Physics 3220 – Quantum Mechanics 1 – Spring 2009
Problem Set #13

Due Wednesday, April 29 at 9am

Problem 13.1: Survey (10 points)

Please take the following survey. **You will not be graded for accuracy for the survey, you get credit just for participating.**

Link is clickable:

http://www.colorado.edu/sei/surveys/Sp09/Clicker_Phys3220_sp09-post.html

Note: cutting and pasting the link makes the underscore characters disappear, so try either clicking directly, or go to the link at the web page.

Problem 13.2: Projection operators. (30 points)

Consider first a Hilbert space spanned by a basis of orthonormal states $\{|n\rangle\}$ labeled by one discrete quantum number, which we will call n. The states $|n\rangle$ are the eigenstates of some operator $\hat{Q}$, and n labels the various eigenvalues, $\hat{Q}|n\rangle = q_n|n\rangle$. (For example, $\hat{Q}$ could be the Hamiltonian, and n could label the allowed energies.)

For each n, we define the *projection operator onto the state* $|n\rangle$ as

$$\hat{P}_n \equiv |n\rangle\langle n| \,. \tag{1}$$

Thus there is a different $\hat{P}_n$ for each state $|n\rangle$.

a) Demonstrate that $\hat{P}_n$ is Hermitian, and that $\hat{P}_n^2 = \hat{P}_n$.

b) What is the result of acting $\hat{P}_n$ on an arbitrary state $|\psi\rangle = \sum_m c_m|m\rangle$? Explain why the name "projection operator" is justified. If there are N distinct values of n, all operators will be $N \times N$ matrices; what does $\hat{P}_n$ look like as such a matrix?

c) In general $\hat{P}_n|\psi\rangle$ is not normalized; show that the state $\hat{P}_n|\psi\rangle/\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}$ is properly normalized.

d) How are the number $\langle\psi|\hat{P}_n|\psi\rangle$ and the state $\hat{P}_n|\psi\rangle/\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}$ related to the result of making a measurement of Q? Relate them to the postulates of quantum mechanics.

Now consider a system where the Hilbert space is labeled by more than one quantum number: the hydrogen atom, with states $|n\,\ell\,m\rangle$ labeled by n, ℓ and m. The projection operator associated to a given value of n now has a sum over all values of the *other* quantum numbers:

$$\hat{P}_n = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} |n\,\ell\,m\rangle\langle n\,\ell\,m| \,. \tag{2}$$

In the following consider the hydrogen atom wavefunction

$$|\psi\rangle = \frac{1}{2}\left(|2\,1\,0\rangle + \sqrt{2}|2\,0\,0\rangle + |1\,0\,0\rangle\right) \,. \tag{3}$$

e) Consider a measurement of energy. Which values of n might be observed? Write down the projection operators associated with each possible result, and use them to calculate the probabilities of each outcome, and the result of the collapse of the wavefunction. Do these results agree with what you would have expected?

f) If you were to make a measurement of L_z instead, how would you define the projection operator(s) you need? Repeat the calculation of e) for this case.

Problem 13.3: Spin angular momentum operators. (20 points)

a) Demonstrate that the 2×2 spin matrices $\vec{\hat{S}} = (\hbar/2)\vec{\sigma}$ with $\vec{\sigma}$ the three Pauli matrices, which operate on spin-1/2 particles, satisfy the correct angular momentum commutation relations,

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \,, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x \,, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \,. \tag{4}$$

b) Consider the spin-1 case. There are three eigenvectors; let them be called

$$|s=1, m_s=1\rangle \equiv (1,0,0) \,, \quad |s=1, m_s=0\rangle \equiv (0,1,0) \,, \quad |s=1, m_s=-1\rangle \equiv (0,0,1) \,, \tag{5}$$

and construct the matrices for $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ for this spin. Hint: as with the spin-1/2 case, $\hat{S}_z$ can be deduced from the eigenvectors, and $\hat{S}_x$ and $\hat{S}_y$ can be deduced from the action of $\hat{S}_+$ and $\hat{S}_-$,

$$\hat{S}_\pm|s\;m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}\,|s\;m_s \pm 1\rangle \,. \tag{6}$$

c) Verify that the matrices you constructed in part b) also satisfy the correct angular momentum commutation relations.

Problem 13.4: Spin of an electron. (20 points)

An electron is in the spin state

$$|\chi\rangle = A \begin{pmatrix} 3 \\ 4i \end{pmatrix} . \tag{7}$$

a) Determine the normalization constant A.

b) Calculate the expectation values of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$.

c) An observation of $\hat{S}_z$ is made. What are the possible values that may be observed, and with what probabilities?

d) Imagine we never made an $\hat{S}_z$ measurement and make an observation of $\hat{S}_y$ instead. What are the possible values that may be observed, and with what probabilities? To determine this, find the eigenvalues and eigenvectors of $\vec{S}_y$ in general, and then use them in this case.