University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#5
due Wed, Sep 23, 2PM at start of class

1. A few more properties of Hermitian operators (total: 10pts)

   a) Show that the sum of two Hermitian operators is a Hermitian operator.

   b) Suppose that $\hat{A}$ is a Hermitian operator and α is a number. Under what condition on α is $\alpha\hat{A}$ a Hermitian operator?

2. A few useful properties of the Dirac delta function (total: 10pts)

   a) By multipying both sides of the following equations by a differentiable function $f(x)$, and integrating over x, verify the following equations:

$$\delta(x) = \delta(-x) \tag{1}$$

$$\frac{d}{dx}\delta(x) = -\frac{d}{dx}\delta(-x) \tag{2}$$

$$x\delta(x) = 0 \tag{3}$$

$$x\frac{d}{dx}\delta(x) = -\delta(x) \tag{4}$$

   b) Prove the following relations

$$\int_{-\infty}^{\infty} \delta(a-x)\delta(x-b)dx = \delta(a-b) \tag{5}$$

$$f(x)\delta(x-a) = f(a)\delta(x-a) \tag{6}$$

3. A few useful properties of Fourier transforms (total: 10pts)

   a) Proove Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |g(k)|^2 dk, \tag{7}$$

   for any regular function

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\exp(ikx)dx$$

   with

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\exp(-ikx)dx$$

   b) The convolution of two functions f_1 and f_2 is defined as the integral

$$F(x) = \int_{-\infty}^{\infty} f_1(y)f_2(x-y)dy$$

   Show that if $G(k)$ is the Fourier transform of $F(x)$ and $g_1(k)$ and $g_2(k)$ are the Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively, then

$$G(k) = \sqrt{2\pi}g_1(k)g_2(k) \tag{8}$$

4. (Total: 15pts) Consider the plane wave characterized by a positive constant k_0

$$\Psi(x,t) = A\exp(ik_0x - i\hbar k_0^2 t/2m) \tag{9}$$

a) Find the Fourier transform $\Phi(k,t)$ of this wavefunction. Describe in words: What is the result telling you about plane waves?

b) Find the probability current

$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$

for the wavefunction given in Eq. (9). Try to simplify your result as much as possible and give a brief interpretation of the expression you get.

c) If you flip the sign of k_0, describe what has changed physically and mathematically about the state. How is this reflected in the results from part b)?

5. (Total: 15pts) Consider the following function

$$\Phi(k) = \begin{cases} A(a-|k|), & |k| \leq a \\ 0, & |k| > a \end{cases}$$

a) Find the normalization factor A and plot(!) the function $\Phi(k)$.

b) Find the Fourier transform of $\Phi(k)$ (Result: $\Phi(x) = \frac{A}{\sqrt{2\pi}}\frac{4}{x^2}\sin^2(ax/2)$). Plot the function $\Phi(x)$.