University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#3
due Wed, Sep 9, 2PM at start of class

1. Probabilities and expectation (average) values (total: 10 pts)
   A box contains 18 small items, of various lenghts. The distribution of lengths in cm of the set of objects is

   3,3,3,3,4,6,6,6,8,8,8,8,9,9,9,11,11,11

   a) What is the probability that an object chosen at random from the box will have length 8 cm, assuming there is an equal probability of selecting any one object?

   b) What is the average length $< L >$ (also called expectation value of length) of an object in the box?

   c) What is the probability that an object chosen at random from the box will have length $< L >$, assuming that there is an equal probability of selecting any one object?

   d) What is the average of the square of the length $< L^2 >$ of an object in the box?

   e) Use your previous result to compute the standard deviation σ for the length of an item drawn from the box?

2. Another wavefunction (total: 20 pts)

   a) Using the known integral (you have calculated it in the previous set)

   $$\int_{-\infty}^{\infty} exp(-az^2)dz = \sqrt{\frac{\pi}{a}} \tag{1}$$

   find an expression for the integral $\int_{-\infty}^{\infty} z^2 exp(-az^2)dz$ by differentiating equation (1) with respect to a.

   b) Consider the following wavefunction for a particle of mass m

   $$\Psi(x,t) = A\left[\exp(-i\alpha\hbar t/m) + \beta x \exp(-3i\alpha\hbar t/m)\right]\exp(-\alpha x^2) \tag{2}$$

   where A, α and β are real constants. Choose A as to normalize the wavefunction. (Hint: You'll find part a) helpful here. Also, think about whether you can argue some terms are zero without explicitly calculating them.)

   c) What is the probability density for this wavefunction? Is the probability density independent of time?

   d) What is the expectation value $< x >$? What kind of motion is the average value of the particle's position executing?

3. Superposition and product forms (total: 20 pts)

   a) Show that if $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are solutions of the Schrödinger equation

   $$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi(x,t) \tag{3}$$

   then the superposition $\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$ is a solution too.

b) Consider now two Schrödinger equations with different potential terms $V_1(x_1, t)$ and $V_2(x_2, t)$

$$i\hbar\frac{\partial}{\partial t}\Psi_1(x_1,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + V_1(x_1,t)\right)\Psi_1(x_1,t) \tag{4}$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2(x_2,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V_2(x_2,t)\right)\Psi_2(x_2,t) \tag{5}$$

Show that if $\Psi_1(x_1, t)$ is a solution of equation (4) and $\Psi_2(x_2, t)$ is a solution of equation (5), then the product form $\Psi(x_1, x_2, t) = \Psi_1(x_1, t)\Psi_2(x_2, t)$ is a solution of the Schrödinger equation composed of the two equations above, namely

$$i\hbar\frac{\partial}{\partial t}\Psi(x_1,x_2,t) = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}\right) + V_1(x_1,t) + V_2(x_2,t)\right]\Psi(x_1,x_2,t) \tag{6}$$

c) Replace in part b) all time derivatives $\partial/\partial t$ by $\partial^2/\partial t^2$ and show that in this case the product form is *not* solution of the composite Schrödinger equation.

d) The composite Schrödinger equation (6) in part b) describes a system of two independent particles. Use the interpretation that $|\Psi(x_1, x_2, t)|^2$ is a probability density to show that the state of such a system must be in the product form $\Psi(x_1, x_2, t) = \Psi_1(x_1, t)\Psi_2(x_2, t)$. (Note: This does disqualify the wave equation considered in part c) as a valid equation of motion for a system of independent particles.)

4. Current density (total: 10 pts)
In class we will show (have shown) that the continuity equation for probability is given by

$$\frac{d}{dt}|\Psi(\mathbf{r},t)|^2 + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0. \tag{7}$$

In deriving the equation we have assumed that the potential $V(\mathbf{r}, t)$ is a real quantity.

a) Prove that if $V(\mathbf{r}, t)$ is complex the continuity equation becomes

$$\frac{d}{dt}|\Psi(\mathbf{r},t)|^2 + \nabla\cdot\mathbf{J}(\mathbf{r},t) = \frac{\hbar}{m}Im(V(\mathbf{r},t))|\Psi(\mathbf{r},t)|^2 \tag{8}$$

b) How do you interpret the term on the right hand side of equation (8)? Discuss in particular the cases $Im(V(\mathbf{r}, t)) > 0$ and $Im(V(\mathbf{r}, t)) < 0$.