Physics 3220 – Quantum Mechanics 1 – Fall 2008
Problem Set #3

Due Wednesday, September 10 at 2pm

Problem 3.1: Practice with complex numbers. (20 points)

Every complex number z can be written in the form $z = x + iy$ where x and y are real; we call x the *real part* of z, written $x = \mathrm{Re}\, z$, and likewise y is the *imaginary part* of z, $y = \mathrm{Im}\, z$. We further define the *complex conjugate* of z as $z^* \equiv x - iy$.

a) Prove the following relations that hold for any complex numbers z, z_1 and z_2:

$$\begin{aligned} \mathrm{Re}\, z &= \frac{1}{2}(z + z^*)\ , & (1)\\ \mathrm{Im}\, z &= \frac{1}{2i}(z - z^*)\ , & (2)\\ \mathrm{Re}\,(z_1 z_2) &= (\mathrm{Re}\, z_1)(\mathrm{Re}\, z_2) - (\mathrm{Im}\, z_1)(\mathrm{Im}\, z_2)\,, & (3)\\ \mathrm{Im}\,(z_1 z_2) &= (\mathrm{Re}\, z_1)(\mathrm{Im}\, z_2) + (\mathrm{Im}\, z_1)(\mathrm{Re}\, z_2)\,. & (4) \end{aligned}$$

b) The modulus-squared of z is defined as $|z|^2 \equiv z^* z$. What is $\mathrm{Im}\,|z|^2$, and what is $\mathrm{Im}\, z^2$? In doing quantum mechanics confusing z^2 and $|z|^2$ is very common; be careful!

c) Any complex number can also be written in the form $z = Ae^{i\theta}$, where A and θ are real and θ is usually taken to be in the range $[0, 2\pi)$; A and θ are called the *modulus* and the *phase* of z, respectively. Use Euler's relation (which is provable using a Taylor expansion),

$$e^{ix} = \cos x + i \sin x\,, \tag{5}$$

to find $\mathrm{Re}\, z$, $\mathrm{Im}\, z$, z^* and $|z|$ in terms of A and θ.

d) Use the above relations on $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ to derive trigonometric identities for $\sin(\alpha+\beta)$ and $\cos(\alpha + \beta)$.

e) The second-order differential equation,

$$\frac{d^2}{dx^2} f(x) = -k^2 f(x)\,, \tag{6}$$

has two linearly independent solutions. These can be written in more than one way, and two convenient forms are

$$f(x) = Ae^{ikx} + Be^{-ikx}\,, \qquad f(x) = a \sin(kx) + b \cos(kx)\,. \tag{7}$$

Verify that both are solutions of (6). Since both are equally good solutions, we must be able determine a and b in terms of A and B; do so.

Because the wavefunction is complex, we must use complex numbers a lot in quantum mechanics. Here we review the math, so you don't have to think about it later when you're focused on the physics.

Problem 3.2: Probability and time. (20 points)

Recall that the time evolution of a wavefunction $\Psi(x,t)$ is determined by the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)\,. \tag{8}$$

a) Consider any two normalizable solutions to the Schrödinger equation, $\Psi_1(x,t)$ and $\Psi_2(x,t)$. We can form their *inner product* by $\int_{-\infty}^{\infty}\Psi_1^*(x,t)\Psi_2(x,t)\,dx$; as we will discuss more in class, this is a generalization of the dot product in vector analysis, where we think of Ψ_1 and Ψ_2 as infinite-dimensional vectors with one number for each choice of x. Prove that the inner product is independent of time,

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*(x,t)\Psi_2(x,t)\,dx = 0\,. \tag{9}$$

Hint: prove the useful intermediate result,

$$\frac{\partial}{\partial x}\left(\Psi_1^*\frac{\partial\Psi_2}{\partial x} - \frac{\partial\Psi_1^*}{\partial x}\Psi_2\right) = \Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2} - \frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\,. \tag{10}$$

b) Say you have a wavefunction at time $t = 0$, $\Psi(x, t = 0)$, which you have normalized to total probability one. Given your results, what happens to this normalization at later times? If this failed to happen, how should we have treated the Schrödinger equation?

c) Consider the total probability of observing the particle between points $x = a$ and $x = b$, which we can call P_{ab}. Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)\,, \tag{11}$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)\,. \tag{12}$$

d) Think about the result for part c). What is the physical meaning of equation (11), and what role does J play? A clue is its name: $J(x,t)$ is called the *probability current*. (In thinking about this question, don't worry about the form of equation (12); it's not very illuminating here.)

Here we think about how probability changes as time evolves. We will return to the idea of the probability current later on.

