We will show that f will depend on quantum-nbrs l, m, so we write it as f_l^m, and that

$$\boxed{\begin{aligned} L^2 f_l^m &= \hbar^2 l(l+1)\cdot f_l^m \qquad \star \\ L_z f_l^m &= \hbar m \cdot f_l^m \\ \text{where } l &= 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \qquad m = -l, -l+1, \ldots l-1, l \end{aligned}}$$

$f_l^m = Y_l^m(\theta, \varphi)$ will be determined later

Notice max eigenvalue of L_z ($= l\hbar$) is smaller than square root of eigenval of $L^2 = \hbar\sqrt{l(l+1)}$

So, in QM $L_{z_{max}} < |L|$ Odd!

Also notice $l = 0$, $m = 0$ state has zero angular momentum ($L^2 = 0$, $L_z = 0$) so, unlike Bohr model, can have electron in state that is "just sitting there" rather than revolving about proton in H-atom

Proof of $\star$: (This proof takes 3 pages.)

Define $L_+ = L_x + iL_y$ = "raising operator"

$L_- = L_x - iL_y$ = "lowering operator"

(Note $L_+^\dagger = L_-$, $L_-^\dagger = L_+$, $A^\dagger$ = hermitean adjoint of A)

Neither L_+ or L_- are hermitean (self-adjoint)

Note $[L^2, L_\pm] = 0$ $\left([L^2, L_+] = \underbrace{[L^2, L_x]}_{0} + i\underbrace{[L^2, L_y]}_{0} = 0\right)$

$\Rightarrow$ Consider f : $L^2 f = \lambda \cdot f$, $L_z f = \mu \cdot f$

Claim: $g = L_+ f$ is eigenfcn of L^2 w/ <u>same</u> eigenvalue λ, and g is eigenfcn of L_z w/ eigenvalue $= (\mu + \hbar)$. So L_+ operator <u>raises</u> eigenvalue of L_z by $1\hbar$.

Proof: $L^2 g = L^2 (L_+ f) = L_+ \underbrace{(L^2 f)}_{\lambda f} = \lambda \cdot L_+ f = \lambda \cdot g$ ✓

To prove $L_z g = (\mu + \hbar) g$, need to show that

$[L_z, L_+] = \hbar L_+$

$$[L_z, L_x + iL_y] = \underbrace{[L_z, L_x]}_{i\hbar L_y} + i \underbrace{[L_z, L_y]}_{-i\hbar L_x} = \hbar \underbrace{(L_x + iL_y)}_{L_+} \quad ✓$$

Now, $L_z g = \underbrace{L_z (L_+}_{L_+ L_z + \hbar L_+} f) = L_+ \underbrace{L_z f}_{\mu f} + \hbar L_+ f = (\mu + \hbar) L_+ f$

$= (\mu + \hbar) g$ ✓

So, operating on f w/ raising operator L_+ <u>raises</u> eigenvalue of L_z by $1\hbar$ but keeps eigenvalue of L^2 unchanged.

(Similarly, L_- <u>lowers</u> eigenvalue of L_z by $1\hbar$)

Operating ~~op~~ repeatedly w/ L_+ raises eigenvalue of L_z by $\hbar$ each time : $L_+ (L_+ f)$ has $(\mu + 2\hbar)$ etc

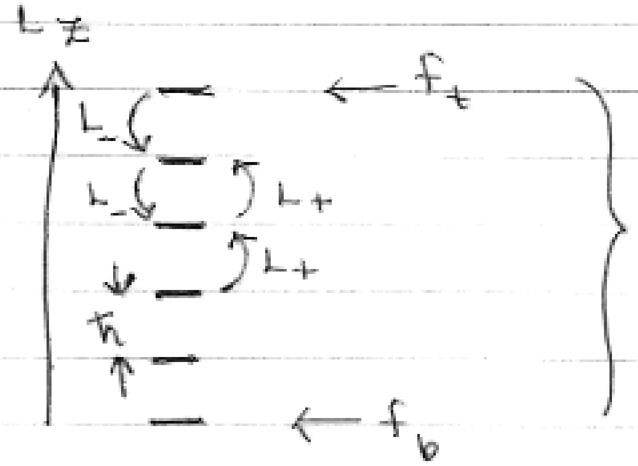
But eigenvalue of L_z cannot increase without limit since $\langle L_z \rangle$ <u>cannot</u> exceed $\sqrt{\langle L^2 \rangle}$

$$\underbrace{\langle L^2\rangle}_{\lambda} = \underbrace{\langle L_x^2\rangle}_{\mu^2} + \underbrace{\langle L_y^2\rangle}_{\geq 0} + \underbrace{\langle L_z^2\rangle}_{\geq 0} \Rightarrow \lambda > \mu^2,$$

$$\lambda > |\mu|$$

There is only one way out. There must be for a given λ a "top state" f_t for which $L_+ f_t = 0$

