Probability distribution example (from Griffiths)
Example 1.1

Alternate Solution

Rock, released from rest at time $t=0$, falls a distance h in a time T.

$$x = \frac{1}{2} g t^2 \quad , \quad h = \frac{1}{2} g T^2$$

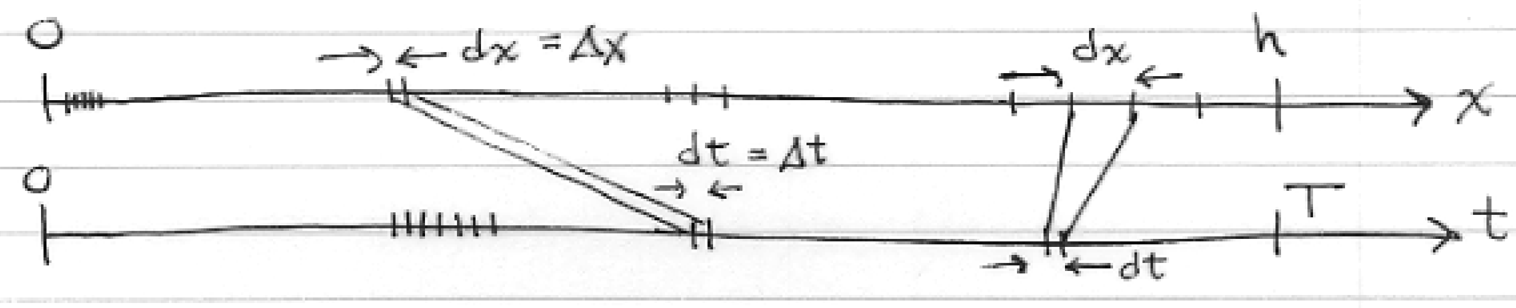
A movie is taken as the rock falls during the period $t = 0 \to T$, at 60 frames per second, resulting in thousands of photos of the rock at regularly-spaced time intervals. The individual frames are cut out from the film and then shuffled. Each frame corresponds to a particular x and t, and a particular dx and dt.
All frames have the same dt, but different frames have different dx's:

$$\frac{dx}{dt} = g \cdot t \quad \Rightarrow \quad dx = g \cdot t \cdot dt$$

We define probability distribution in space, $\rho(x)$ and probability distribution in time, $\tau(t)$

$$\rho(x)\,dx = \text{Prob}\left\{\begin{array}{c}\text{frame chosen at random is} \\ \text{the one at } x \to x + dx\end{array}\right\}$$

$$\tau(t)\,dt = \text{Prob}\left\{\begin{array}{c}\text{frame chosen at random is} \\ \text{the one at } t \to t + dt\end{array}\right\}$$



(To be precise, I should really be writing Δx instead of dx, and Δt instead of dt. In the end, I'll take limit $\Delta t \to 0$.)

Notice all dt's are same size, but dx's start out very short and get longer and longer.

Now $\tau(t)\, dt = \dfrac{dt}{T}$, $\tau(t) = \text{const} = \dfrac{1}{T}$

$\tau(t)$ = constant, since random frame is equally likely to be at any time – early, middle, or late

Notice $$\int_0^T \tau(t)\, dt = \frac{1}{T}\int_0^T dt = \frac{1}{T}\cdot T = 1$$

(Total probability must be 1.)

Pick a <u>particular</u> t and dt (a particular frame). Corresponding to that (t, dt) is a particular (x, dx). The probability that that particular frame will be picked is:

$$\underbrace{\text{Prob}\{t \to t + dt\}}_{\tau(t)\,dt} = \underbrace{\text{Prob}\{x \to x + dx\}}_{\rho(x)\,dx}$$

$$\Rightarrow \rho(x) = \tau(t)\frac{dt}{dx} = \frac{\tau(t)}{(dx/dt)} = \frac{1/T}{g\cdot t}$$

$$T = \sqrt{2h/g}\,, \qquad t = \sqrt{\frac{2x}{g}}$$

$$\Rightarrow \rho(x) = \frac{\left(\frac{1}{T}\right)}{g\cdot t} = \frac{\sqrt{\frac{g}{2h}}}{g\sqrt{\frac{2x}{g}}} = \frac{1}{2\sqrt{h\cdot x}}$$

The key formula in this problem is

$$\tau(t)\,dt = \rho(x)\,dx \Rightarrow \rho(x) = \frac{\tau(t)}{(dx/dt)}$$

It is vital to remember that, when using this formula, the x and t are not independent. The x is the x which corresponds to the particular t and dx is the interval in x which corresponds to dt.

Also you may be uncomfortable about treating the derivative dx/dt as if it was a quotient $\Delta x/\Delta t$

We often "pull apart" $\frac{dx}{dt}$ and write things

like $$\frac{dx}{dt} = f(x) \Rightarrow dx = f(x)\cdot dt$$

or $$dt/dx = \frac{1}{(dx/dt)}$$

This only makes sense if ~~it~~ you remember that

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$\frac{dx}{dt}$ really is a tiny Δx (dx) divided by a tiny Δt (dt).