Problem 3.3: Another wavefunction. (20 points)

a) Using the known definite integral $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$, show directly the related result

$$\int_{-\infty}^{\infty} e^{-az^2} dz = \sqrt{\frac{\pi}{a}} \,. \tag{13}$$

Now find an expression for the integral $\int_{-\infty}^{\infty} z^2 e^{-az^2} dz$ by *differentiating* (13) with respect to the parameter a.

b) Consider the wavefunction for a particle of mass m,

$$\Psi(x,t) = A\left(e^{-i\alpha\hbar t/m} + \beta\, x\, e^{-3i\alpha\hbar t/m}\right) e^{-\alpha x^2} \,, \tag{14}$$

where A, α and β are real constants. What is the probability density $\rho(x)$ for this wavefunction? Use Euler's formula to write all complex exponentials as sines and cosines. Note that when $\beta = 0$ this reduces to a wavefunction on the previous problem set, whose probability density was independent of time; is this probability density independent of time as well?

c) Choose A so as to normalize this wavefunction. You will find part a) helpful; also, think about whether you can argue some terms are zero without explicitly calculating them.

d) What is the expectation value $\langle x \rangle$? What kind of motion is the average value of the particle's position executing?

This problem is more practice with wavefunctions, their probability densities and expectation values, but with a more complicated wavefunction than last problem set, so we can see some interesting physics. You weren't asked to prove it, but this wavefunction solves the Schrödinger equation with the same potential as the one in problem 2.4.

Problem 3.4: The 3-slit experiment. (20 points)

Consider a wavefunction of the *plane-wave* form

$$\Psi(x,t) = \Psi_0 e^{i(kx-\omega t)} \,. \tag{15}$$

While this wavefunction is not normalizable, and is thus not physical, it will still be useful for us as an idealization of a physical wave.

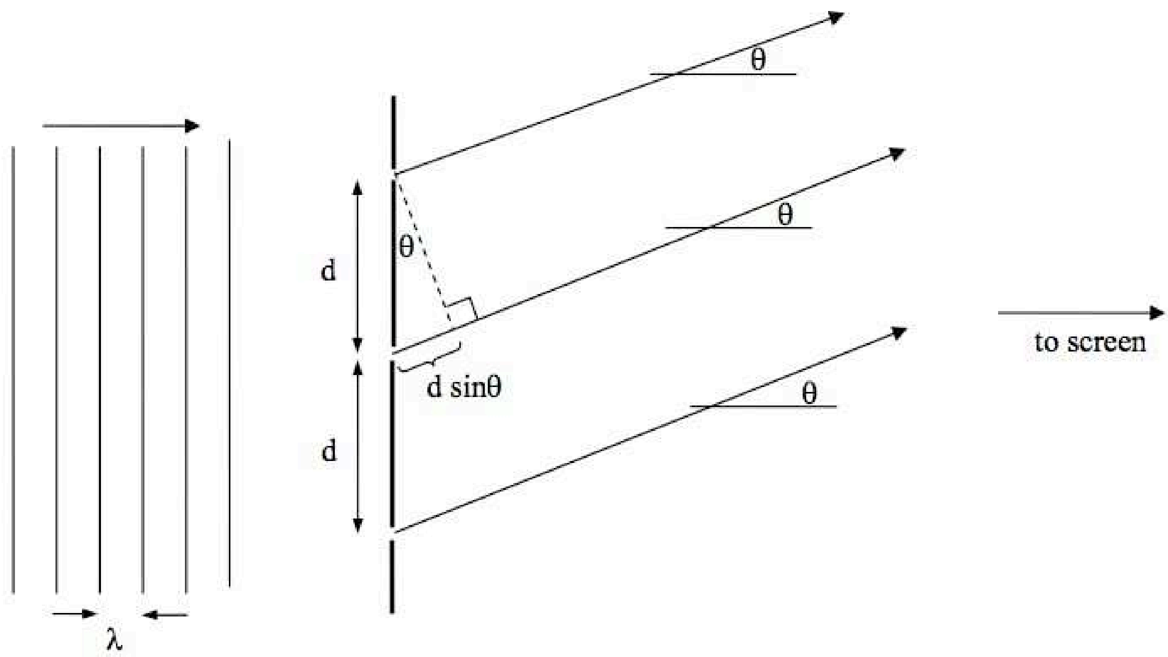

Figure 1: The 3-slit experiment.

Consider the plane wave hitting a wall with three equally spaced slits and going to illuminate a distant screen; the wall and slits will change the wavefunction on the other side. At any point on the screen, there is a phase difference δ between the waves from slit 1 and 2 and a phase difference 2δ between slits 1 and 3, where δ depends on the path difference ℓ according to $\delta = 2\pi\ell/\lambda = k\ell$. The total wavefunction at a point on the screen is

$$\Psi_{tot} = \Psi_1 + \Psi_2 + \Psi_3 = \Psi_0(e^{-i\omega t} + e^{-i(\omega t-\delta)} + e^{-i(\omega t-2\delta)}) = \Psi_0 e^{-i\omega t}(1 + e^{i\delta} + e^{2i\delta})\,. \quad (16)$$

See the figure for the basic set-up.

a) What is the probability density at that point in the screen? Simplify your answer as much as possible.

b) Plot the probability density as a function of δ; this is the 3-slit interference pattern.

c) For adjacent slits separated by a distance d with $d \gg \lambda$, what is the angular separation of global maxima on the screen?

The 2-slit experiment is famous, and here we look at its more complicated cousin. All the "wave-like" features of particles come about because the wavefunction (as the name suggests!) behaves like a wave.