University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#7
due Wed, Oct 7, 2PM at start of class

1. Momentum operator (Total: 10 pts)
   Since the momentum p is an observable, its expectation value $< p >$ should be a real value. However, the complex factor $(-i\hbar)$ of the momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ raises the question whether the expectation value of p in some quantum states could be complex. Show that, in fact, the imaginary part of the expectation value of p is always zero.
   (Hint: How do you write the imaginary part of a number z with the help of z and z^*?)

2. Probability current density (Total: 10 pts)
   Show the following relations for the probability current density

$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial x}\right)$$

   a) $J(x,t) = \frac{\hbar}{m} Im(\Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial x})$
   b) $J(x,t) = \frac{1}{m} Re(\Psi^*(x,t)\hat{p}_x\Psi(x,t))$

3. Griffiths, problem 1.16 (Total: 10 pts)
   Consider any two normalizable solutions of the time-dependent Schrödinger equation, $\Psi_1(x,t)$ and $\Psi_2(x,t)$. Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*(x,t)\Psi_2(x,t)dx = 0$$

   (Hint: We have shown this relation in class for the special case $\Psi_1(x,t) = \Psi_2(x,t) = \Psi(x,t)$. Another approach to solve the problem is to consider that any solution of the time-dependent Schrödinger equation can be expanded using a complete set of energy eigenfunctions.)

4. Energy measurement (total: 15 pts)
   Suppose a particle is constraint living on an unit circle (e.g. an electron in a circular metal ring). This system has the following set of stationary states,

$$\Psi_n(x,t) = \frac{1}{\sqrt{2\pi}}\exp(inx)\exp(-iE_nt)$$

   with $n = 0, \pm1, \pm2, ...$ Due to the constraint the normalization is determined by the condition that

$$\int_0^{2\pi} |\Psi(x,t)|^2 dx = 1$$

   Suppose the normalized wave function of the particle is

$$\Psi(x, t = 0) = N\cos^2(x)$$

   What is the probability that an energy measurement yields the result E_0?

5. Total: 15 pts

a) The state of a system is given by a superposition of stationary states as:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n(t)\chi_n(x)\exp(-iE_n t)$$

Solve for c_2.

b A system is initially in the state

$$\Psi(x, t=0) = \frac{1}{\sqrt{7}}\left(\sqrt{2}\chi_1(x) + \sqrt{3}\chi_2(x) + \chi_3(x) + \chi_4(x)\right)$$

where $\chi_n(x)$ are eigenstates of the system's Hamiltonian such that $\hat{H}\chi_n(x) = n^2 E\chi_x(x)$ Is $\Psi(x,t)$ normalized for $t > 0$? If energy is measured, what values will be obtained and with what probabilities? Suppose that a measurement yields $4E$, write down the wave function immediately after the measurement.