Physics 3220 – Quantum Mechanics 1 – Spring 2009
Problem Set #8

Due Wednesday, March 11 at 9am

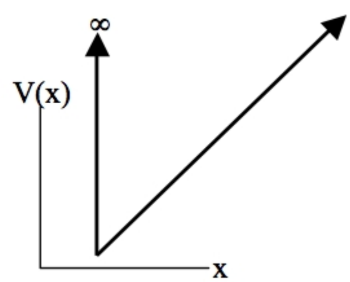
Problem 8.1: Qualitative methods for stationary states. (20 points)

a) The potential energy for a particle is given by

$$V(x) = \begin{cases} V_0\,, & x < 0\,, \\ 0\,, & 0 < x < a\,, \\ V_0/2\,, & a < x < 2a\,, \\ V_0\,, & 2a < x\,. \end{cases} \tag{1}$$

Sketch this potential. Assume V_0 and a have been chosen so that $0 < E_1 < V_0/2 < E_2 < V_0$ for the energies E_1, E_2 of the first two stationary states. Without actually solving the TISE, draw two separate sketches, one for the ground state $u_1(x)$ with energy E_1, and one for the first excited state $u_2(x)$ with energy E_2. Comment on the features of the curves everywhere, including things like wavelength, curvature and amplitude, and describe what happens at $x = 0, a, 2a$, using your qualitative knowledge of wavefunctions. (A sketch with no explanation will receive little credit!)

b) For the asymmetric well pictured below, sketch the states $u_1(x)$ (ground state), $u_2(x)$ (first excited state) and some larger $u_n(x)$ (like $n = 10$ or so). Explain in words the relevant important features of the wavefunctions, including things like wavelength, curvature and amplitude. Also, describe a physical example of a potential energy function that looks like this.



Problem 8.2: Is it a vector? (20 points)

a) Consider ordinary 3D vectors $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$, with complex components. For each of the following three cases, find out whether it constitutes a vector space. If so, what is its dimension? If not, why not?

i) The subset of all vectors with $A_z = 0$.

ii) The subset of all vectors with $A_z = 1$.

iii) The subset of all vectors whose components are all equal.

b) Does the set of all 2×2 matrices form a vector space? Assume the usual rules for matrix addition and multiplication by a scalar:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}, \qquad \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}. \tag{2}$$

If it does not form a vector space, what axiom or axioms are not obeyed? If it does, state the dimensionality and give an example of a set of basis vectors.

c) Does the set of all functions $f(x)$ defined on the range $0 < x < 1$ that vanish at $x = 0$ and $x = 1$ form a vector space? If not, list the things that go wrong.

d) Is the set of all *normalized* functions $f(x)$ a vector space? If not, list the things that go wrong.

Problem 8.3: The Schwarz inequality. (10 points)

a) Prove the Schwarz inequality:

$$|\langle A|B\rangle|^2 \le \langle A|A\rangle\langle B|B\rangle\,. \tag{3}$$

Hint: Let $|C\rangle \equiv |B\rangle - \left(\frac{\langle A|B\rangle}{\langle A|A\rangle}\right)|A\rangle$, and use the fact that for any vector at all, $\langle C|C\rangle \ge 0$.

b) Because of the Schwarz inequality, you can define an "angle" θ between two abstract kets, $|A\rangle$ and $|B\rangle$, as

$$\cos\theta \equiv \sqrt{\frac{\langle A|B\rangle\langle B|A\rangle}{\langle A|A\rangle\langle B|B\rangle}}\,. \tag{4}$$

Explain briefly why the Schwarz inequality allows you to do this, and explain why this definition reduces to the *usual* definition if $|A\rangle$ and $|B\rangle$ are just ordinary 3D vectors.

Problem 8.4: Properties of hermitian operators. (20 points)

a) Show that the sum of two hermitian operators is hermitian.

b) Suppose that $\hat{Q}$ is hermitian, and α is a complex number. Under what conditions on α is $\alpha\hat{Q}$ hermitian?

c) The hermitian conjugate (also called adjoint) of an operator $\hat{Q}$ is denoted $\hat{Q}^\dagger$, and is defined by

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}^\dagger f|g\rangle\,. \tag{5}$$

What are $\hat{Q}^\dagger$ for the two cases $\hat{Q} = i$ ("multiply by i") and $\hat{Q} = d/dx$? You can assume f and g are normalizable.

d) Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$ for any operators (not necessarily hermitian). When is the product of two hermitian operators also hermitian?

e) Show that the position operator $\hat{x} = x$ and the momentum operator $\hat{p} = -i\hbar d/dx$ are hermitian. Which of the following three operators are hermitian: $\hat{x}\hat{p}$, the Hamiltonian operator $\hat{H} = -(\hbar^2/2m)(d^2/dx^2) + V(x)$, the simple harmonic oscillator raising operator a_+? If any are not hermitian, give the hermitian conjugate.

Problem 8.5: Free particle on a circle. (20 points)

Consider the operator

$$\hat{Q} \equiv \frac{d^2}{d\phi^2}\,, \tag{6}$$

living on a circle of radius R, so ϕ is the usual angle in polar coordinates with range $\phi \in [0, 2\pi)$. (Note we are not including a radial direction here, just the one dimension of the angular circle.) In order to be well-defined on the circle, all functions there must acknowledge that angles that are different by 2π are really the same point by taking the same value:

$$f(\phi + 2\pi) = f(\phi)\,. \tag{7}$$

a) Is $\hat{Q}$ hermitian?

b) Find the eigenfunctions and eigenvalues of $\hat{Q}$.

c) Convince yourself that Hamiltonian $\hat{H}$ for a free particle moving on the circle is proportional to $\hat{Q}$. What is the proportionality constant? What is the spectrum of possible energies in this system? What is the main difference between this spectrum and that of the free particle on the infinite line?

d) Is the spectrum of energies degenerate — that is, are there distinct wavefunctions with the same energy? If so, construct a simple hermitian operator that distinguishes the degenerate states physically, by making the states also eigenfunctions of the new operator, but with different eigenvalues for the degenerate states.