.

Work it out: $\int \Psi^* \Psi \, dx = 1$ tells you

$$1 = \sum_m \sum_n \int c_m^* c_n \underbrace{u_m^*(x) u_n(x)}_{} e^{i(E_m - E_n)t/\hbar} \, dx$$

<u>collapses</u> the integral to δ_{mn}, (thus $e^{i\ldots} \to 1$)

$$= \sum_{mn} c_m^* c_n \delta_{mn}$$

$$= \sum_n |c_n|^2 \quad \leftarrow \text{convince yourself.}$$

<u>Similarly</u> (try it!) use $\hat{H} \psi_n = E_n \psi_n$

$$\langle H \rangle = \int \Psi^* \hat{H} \Psi = \sum_n E_n |c_n|^2$$

Interpretation (which we'll develop more formally)

$$\Psi(x,t) = \sum_n C_n\, U_n(x)\, e^{-iE_n t/\hbar} = \sum_n C_n\, \Psi_n(x,t)$$

says "the state Ψ is a linear combination of the special stationary states, the Ψ_n's."

the C_n's carry information,

from $\langle H \rangle = \sum_n |C_n|^2 E_n$ which looks like $\sum_n P_n E_n$

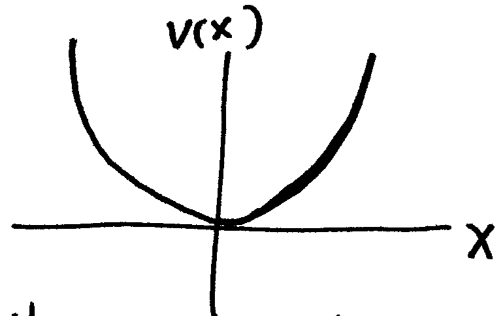
$|C_n|^2$ = Probability that this particle's Energy would be measured to be E_n.

(So $\sum_n |C_n|^2 = 1$ is just "conservation of probability".)

this is a central concept in QM, we'll come back to it often.

The Harmonic Oscillator:

Many objects experience a (classical) force $\vec{F} = -k\vec{x}$, Hooke's Law. This is the Harmonic oscillator, and $V(x) = \frac{1}{2}kx^2$



Note that almost any $V(x)$ with a minimum will look at least approximately like this, at least for small x, so it's physically a very common situation!

Classically, $x(t) = A\sin\omega t + B\cos\omega t$ for Harm. Osc

with $\omega \equiv \sqrt{\frac{k}{m}}$ (so $V = \frac{1}{2}m\omega^2 x^2$)

a defined quantity, but it tells you period $T = 2\pi/\omega$.

So let's consider the Quantum Harmonic oscillator

As before, separate x & t, and look for sol'ns to

$$-\frac{\hbar^2}{2m}u''(x) + \frac{1}{2}m\omega^2 x^2 u(x) = E\,u(x)$$

these u's will give us stationary states

(then, if you find such a u, $\Psi(x,t) = u(x)e^{-iEt/\hbar}$ gives time dependence)

Just like before, we will find that, if we insist $u(x) \xrightarrow[x\to\infty]{} 0$, (i.e. our "boundary condition") we will find only <u>certain</u> E's, called E_n, will work, + the corresponding $u_n(x)$ will be unique. So just like in the box, we'll have $\Psi_n(x,t)$'s, each n corresponding to a stationary state with discrete energy.

This diff e.q. is a 2nd order ODE but that "x^2" term makes it hard to solve. There are <u>tricks</u>. (* See aside, next page)

<u>Trick #1</u>: For <u>large</u> X, x^2 dominates, so

$$-\frac{\hbar^2}{2m} u''(x) = \left(E - \tfrac{1}{2}m\omega^2x^2\right) u \underset{\text{large } x}{\approx} -\tfrac{1}{2}m\omega^2 x^2 u$$

Sol'n (check, it's not familiar)

$$u(x) = A e^{-\frac{m\omega x^2}{2\hbar}} + B e^{+\frac{m\omega x^2}{2\hbar}}$$

[2 undetermined constants for 2nd order ODE ✓]

↑ <u>very</u> nasty as $x \to \infty$, toss as unphysical!

