University of Colorado, Department of Physics
PHYS3220, Spring 11, HW#9
due Fri, Apr 1, in class
- Please note: Late Homeworks will not be accepted -

1. Vectors and vector spaces

   a) Consider ordinary 3D vectors $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ (with complex components) together with the set of scalars. For each of the following three cases, find out whether it constitutes a vector space. If so, what is its dimension? If not, why not?
   (i) The subset of all vectors with $A_z = 0$.
   (ii) The subset of all vectors with $A_z = 1$.
   (iii) The subset of all vectors whose components are all equal.

   b) Does the set of all 2×2 matrices form a vector space? Assume the usual rules for matrix addition and multiplication by a scalar:
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$
   If it does not form a vector space, why not? If it does, state the dimensionality and give an example of a set of basis vectors.

   c) Does the set of all functions $f(x)$ defined on the range $0 < x < 1$ that vanish at $x = 0$ and $x = 1$ together with the set of scalars form a vector space? If not, why not? If it does, state the dimensionality.

2. Properties of Hermitian operators

   a) Show that the sum of two Hermitian operators is a Hermitian operator.

   b) Suppose that $\hat{A}$ is a Hermitian operator and α is a number. Under what condition on α is $\alpha\hat{A}$ a Hermitian operator?

   c) When is the product of two Hermitian operators a Hermitian operator?

   d) Show that the Hamilton operator $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$ is a Hermitian operator.

3. Projection operators
   An operator is said to be *idempotent*, if $\hat{P}^2 = \hat{P}$. If, in addition, $\hat{P}$ is an Hermitian operator, then $\hat{P}$ is called a projection operator.

   a) Show that $\left(\hat{I} - \hat{P}\right)$ is a projection operator, if $\hat{P}$ is a projection operator ($\hat{I}$ is the identity operator with $\hat{I}|\Psi\rangle = |\Psi\rangle$).

   b) Assume that a state vector $|\Psi\rangle$ can be written as
   $$|\Psi\rangle = |\Phi\rangle + |\chi\rangle$$
   with $|\Phi\rangle = \hat{P}|\Psi\rangle$ and $|\chi\rangle >= \left(\hat{I} - \hat{P}\right)|\Psi\rangle$ where $\hat{P}$ is a projection operator and $\hat{I}$ is the identity operator. Show that $|\Phi\rangle$ and $|\chi\rangle$ are orthogonal.
   (In other words, any state can be written in terms of two orthogonal states by means of a projection operator.)

- There is another problem on the back -

4. Hermitian conjugate (or adjoint) of an operator
   The Hermitian conjugate or Hermitian adjoint of an operator $\hat{A}$ is denoted as $\hat{A}^\dagger$ and is defined by (see Griffiths, Eq. 3.20)

$$\langle f|\hat{A}g\rangle = \langle \hat{A}^\dagger f|g\rangle$$

   a) In other books you find instead the following definition of the Hermitian conjugate:

$$\langle f|\hat{A}^\dagger g\rangle = \langle \hat{A} f|g\rangle$$

   Show that this definition is equivalent to the one above.

   b) Show that a Hermitian operator is self-adjoint, i.e. $\hat{A}^\dagger = \hat{A}$.

   c) Show the following properties of Hermitian conjugates of operators:
   (i) $\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$, (ii) $\left(\alpha\hat{A}\right)^\dagger = \alpha^*\hat{A}^\dagger$, and (iii) $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger$

   d) Find the Hermitian conjugates of i, $\partial/\partial x$ and the raising operator $\hat{a}_+$.

   e) An operator $\hat{A}$ can be represented by a matrix $(A)_{ij}$. Show that the Hermitian conjugate of $\hat{A}$ is then represented by $(A^*)_{ji}$.