Dirac Bra - Ket Notation

bracket or inner-product:

$\langle f|g\rangle = \int dx\, f^*(x)\, g(x)$ or $\int dV\, f^*(\vec{r})\, g(\vec{r})$ or $d\Omega(\theta,\varphi)$ ...

Which integral depends on configuration space of problem.

Key (defining) Properties of bracket:

- $\langle f|g\rangle^* = \langle g|f\rangle$
- $\langle f|c\cdot g\rangle = c\langle f|g\rangle$, $\langle cf|g\rangle = c^*\langle f|g\rangle$ ← c = constant
- $\langle\alpha|\left(b|\beta\rangle + c|\gamma\rangle\right) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$

Dirac proclaims: $\langle g|f\rangle = \langle g|$ next to $|f\rangle$
bracket = "bra" and "ket"

ket $|f\rangle$ represents vector in H-space (Hilbert Space)

$|\Psi\rangle$ ("ket") is to $\Psi(x)$ ("wavefunction") as $\vec{R}$ is to (R_x, R_y, R_z)

Both $|\Psi\rangle$ and $\Psi(x)$ describe same state, but $|\Psi\rangle$ is more general

$$\Psi(x_0) = \langle x_0|\Psi\rangle \quad \leftarrow g_{x_0} = \delta(x-x_0)$$

$$\Phi(p) = \langle p|\Psi\rangle \quad \leftarrow f_p = \frac{1}{\sqrt{h}}e^{ipx/\hbar}$$

$$\{c_n\} = \{\langle u_n|\Psi\rangle\} \quad \leftarrow \hat{H}u_n = E_n u_n$$

Different "representations" of same H-space vector $|\Psi\rangle$

~ position-representation, momentum-rep, energy-rep

What is a "bra"? $\langle g|$ is a new kind of mathematical object, called a "functional"

$$\langle g| \_\_ = \int dx\, g^*(x)\, \_\_$$

insert state function here

	input	output
function :	nbr	nbr
operator :	fcn	fcn
functional :	fcn	nbr

$\langle g|$ wants to bind w/ $|f\rangle$ to produce inner product $\langle g|f\rangle$

For every ket $|f\rangle$ there is a corresponding bra $\langle f|$. Like the kets, the bra's form a vector space

$$|cf\rangle \longrightarrow \langle cf| = c^*\langle f|$$

$$|\alpha f + \beta g\rangle \rightarrow \langle \alpha f + \beta g| \overset{(?)}{=} \alpha^*\langle f| + \beta^*\langle g|$$

$$\langle \alpha f + \beta g | h\rangle = \alpha^* \langle f|h\rangle + \beta^* \langle g|h\rangle \checkmark$$

$$\left.\begin{array}{l} \text{complex nbr} \times \text{bra} = \text{another bra} \\ \text{any linear combo of bra's} = \text{another bra} \end{array}\right\} \Rightarrow \text{bra's forms vector space}$$

The vector space of bras is called the "dual space". It's the dual of the ket vector space.

$\hat{A}|f\rangle = |\hat{A}f\rangle$ is a ket. What is the corresponding bra?

$$|\hat{A}f\rangle \longrightarrow \langle \hat{A}f| = \langle f|A^\dagger \quad \leftarrow \text{Defin of } A^\dagger$$

Def'n: hermitean conjugate or adjoint $\hat{A}^+$ of operator $\hat{A}$ ($\hat{A}^+$ = "A - dagger")

$$\boxed{\langle \hat{A} f | g \rangle = \langle f | A^+ g \rangle} \quad \text{for all } f, g$$

(If $\hat{A} = \hat{A}^+$, then $\hat{A}$ is hermitean or self-adjoint.)

Some properties:

- $(\hat{A}\hat{B})^+ = \hat{B}^+ \hat{A}^+$

- $(\hat{A}^+)^+ = \hat{A}$ [Proof: $\langle f | (A^+)^+ g \rangle = \langle A^+ f | g \rangle = \langle g | A^+ f \rangle^* = \langle A g | f \rangle^* = \langle f | A g \rangle$ ✓]

The adjoint of an operator is analogous to complex conjugate of a complex nbr: $c^{**} = c$, $\hat{A}^{++} = \hat{A}$

$c^* = c \Rightarrow c$ real, $\hat{A}^+ = \hat{A} \Rightarrow$ $\hat{A}$ hermitean

The "ket-bra" $|f\rangle\langle g|$ is an operator. It turns a ket (function) into another ket (function):

$$(|f\rangle\langle g|) |h\rangle = |f\rangle \langle g | h \rangle$$

<u>Projection Operators</u>

$$\hat{H} u_n(x) = E_n u_n(x) \longrightarrow \hat{H} |n\rangle = E_n |n\rangle$$

$$\Psi(x) = \sum_n c_n u_n(x) = \sum_n \langle u_n | \Psi \rangle u_n(x) \longrightarrow$$

$$|\Psi\rangle = \sum_n c_n |n\rangle = \sum_n \langle n | \Psi \rangle |n\rangle = \sum_n |n\rangle\langle n | \Psi \rangle$$