Now comes trick #1 part 2:

 – An aside

Another Method to Know of: Numerical!

How to solve $u'' = -\frac{2m}{\hbar^2}(E - V(x))\,u(x)$?

(without yet knowing E !)

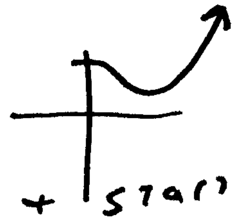
Even sol'ns: Guess an E (use a similar situation, or just units? For Harmonic oscillator, $\hbar\omega$ has units of energy, so maybe just try it as a starting guess, $\tilde{E}$.)

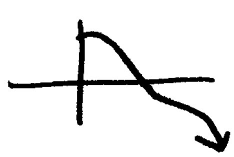
Let $u(0) = 1$, $u'(0) = 0$ ← flat, I'm looking for the even sol'n!

we'll normalize later! Pick a "step-size" ε (tiny!)

- Use S.E. to tell you $u''(0)$. $\left[u''(0) = -\frac{2m}{\hbar^2}(\tilde{E} - V(0))\,u(0)\right]$
- Use $u'(0) = \frac{u(\varepsilon) - u(0)}{\varepsilon}$ to compute $u(\varepsilon)$
- Use $u''(0) = \frac{u'(\varepsilon) - u'(0)}{\varepsilon}$ to compute $u'(\varepsilon)$

Now Repeat, stepping across x in steps of ε.

If u starts to blow up,  you need more curvature @ start, so raise $\tilde{E}$, + start again.

If u blows down  you need less curvature, lower $\tilde{E}$

For odd sol'n, start with $u(0)=0$, $u'(0)=1$

we'll normalize later!

(Same game....)

So basically you pick an E, numerically "shoot" from the origin, + keep fixing E till you get a sol'n which behaves well at large x.

In this way, you can plot/compute $u(x)$ and find the energies, for essentially any $V(x)$. (In this sense, you don't have to feel bad that only a few $V(x)$'s yield analytic sol'ns for $u_n(x)$ and E_n. All V's can be "solved" numerically)

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Assume (hope, wonder if?) $u(x) = h(x)\, e^{-\frac{m\omega^2 x^2}{2\hbar^2}}$

maybe $h(x)$ will be "simple". Indeed, use the (love this name!) Method of Frobenius : try

$h(x) \overset{?}{=} a_0 + a_1 x + a_2 x^2 + \ldots$ Plug in, see what happens ...

For now, I'm going to skip this algebraically tedious exercise, + skip to the punchline:

If you insist that $u(x)$ stays normalizable, you find that series for $h(x)$ needs to end, it must be a finite order polynomial. And that will happen if and only if E takes on special values, namely

$$E_n = (\underset{\substack{\uparrow \\ \text{integer.}}}{n} + \tfrac{1}{2})\hbar\omega$$

If E is this, then the polynomial is "n^{th} order", the $h_n(x)$ functions that come out are Hermite Polynomials. (See Griffiths p. 56.)

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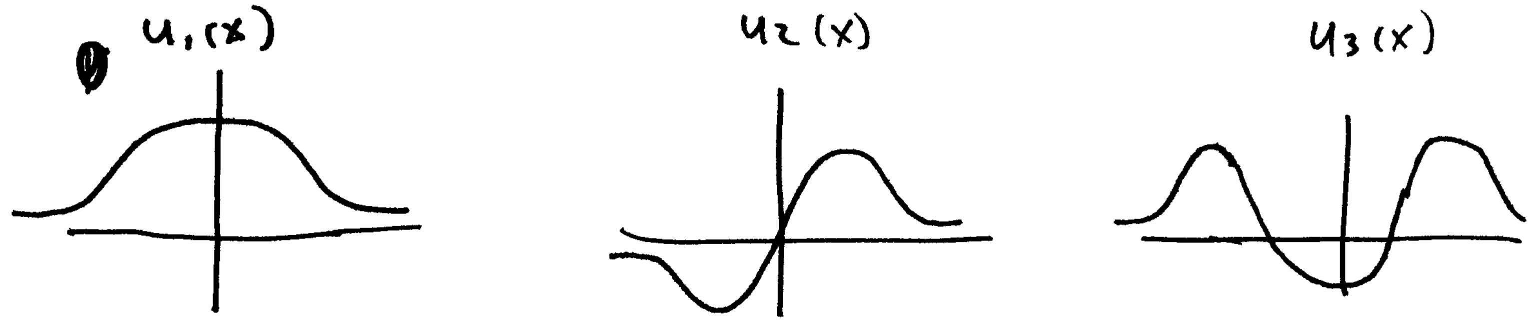


