I: Thinking about the wave function

In quantum mechanics, the term wave function usually refers to a solution to the Schrödinger equation,

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t),$$

where $V(x)$ is the potential energy experienced by a particle of mass m and $\Psi(x,t)$ is the wave function in this one-dimensional example.

A. Let's say you have a system where the wave function is of the form:

$$\Psi_1(x,t) = f(x)e^{i\omega t}$$

where $f(x)$ is some real-valued function of x.

1. Is $|\Psi_1(x,t)|^2$ real? Is it positive? Do your answers make sense given the physical meaning (as discussed in class) of $|\Psi_1(x,t)|^2$?

2. Does $\Psi_1(x,t)$ depend on time? Does $|\Psi_1(x,t)|^2$ depend on time?

3. Write down an expression for $\langle x\rangle$. Does it depend on time? Is it real?

   Describe in words how you interpret this quantity. Precisely what information do you get from $\langle x\rangle$?

4. Write down an expression for $\langle g(x)\rangle$ where $g(x)$ is any real-valued function of x. Does it depend on time? Again, how would you physically interpret $\langle g(x)\rangle$ (hint: think about what you would actually measure)?

B. Now let's say your system is a bit more complex (pun intended):

$$\Psi_2(x,t) = f(x)e^{i\omega t} + g(x)e^{2i\omega t}$$

where $f(x)$ and $g(x)$ are real functions of x which are orthogonal to each other.

1. Is $|\Psi_2(x,t)|^2$ real? Is it positive? Do your answers make sense given the physical meaning of $|\Psi_2(x,t)|^2$?

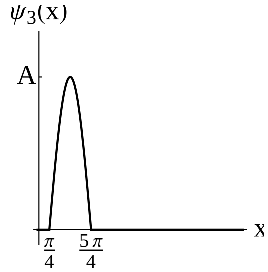
2. Does $|\Psi_2(x,t)|^2$ depend on time?

3. Write down an expression for $\langle x\rangle$. Does it depend on time? Describe the difference(s) between this result and the result for section A.3 above.

   Even though f and g are unknown functions of x, do your best to give a physical description or interpretation of this new result for $\langle x\rangle$ for the state Ψ_2.

✓ Check your results with a tutorial instructor.

C. Now, we will deal with a new wave function at a single moment in time, $\psi_3(x) = \Psi_3(x, t = t_0)$, represented by the graph below (a sine curve from $\pi/4$ to $5\pi/4$ and zero everywhere else).



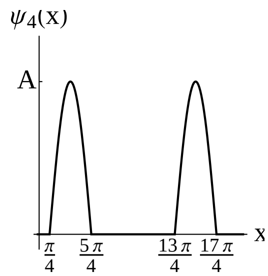
1. Find a value of A which will normalize $\psi_3(x)$.

2. Using physical arguments (*i.e.*, without doing the integral), what do you think $\langle x \rangle$ is? (If you feel uncertain, you can check by doing the integral)

3. We want to find the standard deviation for x for this system. First, do you think that $\langle x^2 \rangle$ is larger/the same/smaller (circle one) than $\langle x \rangle^2$? Now, actually calculate $\langle x^2 \rangle$.

4. What is σ_x^2? What is the probability that you will find the particle represented by $\psi_3(x)$ in the range $\langle x \rangle \pm \sigma_x$? (Recall that $\sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$).

D. Now we somehow create a system where for an instant, the wave function, $\psi_4(x) = \Psi_4(x, t = t_0)$, looks like the graph below.



1. Find the value of A which will normalize $\psi_4(x)$.

2. Using physical arguments (*i.e.*, without doing the integral), what do you think $\langle x \rangle$ is? (If you feel uncertain, you can check by doing the integral)

3. Estimate $\langle x^2 \rangle$ and σ_x. Indicate on the graph above the range which you think represents $\langle x \rangle \pm \sigma_x$.
   Bonus (*i.e.*, come back to this if you have time after finishing the rest of the tutorial), calculate $\langle x^2 \rangle$ and σ_x^2.

