Physics 3220 – Quantum Mechanics 1 – Spring 2009
Problem Set #11

Due Wednesday, April 15 at 9am

Problem 11.1: Modeling molecules: the Quantum Rigid Rotor. (20 points)

Simple diatomic molecules can be modeled as two particles of mass m (representing the atoms), attached to the ends of a massless rod of total length a. The system is free to rotate in three dimensions, but we will assume the center of mass is not moving.

a) Show that, classically, the total angular momentum of the rigid object described above, rotating about a fixed axis *through its center-of-mass*, is independent of the choice of origin. Hence, the origin in this problem can be chosen to be on the axis without loss of generality.

b) The energy of this system is rotational kinetic energy. Express the classical energy in terms of the angular momentum of the system, and correspondingly deduce the quantum Hamiltonian.

c) Show that the allowed energies of the quantum system are

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2}\,, \qquad n = 0, 1, 2, \ldots . \tag{1}$$

d) What are the normalized eigenfunctions for this system? What is the degeneracy of energy level n?

Problem 11.2: Measurement of atomic angular momentum. (10 points)

Individual atoms often have a total angular momentum of $(1/2)\hbar$ or $1\hbar$. In most materials, the expectation values of the angular momenta of individual atoms point in random directions, so the net angular momentum is zero. We will see in lecture that the angular momentum of an atom is parallel to the magnetic moment of the atom. By applying a strong magnetic field to a magnetic material, one can force alignment of the magnetic moments, and hence alignment of the angular momenta.

Imagine that you have a cylinder of unmagnetized magnetic material (such as iron) suspended from a thread, which is initially at rest. If you apply a strong vertical magnetic field, all the angular momenta of the atoms will align, and the cylinder should start rotating.

a) Why should the cylinder start rotating?

b) Assume an angular momentum per atom of (one) $\hbar$. Derive a formula for the final angular speed ω of the cylinder. Using reasonable values for the dimensions, mass, etc of the cylinder, compute a value for the angular speed. Is this speed big enough to be measurable in the lab? (If it is big enough to measure, then this would be one way to experimentally verify that the angular momentum per atom is $\hbar$.)

Problem 11.3: Angular momentum operators. (30 points)

The x, y and z components of angular momentum expressed in spherical coordinates are

$$\begin{aligned} L_x &= -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) && (2)\\ L_y &= -i\hbar\left(+\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) && (3)\\ L_z &= -i\hbar\frac{\partial}{\partial\varphi}\,. && (4)\end{aligned}$$

a) Prove the above expression for L_z, by showing it is equivalent to the expression for L_z in Cartesian coordinates. Recall that $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. *Hint:* work backwards from the desired answer using the chain rule:

$$\frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}\,. \tag{5}$$

b) Use the generalized uncertainty formula to find an uncertainty relation between L_z and the angle φ. What does the result remind you of?

c) Using the definitions of L_+ and L_-, verify that

$$L_\pm = \pm\hbar e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right)\,. \tag{6}$$

d) Verify the expression given in class for $L^2 \equiv L_x^2 + L_y^2 + L_z^2$:

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\,. \tag{7}$$

It may be useful to have the operators act on a test function to help keep track of the derivatives. There are several ways to do it, but one way is to start with one of the formulas proved in class:

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z\,. \tag{8}$$

e) Using the formulas you just proved, calculate $L_+Y_1^1$, $L_-Y_1^1$, $L_zY_1^1$ and $L^2Y_1^1$, where the spherical harmonic Y_1^1 (which has $\ell = 1$, $m = 1$) is

$$Y_1^1(\theta, \varphi) = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \,. \tag{9}$$

Write the results that are not zero in terms of other spherical harmonics. (Helpful fact: $Y_1^0 = \sqrt{3/4\pi}\cos\theta$.) Comment on each result — is it what you expect?

Problem 11.4: Ground state of the hydrogen atom. (20 points)

In studying the hydrogen atom, we will take the proton fixed at the origin and study the motion of the single electron in the resulting Coulomb potential, $V(r) = ke^2/r$ where k is the usual "Coulomb's constant" from freshman E&M. The spherically-symmetric wavefunction

$$\psi(r, \theta, \varphi) = Ae^{-r/a} \,, \tag{10}$$

solves the associated time-independent Schrödinger equation, but ONLY if the constant "a" is cleverly chosen.

a) Starting from the TISE in spherical coordinates, verify that this is true, and thus solve for the constant a, and the unique energy eigenvalue E of this state. Your answer should be expressed in terms of fundamental constants of nature like k, e, m (the mass of the electron), and $\hbar$. This is the ground state wavefunction and energy of the hydrogen atom!

b) What is the angular momentum of this state? Justify your answer.

c) Solve for the normalization constant A.

d) Plot the radial hydrogenic wavefunctions $R(r)$ vs r for the states $(n, \ell) = (1, 0)$, $(2, 0)$, and $(2, 1)$. Use units of a on your r-axis. (See p.154 in Griffiths for the wavefunctions. If we havent derived these yet, we (or you) will soon, but for now just plot them so you have a feel for them.) Are you happy with the behaviour of these plots near the origin? Briefly, explain the basic features you see.