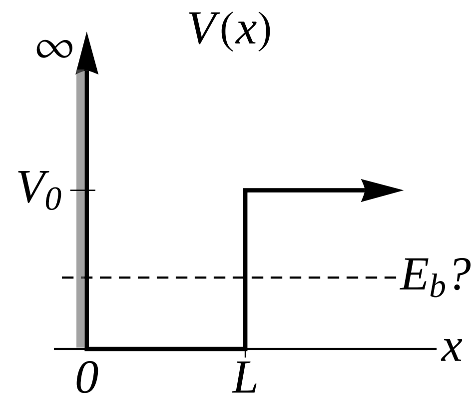
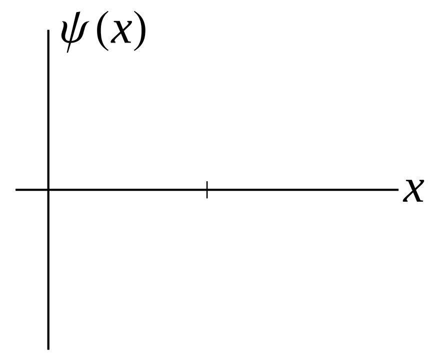
I: Scattering

A. Half-Infinite Square Well

Consider the potential shown at the right where the potential is infinite for $x < 0$, zero in the region, $0 \leq x < L$ and V_0 in the region $x \geq L$.



1. Are there any bound energy eigenstates for this well? How can you tell and how many are there?



2. On the graph to the right, assume that there is at least one bound state and sketch that state, including all the relevant properties.

Now, consider the case where $E > V_0$:

3. Are there any states for this energy range? How many?

4. For any allowable energy in this range, what is the general solution of the time independent Schrödinger equation?

5. Imagine that the solution to part 4 represents a particle coming in from the right and scattering off the well. What is the ratio of the amplitudes of the incoming and outgoing waves? Is this surprising?

II: Measurement and Quantum Mechanics Formalism

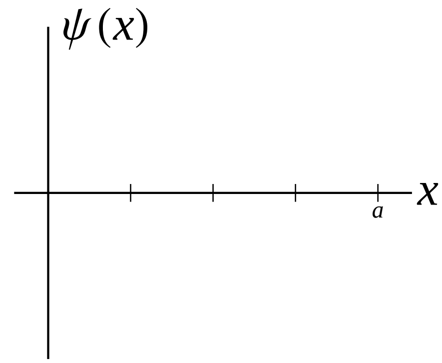
A. Measurement

Consider a particle of mass m in a 1D infinite square well with $V(x) = 0$ for $0 < x < a$, and $V = \infty$ elsewhere. Recall that the energy eigenstates and eigenvalues are

$$u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \ , E_n = n^2 \frac{\pi^2 \hbar^2}{2ma^2}$$

Suppose that, at $t = 0$, the particle is in the state

$$\Psi(x, t = 0) = \begin{cases} A & \text{for } \frac{a}{4} < x < \frac{3a}{4} \\ 0 & \text{elsewhere} \end{cases} .$$



1. Sketch the wave function at $t = 0$.

2. Solve for the constant A that normalizes the wave function.

3. Without doing any integrals or lengthy calculations, estimate the uncertainty in the momentum at $t = 0$. Justify your answer, briefly.

Suppose that at time $t = 0$, a measurement of the energy is made.

4. Write an expression for the probability that the measured energy will be E_n. If your expression involves an integral or integrals, you need not perform the integral(s); just write a correct expression.

5. What is the probability that the measured energy will the first-excited state energy E_2? (Give a numerical answer.)

6. What is the probability that the measured energy will be the ground state energy E_1? (Give a numerical answer.)

7. Suppose the energy measured at time $t = 0$ is E_1. And suppose a second measurement of the energy is made at time $t > 0$. What is the probability that the second measurement will yield the same result E_1? Justify your answer.

B. Operators

Let's assume we have two new Hermitian operators, $\hat{P}$ and $\hat{N}$. These operators are associated with observables in systems with a two-dimensional Hilbert space which means that they each have a basis of two eigenvectors. The eigenvectors of the $\hat{P}$ operator are $|s\rangle$ and $|x\rangle$ while the eigenvectors of the $\hat{N}$ operator are $|n\rangle$ and $|w\rangle$. These states obey the following eigenvalue equations:

$$\hat{P}\,|s\rangle = |s\rangle \qquad \hat{P}\,|x\rangle = -\,|x\rangle$$
$$\hat{N}\,|n\rangle = |n\rangle \qquad \hat{N}\,|w\rangle = -\,|w\rangle$$

1. We place a series of particles into the $|s\rangle$ state and quickly measure the observable associated with the $\hat{P}$ operator. What value(s) will we get and what is the probability with which we will obtain each value?

2. Now, we place a series of particles into the $|s\rangle$ state and quickly measure the observable associated with the $\hat{N}$ operator. 40% of the time, we measure 1 and 60% of the time, we measure -1. Write a plausible representation of the $|s\rangle$ state in terms of the eigenvectors of the $\hat{N}$ operator.

3. Recall that the basis states are orthogonal, *e.g.*, $\langle s|\, x\rangle = 0$ and $\langle n|\, w\rangle = 0$. How would you write the $|x\rangle$ state in terms of the eigenvectors of the $\hat{N}$ operator?

4. If we placed a series of particles into the $|x\rangle$ state and quickly measured the observable associated with the $\hat{N}$ operator, how often would you expect to get a measurement of -1? (work this out)

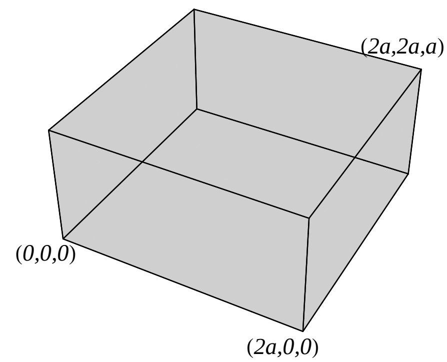
III: Quantum Mechanics in Three Dimensions

A. 3D Infinite Square Well (3D Box)
Consider a three-dimensional infinite square well (shown at right) where the potential energy is zero in the region ($0 < x < 2a$, $0 < y < 2a$ and $0 < z < a$) and infinite everywhere else. Let's write the normalized energy eigenstates of this system in the notation $|n_x n_y n_z\rangle$ where the equivalent position space wave function is:

$$\psi_{n_x,n_y,n_z}(x,y,z) = \sqrt{\frac{2}{a^3}} \sin\left(\frac{n_x \pi x}{2a}\right) \sin\left(\frac{n_y \pi y}{2a}\right) \sin\left(\frac{n_z \pi z}{a}\right)$$

and the energies are:

$$E_{n_x,n_y,n_z} = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_x^2}{4a^2} + \frac{n_y^2}{4a^2} + \frac{n_z^2}{a^2}\right)$$



A particle in this potential is placed into the initial state:

$$|\psi_0\rangle = A\left(3\,|212\rangle - 4\,|141\rangle + i\sqrt{11}\,|222\rangle\right)$$

1. What constant A will properly normalize this state?

2. Is this state an energy eigenstate? Why or why not?

3. If we were to measure the energy of this state, what would be the result(s) (along with the respective probabilities if more than one result is possible)?

4. An energy measurement is made and the lowest possible result is obtained. Write the resulting normalized quantum state.