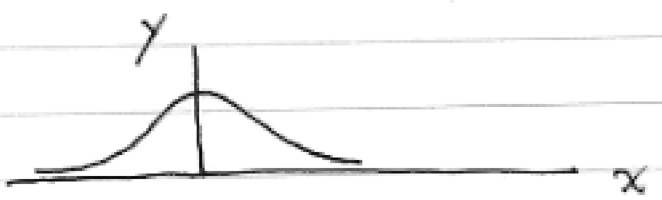
Classical Waves

Before describing QM waves $\Psi(x,t)$, let's review classical waves:

wave = self-propagating disturbance in a medium

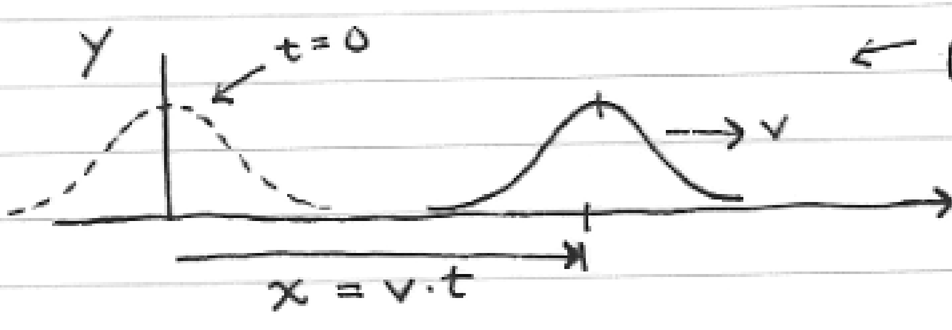
$y = f(x)$ = displacement of medium from equilibrium position



Claim: For any function $y = f(x)$, the function $y(x,t) = f(x - v \cdot t)$ is a (1D) traveling wave moving rightward w/ speed v

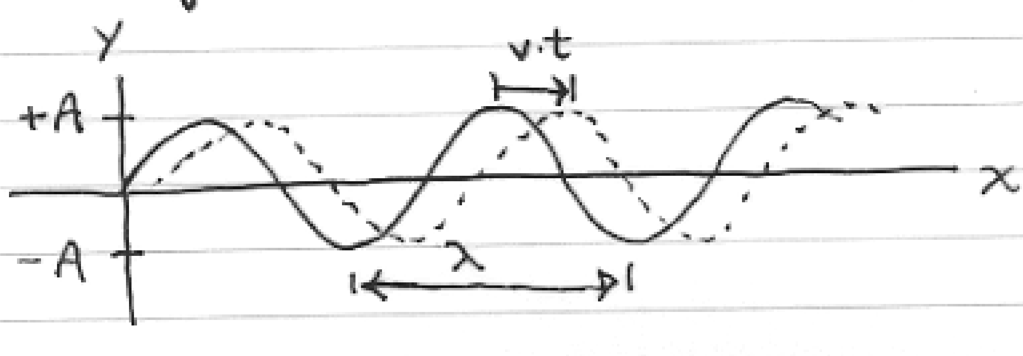
Example: gaussian pulse $y = f(x) = A\, e^{-x^2/2\cdot\sigma^2}$

Traveling pulse: $y(x,t) = f(x - v\cdot t) = A\, e^{-(x - v\cdot t)^2/2\sigma^2}$



← peak at $x - v\cdot t = 0 \Rightarrow$ peak at $x = v\cdot t$

Example: Sinusoidal wave $y(x) = A \sin \underbrace{\left(2\pi \frac{x}{\lambda}\right)}_{\text{argument changes by } 2\pi \text{ when } x \text{ changes by } \lambda}$



Define $k = \frac{2\pi}{\lambda}$ = wave number = rads/length

$y(x) = A\sin(kx) \longrightarrow y(x,t) = A\sin[k(x - v\cdot t)]$

Relate speed v to λ, frequency $f = \frac{1}{T}$ ← period

$$v = \frac{\text{dist}}{\text{time}} = \frac{\lambda}{T} = \lambda \cdot f$$

Aside: $f = \frac{\#\text{ cycles}}{\text{time}} = \frac{1\text{ cycle}}{(\text{time for } 1\text{ cycle})} = \frac{1}{T}$

So, $k\cdot(x - v\cdot t) = \frac{2\pi}{\lambda}\left(x - \frac{\lambda}{T}\cdot t\right) = 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$

$$= (kx - \omega t)$$

$\omega = \frac{2\pi}{T} = 2\pi f$ = angular frequency = rads/time

$$y(x,t) = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] = A\sin(kx - \omega t)$$

argument [ ] changes by 2π when x changes by λ or t changes by T

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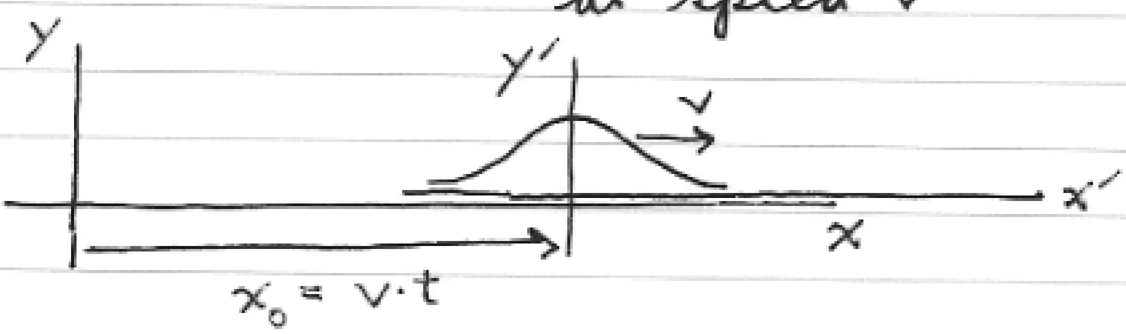
Claim: A rigidly shaped ("dispersionless") traveling wave has the form