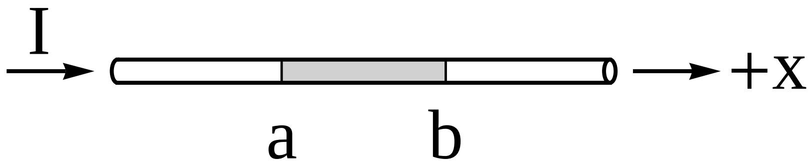
4. How do you physically interpret σ_x?

5. What are the possible values of a measurement of x on any of these identical systems? Do you "expect" to measure x equal to the expectation value of x?

✓ Check your results with a tutorial instructor.

II: Classical current

A. Consider a thin, insulated wire with a current which depends on the position along the wire. Let the current be given as $I(x)$, where a positive value of I represents current flowing to the right.



Student A defines $Q_{\rm ab}(t)$ to be the total electric charge in the wire between points a and b (see figure above). Student B points out that since charge cannot be created or destroyed (*i.e.*, charge is conserved), $Q_{\rm ab}$ cannot be a function of time. You are called in to settle the dispute. Could $Q_{\rm ab}$ depend on time? What is your reasoning?

B. No matter what you said above, suppose we told you we had set up a situation where at an instant of time, t_0, we had measured $I(\text{b}) > I(\text{a})$.

1. What does this situation imply about the time dependence of $Q_{\rm ab}$?

2. Construct a formula for the time derivative of $Q_{\rm ab}$ in terms of $I(\text{a})$ and $I(\text{b})$

Useful Formulas

$$\int_a^b \sin^2(x-x_0)\mathrm{d}x = \left(\frac{x-x_0}{2} - \frac{\sin(2(x-x_0))}{4}\right)\Bigg|_a^b$$

$$\int_a^b x\sin^2(x-x_0)\mathrm{d}x = \left(\frac{x^2-x_0^2}{4} - \frac{\cos(2(x-x_0))}{8} - \frac{x\sin(2(x-x_0))}{4}\right)\Bigg|_a^b$$

$$\int_a^b x^2\sin^2(x-x_0)\mathrm{d}x = \frac{1}{24}\left(4x^3 - 6x\cos(2(x-x_0)) + (3-6x^2)\sin(2(x-x_0))\right)\Bigg|_a^b$$

Useful Formulas

$$\int_a^b \sin^2(x-x_0)\mathrm{d}x = \left(\frac{x-x_0}{2} - \frac{\sin(2(x-x_0))}{4}\right)\Bigg|_a^b$$

$$\int_a^b x\sin^2(x-x_0)\mathrm{d}x = \left(\frac{x^2-x_0^2}{4} - \frac{\cos(2(x-x_0))}{8} - \frac{x\sin(2(x-x_0))}{4}\right)\Bigg|_a^b$$

$$\int_a^b x^2\sin^2(x-x_0)\mathrm{d}x = \frac{1}{24}\left(4x^3 - 6x\cos(2(x-x_0)) + (3-6x^2)\sin(2(x-x_0))\right)\Bigg|_a^b$$

Useful Formulas

$$\int_a^b \sin^2(x-x_0)\mathrm{d}x = \left(\frac{x-x_0}{2} - \frac{\sin(2(x-x_0))}{4}\right)\Bigg|_a^b$$

$$\int_a^b x\sin^2(x-x_0)\mathrm{d}x = \left(\frac{x^2-x_0^2}{4} - \frac{\cos(2(x-x_0))}{8} - \frac{x\sin(2(x-x_0))}{4}\right)\Bigg|_a^b$$

$$\int_a^b x^2\sin^2(x-x_0)\mathrm{d}x = \frac{1}{24}\left(4x^3 - 6x\cos(2(x-x_0)) + (3-6x^2)\sin(2(x-x_0))\right)\Bigg|_a^b$$