Physics 3220 – Quantum Mechanics 1 – Spring 2009
Problem Set #6

Due Wednesday, February 18 at 9am

Problem 6.1: Coherent states for the simple harmonic oscillator. (30 points)

In the last problem set we found that the ground state of the SHO has the minimum possible product of uncertainties $\sigma_x \sigma_p$, while the other stationary states have larger uncertainty. There exists another kind of state — not stationary — which also minimizes the uncertainty, called a *coherent state*.

We will take as the defining characteristic of a coherent state $\psi_\alpha(x)$ that when acted on by the lowering operator a_-, we get the wavefunction back times a constant:

$$a_- \psi_\alpha(x) = \alpha \, \psi_\alpha(x) \, , \tag{1}$$

or in linear algebra language, $\psi_\alpha(x)$ is an "eigenvector" of a_- with "eigenvalue" α. Different coherent states have different values of α. Do *not* in general assume that the constant α is real.

a) Using the result derived on the last problem set,

$$\int_{-\infty}^{\infty} dx \, f^*(x) \, (a_\pm g(x)) = \int_{-\infty}^{\infty} dx \, (a_\mp f(x))^* \, g(x) \, , \tag{2}$$

along with the eigenvector equation (1), evaluate $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$ and $\langle p^2 \rangle$ for the coherent state wavefunction ψ_α in terms of α and constants. You may assume $\psi_\alpha(x)$ is normalized. *Hint:* Remember how $\hat{x}$ and $\hat{p}$ may be written in terms of a_+ and a_-.

b) Calculate σ_x and σ_p, and check whether a coherent state minimizes the Heisenberg uncertainty relation, that is, check whether $\sigma_x \sigma_p = \hbar/2$.

c) Is the SHO ground state $u_0(x)$ a coherent state? What is the value of α?

d) Any wavefunction in the SHO can be expressed as a linear combination of SHO stationary states $u_n(x)$. Assume therefore that

$$\psi_\alpha(x) = \sum_{n=0}^{\infty} c_n u_n(x) \, , \tag{3}$$

for some constants c_n (which may depend on the value of α). Show that the c_n are given by

$$c_n = \frac{\alpha^n}{\sqrt{n!}} \, c_0 \, . \tag{4}$$

Another interesting property of coherent states is how their expectation values evolve in time. Recall that stationary states have time-independent expectation values; coherent states are different. To study this we find the time-dependence:

e) Assume that $\Psi_\alpha(x, t = 0) = \psi_\alpha(x)$ and show that $\Psi_\alpha(x, t)$ is still a coherent state — that is, show it satisfies

$$a_- \Psi_\alpha(x, t) = \alpha(t) \Psi_\alpha(x, t) \,. \tag{5}$$

What is $\alpha(t)$ in terms of α and other quantities?

f) In this part, for simplicity assume α is real (but $\alpha(t)$ might not be real). Take the results for $\langle x \rangle$ and $\langle p \rangle$ from part a) and put the value of $\alpha(t)$ into them to find $\langle x \rangle(t)$ and $\langle p \rangle(t)$ for $\Psi_\alpha(x, t)$. How does the result compare to the classical motion of a particle in a simple harmonic oscillator?

Coherent states have many applications in atomic, molecular, and optical physics. For instance, lasers and Bose-Einstein condensates are examples of coherent states.

Problem 6.2: Evolution of the gaussian wave packet for a free particle. (20 points)

a) First, a mathematical digression. We've already used the simple Gaussian integral formula,

$$\int_{-\infty}^{\infty} dx \, e^{-x^2} = \sqrt{\pi} \,. \tag{6}$$

Using this, prove the more general expression which will be useful in this problem,

$$\int_{-\infty}^{\infty} dx \, e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} \, e^{b^2/(4a)} \,. \tag{7}$$

To do this, define a new variable $y \equiv \sqrt{a}[x + (b/2a)]$ and substitute it in; for reasons you will see this is called "completing the square".

Particles can be described by approximately localized "lumps" of probability. Because they are in general built out of many stationary states, we think of them as a "packet" of plane waves and call such wavefunctions "wave packets."

For example, consider a free particle ($V = 0$, not the SHO anymore) with the initial wave function

$$\Psi(x, 0) = A \, e^{-ax^2} \,, \tag{8}$$

where a and A are constants, with a real and positive.

b) Normalize $\Psi(x, 0)$ and calculate the Fourier transform distribution $\phi(k)$ which tells you how Ψ is built out of plane waves. The integral expression from part a) will be useful here.

c) Now calculate the wavefunction $\Psi(x,t)$ at all times. The integral expression is again useful. The answer will be

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+i\Omega t]}}{\sqrt{1+i\Omega t}}, \tag{9}$$

where we have defined a new quantity, $\Omega \equiv 2a\hbar/m$.

Problem 6.3: Evolution of the gaussian wave packet, continued. (20 points)

a) Continuing from the last problem, calculate the probability density $|\Psi(x,t)|^2$. Express your answer in terms of the quantity $w \equiv \sqrt{a/(1+\Omega^2 t^2)}$. Sketch $|\Psi|^2$ (as a function of x) at $t = 0$ and then at some much larger t. What happens to $|\Psi|^2$ as time goes on?

b) Find $\langle x \rangle$ and $\langle p \rangle$.

c) Find $\langle x^2 \rangle$ and $\langle p^2 \rangle$. (Partial answer: $\langle p^2 \rangle = a\hbar^2$.) *Hint:* an integral you found in the old problem 3.3 may be useful.

d) What are σ_x and σ_p? Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

Problem 6.4: Plane waves and probability current. (10 points)

a) Consider the wavefunction characterized by a positive constant k_0,

$$\Psi(x,t) = Ae^{ik_0 x - i\hbar k_0^2 t/2m}. \tag{10}$$

Find the Fourier transform $\phi(k,t)$ of this wavefunction. What is this telling you about how $\Psi(x,t)$ is built out of plane waves? In words, how would you describe such a state?

b) Find the probability current $J(x,t)$ for this wavefunction. Recall from HW#3 this was defined to be,

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right). \tag{11}$$

Try to simplify your result as much as possible, and briefly interpret or make sense of the (hopefully brief) expression you get – that is, explain what this result is telling us about the physical state.

c) If you flip the sign of k_0, describe what has changed physically and mathematically about the state. How is this reflected in the result from part b)?