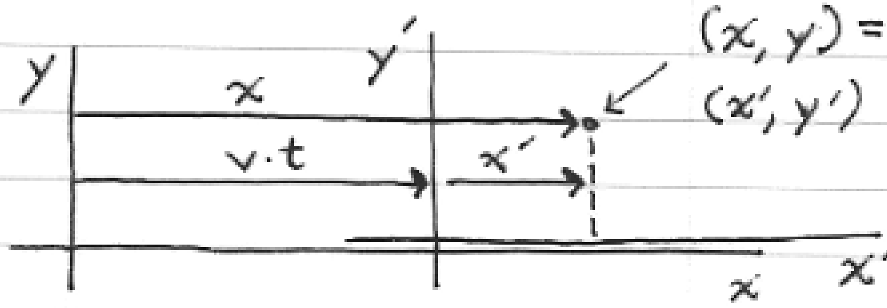
$$y(x,t) = f(x \mp v\cdot t)$$

rt-going ($-$), left-going ($+$)

Proof of Claim: Consider moving coord. system $x'y'$ moving along w/ wave at speed v

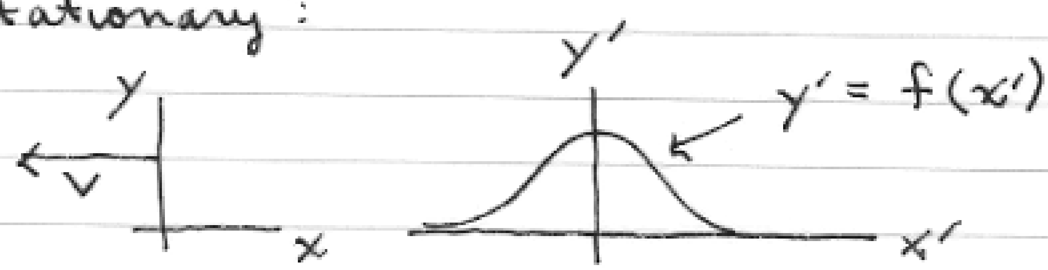


Coordinate transform:



$x = x' + v \cdot t \Rightarrow$ $\boxed{\begin{array}{l} x' = x - v \cdot t \\ y' = y \end{array}}$

In moving $x'y'$ frame, the moving wave is stationary:



$y' = f(x')$ transform: $y' = y$, $x' = x - v \cdot t \Rightarrow$

$y' = f(x - v \cdot t)$ ✓ End of proof

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Any (1D) traveling wave of the form

$y(x,t) = f(x \mp v \cdot t)$ is a sol'n of the Wave Equation

The (Classical) Wave Eq'n:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

Note units: $[v \cdot t] = [x]$

Proof: $y(x,t) = f(\varphi)$, $\varphi = \varphi(x,t) = x - v\cdot t$

$$\frac{\partial y}{\partial x} = \frac{df}{d\varphi}\cdot\underbrace{\frac{\partial \varphi}{\partial x}}_{1} = \frac{df}{d\varphi}$$

(Chain Rule)

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{df}{d\varphi}\right) = \frac{d}{d\varphi}\left(\frac{df}{d\varphi}\right)\underbrace{\frac{\partial \varphi}{\partial x}}_{1} = \frac{d^2 f}{d\varphi^2} \qquad (1)$$

$$\frac{\partial y}{\partial t} = \frac{df}{d\varphi}\underbrace{\frac{\partial \varphi}{\partial t}}_{-v} = -v\frac{df}{d\varphi}$$

$$\frac{\partial^2 y}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{df}{d\varphi}\right) = -v\frac{d^2 f}{d\varphi^2}\underbrace{\frac{\partial \varphi}{\partial t}}_{-v} = +v^2\frac{d^2 f}{d\varphi^2} \qquad (2)$$

$$(1) + (2) \Rightarrow \frac{d^2 f}{d\varphi^2} = \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$ ✓ Done

Examples:

$E = E_y$ or E_z

• Maxwell's Eq'ns $\Rightarrow$ $\frac{\partial^2 E}{\partial x^2} = \underbrace{(\mu_0 \varepsilon_0)}_{1/v^2}\frac{\partial^2 E}{\partial t^2}$

$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times 10^8 \ \mathrm{m/s}$

• wave on string: $v = \sqrt{\frac{F_T}{\mu}}$ ← tension (F_T), ← mass/length (μ)

Superposition Principle: If $y_1(x,t)$ and $y_2(x,t)$ are solutions of the wave eq'n, then $(y_1 + y_2)$ is also a sol'n.

Follows from fact that Wave Eq'n is <u>linear</u> D.E.

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \Rightarrow \mathbb{L}(y(x,t)) = 0$$

<u>operator</u> $\mathbb{L}(\ ) = \dfrac{\partial^2(\ )}{\partial x^2} - \dfrac{1}{v^2}\dfrac{\partial^2(\ )}{\partial t^2}$

Notice that $\mathbb{L}$ is a <u>linear</u> operator:

$$\mathbb{L}(y_1 + y_2) = \mathbb{L}(y_1) + \mathbb{L}(y_2)$$

$$\mathbb{L}(c \cdot y) = c \cdot \mathbb{L}(y) \quad , \quad c = \text{const}$$

function: $f(x) = y$

nbr in ↑ ↑ number out

operator: $\mathbb{L}(f(x)) = g(x)$ ← function out

function in ↗

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Because of Superposition Principle, waves <u>add</u> just like we expect ⇒

constructive or destructive interference