$$u_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}} e^{-\frac{m\omega^2 x^2}{2\hbar^2}}, \text{ with } E_1 = \tfrac{1}{2}\hbar\omega$$

Note, here $h_1(x) = 1$, basically, the stuff out front is just to normalize!

$$u_2(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{2}{\sqrt{8}} \sqrt{\frac{m\omega}{\hbar}}\, x\, e^{-m\omega^2 x^2/2\hbar^2}, \quad E_2$$

↳ there's my 1st order polynomial.

(You could just verify that, at least for these 1st 2

$$-\frac{\hbar^2}{2m} u_n''(x) + \tfrac{1}{2} m\omega^2 x^2 u_n(x) = E_n u_n(x).$$)

Remember, these give $u_n(x)$, time dependence is simple:

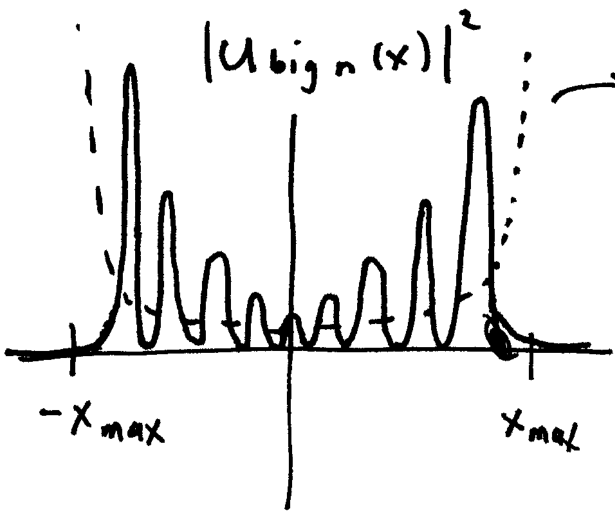
Stationary state $\Psi_n(x,t) = u_n(x)\, e^{-iE_n t/\hbar}$

You can then form any combination of these to build any "particle in a well" state, + the time dependence follows $\Psi = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}$.

SJP 3220 QM 2.21

observations: Like particle in box, $u_n(x)$ are

- even or odd
- oscillatory, with u_n having $n-1$ zero's
- orthogonal $\int_{-\infty}^{\infty} u_n(x)\, u_m(x)\, dx = \delta_{nm}$
- Complete: Any fn can be expanded via fourier's trick as $\sum_{n=1}^{\infty} c_n u_n(x)$ (if it vanishes properly at ∞, anyway)
- Energy of each u_n is discrete, grows like n here.
- Lowest energy is not 0, it's $\frac{1}{2}\hbar\omega$. (You cannot "stop" a mass on a quantum spring!)



→ this dashed curve is the classical $\rho(x)$ function. Note that classically E can be anything, but given E, you "turn around" at $\frac{1}{2}Kx_{max}^2 = E$

For small n, $|u_n(x)|^2$ doesn't look classical at all, for large n, it sort of does if you "average over short wiggles"!

<u>Trick #2</u> for solving the Harm. Osc.

This trick is way out there! I would never have come up with it (?) but it's cool, + turns out to be more general than you can imagine right now. It's the basis for a similar trick to understand angular momentum in 3-D, and then spin, + moving in to quantum field theory. So it's worth learning! It will also teach us some "operator methods" that are deeply central to QM. (So enjoy this "trickery" the math is fun!)

