University of Colorado, Department of Physics
PHYS3220, Spring 11, HW#8
due Fri, Mar 11, in class (extended)
- Please note: Late Homeworks will not be accepted -

1. Symmetric potentials (Griffiths, Problem 2.1(c))
   Prove the following statement: If $V(x)$ is an even function (that is, $V(-x) = V(x)$), then the stationary state wave functions $\chi(x)$ can always be taken to be either even or odd.

2. Attractive delta potential
   Consider a particle of mass m subject to an attractive delta potential $V(x) = -V_0\delta(x)$, where $V_0 > 0$. Show that this particle has only one bound state. Find the energy and the wave function of the state.

3. An expanding infinite square well
   Consider a particle of mass m in an infinite square well, that extends from 0 to a. Assume that the particle is in the ground state of the square well for $t < 0$. At $t = 0$ the size of the square well is suddenly expanded, so that it extends from 0 to $2a$ leaving the wave function of the state undisturbed.

   a) Write down the wave function $\Psi(x, t = 0)$ in the new larger well (Think carefully about the different regions in the new well).
   b) The wave function $\Psi(x, t = 0)$ can be expanded as a linear combination of the stationary states, $\chi_n(x)$, of the new larger well, i.e. $\Psi(x, t = 0) = \sum_{n=1}^{\infty} c_n \chi_n(x)$. Determine a formula for c_n and show by summing the first terms that the sum over $|c_n|^2$ is approaching 1. Does this make sense? Explain.
   c) Assume that the energy of the particle is measured at some time $t > 0$. What is the expectation value $< E >$ of the energy? What is the probability that the energy of the first excited state ($n = 2$) of the new well is measured?
   d) Calculate the smallest period of time, $\tau > 0$, at which $\Psi(x, \tau) = \Psi(x, t = 0)$.
   e) Draw a picture of $\Psi(x, t)$ at $t = \tau/2$.

- There is another problem on the back -

4. Coherent states for the harmonic oscillator
   In a harmonic oscillator a coherent state $\psi_\alpha(x)$ is defined as follows: When acted on by the lowering operator $\hat{a}_-$, we get the wave function back times a constant:

$$\hat{a}_- \psi_\alpha(x) = \alpha\, \psi_\alpha(x)\,, \tag{1}$$

   or in linear algebra language, $\psi_\alpha(x)$ is an eigenvector of $\hat{a}_-$ with eigenvalue α. Different coherent states have different values of α. Do *not* in general assume that the constant α is real.
   (*Coherent states have many applications in atomic, molecular, and optical physics. For instance, lasers and Bose-Einstein condensates are examples of coherent states.*)

   a) Show that for any square-integrable functions $f(x)$ and $g(x)$

$$\int\limits_{-\infty}^{\infty} f^*(x)(\hat{a}_\pm g(x))dx = \int\limits_{-\infty}^{\infty} (\hat{a}_\mp f(x))^* g(x))dx \tag{2}$$

   b) Use Eq. (2) along with the eigenvector equation (1) to evaluate $\langle x\rangle$ and $\langle p\rangle$ for the coherent state wave function ψ_α in terms of α and constants. You may assume $\psi_\alpha(x)$ is normalized. (Hint: How can $\hat{x}$ and $\hat{p}$ be written in terms of a_+ and a_-?)

   c) Is the ground state $\chi_0(x)$ of the harmonic oscillator a coherent state? What is the value of α?

   d) Any wave function of the harmonic oscillator can be expressed as a linear combination of stationary states $\chi_n(x)$ of the harmonic oscillator. Assume therefore that

$$\psi_\alpha(x) = \sum_{n=0}^{\infty} c_n \chi_n(x)\,, \tag{3}$$

   and show that the c_n are given by

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0\,. \tag{4}$$

   (Hint: Operate with $\hat{a}_-$ on Eq. (3).)

   e) Another interesting property of coherent states is how their expectation values evolve in time. Recall that stationary states have time-independent expectation values; coherent states are different. Assume that $\Psi_\alpha(x, t = 0) = \psi_\alpha(x)$ and show that $\Psi_\alpha(x, t)$ is still a coherent state — that is, show it satisfies

$$\hat{a}_- \Psi_\alpha(x,t) = \alpha(t)\Psi_\alpha(x,t)\,. \tag{5}$$

   What is $\alpha(t)$ in terms of α and other quantities?

   f) In this part, for simplicity assume α is real (but $\alpha(t)$ might not be real). Take the results for $\langle x\rangle$ and $\langle p\rangle$ from part b) and put the value of $\alpha(t)$ into them to find $\langle x\rangle(t)$ and $\langle p\rangle(t)$ for $\Psi_\alpha(x,t)$. How does the result compare to the classical motion of a particle in a harmonic oscillator?