Likewise, there must be a "bottom state" f_b, for a given λ for which $L_- f_b = 0$



all with same λ = eigenvalue of L^2

Write $L_z f = m\hbar \cdot f$, m changes by integers only

$L_z f_t = l\hbar f_t$, l = <u>max</u> value of m

$L^2 f_t = ?$ Want to write L^2 in terms of L_+, L_z :

$$L_- L_+ = (L_x - iL_y)(L_x + iL_y) = \underbrace{L_x^2 + L_y^2}_{L^2 - L_z^2} + i\underbrace{[L_x, L_y]}_{\underbrace{i\hbar L_z}_{-\hbar L_z}}$$

$$\Rightarrow L^2 = L_- L_+ + L_z^2 + \hbar L_z$$

(Also, $L^2 = L_+ L_- + L_z^2 - \hbar L_z$)

$$\Rightarrow L^2 f_t = \underbrace{L_- L_+ f_t}_{0} + \underbrace{L_z^2 f_t}_{\hbar^2 l^2 f_t} + \underbrace{\hbar L_z f_t}_{\hbar^2 l f_t} = \hbar^2 l(l+1) f_t$$

So, $L^2 f = \underbrace{\hbar^2 \ell(\ell+1)}_{\lambda} f$ where $\ell =$ max m

same λ for all m's

Repeat for f_b: $L_z f_b = \hbar \bar{\ell} f_b$, $\bar{\ell} =$ min value of m

$$L^2 f_b = \underbrace{L_+ L_- f_b}_{0} + \underbrace{L_z^2 f_b}_{\hbar^2 \bar{\ell}^2 f_b} - \hbar L_z f_b = \underbrace{\hbar^2 \bar{\ell}(\bar{\ell}-1)}_{\lambda} f_b$$

$$ - \hbar^2 \bar{\ell} f_b$$

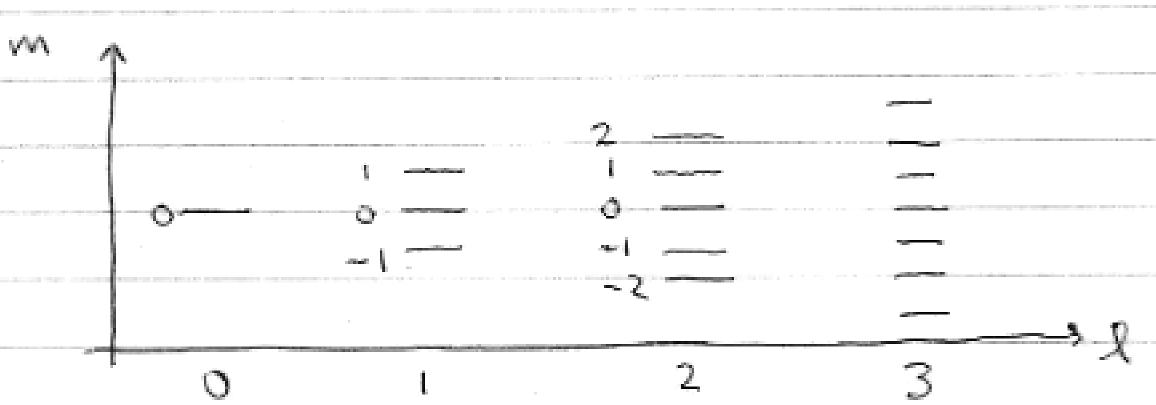
$\lambda = \lambda \Rightarrow \ell(\ell+1) = \bar{\ell}(\bar{\ell}-1) \Rightarrow \bar{\ell} = -\ell$ (try it!)

So $m_{min} = -m_{max}$ and m changes only in units of 1

$$\Rightarrow m = \underbrace{-\ell, -\ell+1, \ldots \ell-2, \ell-1, \ell}_{N \text{ integer steps}}$$

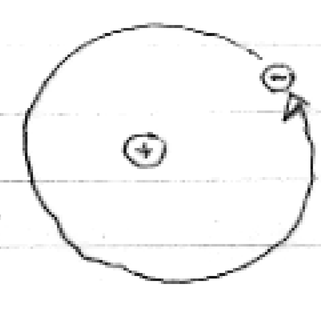
$\Rightarrow 2\ell = N$, $\ell = N/2 \Rightarrow \ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

End of proof of ★



We'll see later that there are 2 flavors of angular momentum:

1. Orbital
Ang. Mom.
(integer ℓ only)



2. Spin
Ang. Mom.
(integer or $\frac{1}{2}$ integer OK)