Here again is the 1-D Schrod eq'n we're studying

$$\frac{1}{2m}\left[p^2 + (m\omega x)^2\right]u = E\,u \qquad \left(\text{with } p = \frac{\hbar}{i}\frac{\partial}{\partial x} \text{ of course!}\right)$$

For <u>numbers</u>, $a^2+b^2 = (ia+b)(-ia+b)$,

so this tempts us to rewrite the equation on left

as $\frac{1}{2m}(ip + m\omega x)(-ip + m\omega x)$

But this isn't <u>right</u>, because $\hat{p}$ <u>operates</u> on any x's it hits ...

Let's define a new operator, two in fact:

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})$$

← I know the signs look funny, the name $\hat{a}_+$ will make more sense in a sec

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(+i\hat{p} + m\omega\hat{x})$$

Now try writing what I had before ?

$$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}(i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x})$$

$$= \frac{1}{2\hbar m\omega}\left(\hat{p}^2 + (m\omega\hat{x})^2 - im\omega(\hat{x}\hat{p} - \hat{p}\hat{x})\right)$$

For numbers, $xp - px = 0$, but not for operators!

We call $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ the commutator of A and B.

my expression has $[\hat{x}, \hat{p}]$ in it. what's that do?

Consider acting it on any old function f

$$[\hat{x}, \hat{p}]f(x) = (\hat{x}\hat{p} - \hat{p}\hat{x})f = x\frac{\hbar}{i}\frac{\partial}{\partial x}f - \underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}}_{\text{This operates on } (xf)!}xf$$

but $\frac{\partial}{\partial x}(xf) = f + x\frac{\partial f}{\partial x}$. The $x\frac{\partial f}{\partial x}$ term cancels, Leaving (please check this for yourself.

$[x, p] f = -\frac{\hbar}{i} f$. So we say $[\hat{x}, \hat{p}] = -\frac{\hbar}{i}$

$= i\hbar$

This is very important, $\boxed{[x, p] = i\hbar}$, it's <u>not</u> zero!

↳ (we'll talk much more about this!)

Back to our problem:

$$\hat{a}_- \hat{a}_+ = \frac{1}{2\hbar m\omega}\left(\hat{p}^2 + (m\omega\hat{x})^2\right) - \frac{im\omega}{2\hbar m\omega}\cdot(i\hbar)$$

$$= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}$$

or $$\hat{H} = \hbar\omega\left(\hat{a}_- \hat{a}_+ - \frac{1}{2}\right)$$

Now go through yourself + convince yourself that

$$\hat{H} = \hbar\omega\left(\underbrace{\hat{a}_+ \hat{a}_-}_{\text{opposite order}} + \frac{1}{2}\right)$$

↳ other sign

Thus, $$[\hat{a}_-, \hat{a}_+] = \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right) - \left(\frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\right)$$

$$= 1$$

(you could get this <u>directly</u> too)

<u>Why</u> have we done all this?? Here comes the crux #1...

SJP QM 3220 2.25

Suppose $u(x)$ is a sol'n to $\hat{H} u = E u$

i.e. suppose we <u>knew</u> the u's (which is what we're after!)

Consider $\hat{H}(\hat{a}_+ u) = \underbrace{\hbar\omega(a_+ a_- + \tfrac{1}{2})}_{\text{this is just } \hat{H}} (a_+ u)$

$$= \hbar\omega \left(a_+ a_- a_+ + \tfrac{1}{2} a_+\right) u$$

$$= \hbar\omega\, a_+ \left(a_- a_+ + \tfrac{1}{2}\right) u$$

$$= a_+ \hbar\omega \left(a_+ a_- + 1 + \tfrac{1}{2}\right) u$$

constants slide past a_+, right?