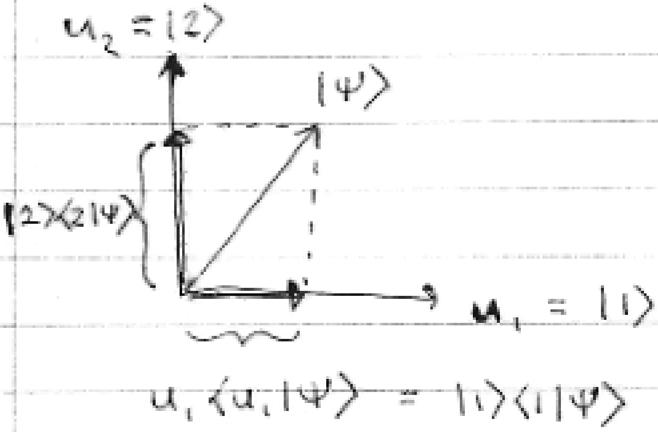
$$\Rightarrow \boxed{\sum_n |n\rangle\langle n| = \hat{\mathbb{1}}}$$

"Completeness relation" (discrete spectrum case)

$$\boxed{\hat{P}_n = |n\rangle\langle n|} = \text{"projection operator"}$$

$\hat{P}_n$ picks out ~~px~~ portion of vector $|\psi\rangle$ that "lies along" $|n\rangle$

$$\hat{P}_n|\psi\rangle = |n\rangle\langle n|\psi\rangle = c_n|n\rangle$$



$|n\rangle\langle n|$ is like $\hat{x}(\hat{x}\cdot\_\_)$

$$\hat{x}(\hat{x}\cdot\vec{R}) = \hat{x}\,R_x$$

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle \quad \text{like} \quad \vec{R} = \hat{x}(\hat{x}\cdot\vec{R}) + \hat{y}(\hat{y}\cdot\vec{R})$$

$$\hat{1} = \sum_n |n\rangle\langle n| \quad \text{like} \quad 1 = \hat{x}(\hat{x}\cdot\_\_) + \hat{y}(\hat{y}\cdot\_\_)$$

Anywhere there is a vertical bar ~~is~~ in a bracket, or a ket or a bra, we can replace bar w/

$$1 = \sum_n |n\rangle\langle n|$$

Example: $\langle\psi|\psi\rangle = 1 \Rightarrow \cancel{\langle\psi|\left(\sum_n |n\rangle\langle n|\right)|\psi\rangle}$?

$$\Rightarrow \sum_n \langle\psi|n\rangle\langle n|\psi\rangle = 1 \Rightarrow \sum_n c_n^* c_n = \sum_n |c_n|^2 = 1$$

If eigenvalue spectrum is continuous (as for $\hat{x}$ or $\hat{p}$) then must use integral, rather than sum, over states

$$\boxed{\int dx\, |x\rangle\langle x| = \hat{\mathbb{1}}}$$

Completeness Relation (continuous spectrum)

Example: $\Phi(p) = \langle f_p|\psi\rangle = \int dx\, \langle f_p|x\rangle\langle x|\psi\rangle =$

$$\frac{1}{\sqrt{2\pi\hbar}}\int dx\, e^{-ipx/\hbar}\,\psi(x)$$

The Measurement Postulates 3 and 4 can be restated in terms of the projection operator:

Starting w/ state $|\Psi\rangle = \sum_n c_n |n\rangle = \sum_n |n\rangle\langle n|\Psi\rangle$,

~~if~~ where sum over $\{n\}$ is over any complete set of states.
if we measure observable associated w/ n, ~~and find~~ then we will find value n_0 w/ probability

$$P(n_0) = |c_{n_0}|^2 = \langle\Psi|n_0\rangle\langle n_0|\Psi\rangle = \langle\Psi|\hat{P}_{n_0}|\Psi\rangle$$

$$\boxed{P(n_0) = \langle\hat{P}_{n_0}\rangle = \langle\Psi|\hat{P}_{n_0}|\Psi\rangle}$$

Probability of finding eigenvalue n_0 = expectation value of projection operator $\hat{P}_{n_0}$

And as result of measurement state $|\Psi\rangle$ collapses to state $|n_0\rangle = \hat{P}_{n_0}|\Psi\rangle$

(apart from Normalization)

Can now generalize to case of states described by more than one eigenvalue, such as H-atom

$$\Psi = \sum_{nlm} c_{nlm}\Psi_{nlm} \longrightarrow |\Psi\rangle = \sum_{nlm} |nlm\rangle\langle nlm|\Psi\rangle$$

If we measure energy (but not also $|\vec{L}|$, L_z), find n_0, then we are projecting onto subspace spanned by $\{l, m\}$ with same n_0

$$\hat{P}_{n_0} = \sum_{lm} |n_0 lm\rangle\langle n_0 lm|$$

$$P(n_0) = \langle\Psi|\hat{P}_{n_0}|\Psi\rangle = \sum_{lm} |c_{n_0 lm}|^2$$

$l = 0, 1 \ldots (n_0-1)$
$m = -l \ldots +l$

State collapses to $\hat{P}_{n_0}|\Psi\rangle = \sum_{lm} |n_0 lm\rangle\langle n_0 lm|\Psi\rangle$

must re-normalize