Here I used $[a_-, a_+] = 1$

so $a_- a_+ - a_+ a_- = 1$

$$= a_+ \left(\underbrace{\hbar\omega(a_+ a_- + \tfrac{1}{2})}_{\hat{H} \text{ again!}} + \hbar\omega\right) u$$

But $\hat{H} u = E u$!)

thus

$$\hat{H}(\hat{a}_+ u) \overset{*}{=} a_+ (E + \hbar\omega) u$$

or $\hat{H}(\hat{a}_+ u) = (E + \hbar\omega)(\hat{a}_+ u)$

so $\hat{a}_+ u$ is an eigenfunction of H with eigenvalue $E + \hbar\omega$

* Given u, I found a <u>new</u> <u>different</u> eigenfunction + eigenvalue.

SJP QM 3220 2.26

Now you try it, this time with

$\hat{H}(\hat{a}_- u)$. I claim you'll get $(E-\hbar\omega)(\hat{a}_- u)$

So now we see the <u>use</u> of $\hat{a}_+$ or $\hat{a}_-$. They ~~are~~ generate <u>new</u> sol'ns, different eigenvalues + eigenfunctions.

$\hat{a}_+$ = "raising operator" because it <u>raises</u> E by $\hbar\omega$

$\hat{a}_-$ = "lowering " " " " <u>lowers</u> E by $\hbar\omega$.

They are "ladder operators" because E changes in <u>steps</u> of $\hbar\omega$.

Now for crux #2...

If you apply $\underbrace{a_- a_- a_- \dots a_-}_{m} u$ you create a state with $E - m\hbar\omega$. You can always pile up enough a_-'s to get a negative energy... but there's a theorem that says E can <u>never</u> be lower than the minimum of $V(x)$, which is 0. ACK! This is bad... <u>unless</u> there is a <u>bottom</u> <u>state</u> u_0, such that $a_- u_0 = 0$

Note that 0 "solves" the S.Eq, it's just not interesting and from then on $a_- a_- u_0 = a_- 0 = 0 \dots$

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OK, so $\hat{a}_- u_0(x) = 0 \Rightarrow \frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right) u_0 = 0$

That's no problem to solve! See Griffiths, or just check

$u_0(x) = A e^{-m\omega x^2/2\hbar}$ works (for any A)

Pick A to normalize u (use $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$)

so $u_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar}$

This solves S.E, just plug it in! On the right, you'll see E_0 pop out, $E_0 = \frac{1}{2}\hbar\omega$.

$$\left(\begin{array}{l} \hat{H} u_0 = E_0 u_0 \Rightarrow \hbar\omega(\underset{\substack{\uparrow \\ \text{this "kills" } u_0}}{a_+ a_-} + \tfrac{1}{2}) u_0 = E_0 u_0 \\ \qquad\qquad\qquad \text{so } \tfrac{1}{2}\hbar\omega\, u_0 = E_0 u_0 \quad \checkmark \end{array}\right)$$

And now, $\hat{a}_+ u_0$ gives a state (call it u_1)

with $E_1 = E_0 + \hbar\omega = \frac{3}{2}\hbar\omega$. Though, you may have to normalize again!

$u_n(x) = A_n (a_+)^n u_0(x)$ with $E_n = (n+\frac{1}{2})\hbar\omega$

($\hookrightarrow$ must normalize each one separately)

Could we have <u>missed</u> some sol'ns this way?

Might there be a $\tilde{u}(x)$ which is <u>not</u> "discovered" by repeated applications of $\hat{a}_+$ to u_0? Fortunately no.

If there were, $(a_-)^n\,\tilde{u}(x)$ would still have to give 0 for <u>some</u> n, and thus there'd be a "lowest" $\tilde{u}$ ... but we <u>found</u> <u>the unique</u> lowest state already!

Griffiths has a cute trick to normalize the states, see p.47–48. He <u>also</u> has an elegant trick to prove $\int_{-\infty}^{\infty} u_m^*(x)\, u_n(x)\, dx = \delta_{mn}$ (see p.49)

But let's leave this for now:

Bottom line: By "Frobenius" or "ladder operators", we found all the stationary states $u_n(x)$, with $E_n = (n+\frac{1}{2})\hbar\omega$

(In general, potentials $V(x)$ give "bound states", with discrete energies.) But let's move on to a qualitatively different problem, where we're <u>not</u> bound!