Physics 3220 – Quantum Mechanics 1 – Spring 2009
Problem Set #10

Due Wednesday, April 1 at 9am

Problem 10.1: Uncertainty. (10 points)

a) For hermitian operators $\hat{A}$ and $\hat{B}$, what must be true about a constant α such that the operator

$$\alpha[\hat{A}, \hat{B}] \tag{1}$$

is also hermitian? Use your result to show that the right-hand-side of the generalized uncertainty principle (Griffiths Eq. 3.62) is always real and nonnegative.

b) Show that the generalized uncertainty principle for the operators $\hat{x}$ and $\hat{H}$ is

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle p \rangle| \,. \tag{2}$$

What can we deduce about the value of $\langle p \rangle$ for a stationary state?

Problem 10.2: Uncertainty principle for small but still macro objects. (10 points)

We never notice the Heisenberg uncertainty principle $\sigma_x \sigma_p \geq \hbar/2$ for macroscopic objects, because Planck's constant is so small. Let's see how big an effect the uncertainty principle produces for an object that is very small, but still large compared to atoms. Consider a 1-micrometer (= 1 micron, 10^{-6} m) diameter droplet of oil suspended in a vacuum chamber; this is about the size of a bacterium. (You can suspend an oil droplet by charging it and applying an electric field, like Millikan did in his famous experiment.)

With care, it is possible to determine the droplet's position to within an uncertainty of about 10% of its diameter. (Not easy, because this uncertainty turns out to be a little smaller than the wavelength of visible light: $\lambda \approx 0.5 \mu m$.) Make an order-of-magnitude estimate of the corresponding minimum uncertainty in the speed of this particle. Compare the speed uncertainty to the size of the particle by calculating how long it would take because of this speed uncertainty to generate a new position uncertainty the size of the particle. Would this uncertainty in the speed be easy or hard to measure? Briefly, explain.

Note: This is an estimation/order of magnitude problem, dont worry about factors of 2. What would be a reasonable density of oil to use?

Problem 10.3: Expansion of functions in bases. (20 points)

Suppose there are two observables A and B, with corresponding hermitian operators $\hat{A}$ and $\hat{B}$, and that each possesses a *discrete* spectrum — that is, their eigenvalues are discrete rather than continuous. (You are very familiar with the Hamiltonian having a discrete spectrum for the infinite square well or the SHO, but there are plenty of other operators with a discrete spectrum — later, we will see that the angular momentum operator has a discrete spectrum.)

Since eigenfunctions form complete sets, these operators each must have a complete orthonormal set of eigenvectors and corresponding eigenvalues, satisfying the eigenvalue equations,

$$\hat{A}|a_n\rangle = a_n|a_n\rangle\,, \qquad \hat{B}|b_n\rangle = b_n|b_n\rangle\,. \tag{3}$$

Here we are adopting the common practice of *labeling an eigenstate by its eigenvalue* — so $|a_n\rangle$ is "the eigenvector of $\hat{A}$ with eigenvalue a_n" and similar for $|b_n\rangle$. Don't confuse the eigenvalues (which are numbers) with their eigenvectors (which are vectors in Hilbert space)!

Since the eigenfunctions form complete sets, any arbitrary evolving wavefunction $|\Psi\rangle(t)$ can be written as an expansion in *either* basis :

$$|\Psi\rangle(t) = \sum_n c_n(t)|a_n\rangle = \sum_n d_n(t)|b_n\rangle\,, \tag{4}$$

for some constants c_n and d_n that change with time. Note by using the letter c_n we're not assuming $\hat{A}$ is the Hamiltonian; it's just a good letter.

a) Prove that the expectation value of $\hat{A}$ is given by

$$\langle A\rangle(t) = \sum_n a_n |c_n(t)|^2\,. \tag{5}$$

b) Show how the d_ns are related to the c_ns. That is, derive a formula that expresses any particular d_n in terms of the set of c_ns. *(Hint: your answer will involve inner products of the basis vectors.)*

c) Consider the object

$$\hat{P} = |a_1\rangle\langle a_1|\,. \tag{6}$$

Show that this is an *operator*, meaning that it can act on a vector in Hilbert space to give another vector. (You don't have to show every axiom of linear operators, just demonstrate how it works.) What is $\hat{P}|\Psi\rangle$ in terms of c_n and $|a_n\rangle$?

d) Show that the operator $\hat{A}$ can be written in terms of its *spectral decomposition*, which means

$$\hat{A} = \sum_n a_n |a_n\rangle\langle a_n|\,. \tag{7}$$

This may look formal, but it's a commonly-used trick. *Hint: an operator is characterized by its action on all possible vectors, so what you must show here is that* $\hat{A}|f\rangle = \left\{\sum_n a_n|a_n\rangle\langle a_n|\right\}|f\rangle$ *for any vector* $|f\rangle$.

Problem 10.4: Particle in a 3D box. (20 points)

a) Consider the energies of a particle of mass m in a three-dimensional box which is a cube of edge-length a.

What are the energies and degeneracies of the first 6 energy levels ($E_1, E_2, \ldots E_6$) counting up from lowest energy? (The degeneracy of an energy level is the number of distinct states that share that energy.)

Stare at the spectrum you get and comment on it — is there a simple pattern, or is it more complicated than we saw for a particle in a 1D box?

b) What are the energies associated to the states with $(n_x, n_y, n_z) = (3, 3, 3)$ and $(5, 1, 1)$? What is the total degeneracy of the $(3, 3, 3)$ energy? (Note the question is not asking how many $(3, 3, 3)$ states there are, it is asking how degenerate is the energy that goes with that state.)

c) Consider now the energies of a particle in a *rectangular* 3D box with edge lengths a, b, and c (which don't have to be the same) along the x, y, and z axes respectively. In other words, the potential is:

$$
\begin{aligned}
V(x) &= 0\,, \qquad 0 < x < a \quad \text{AND} \quad 0 < y < b \quad \text{AND} \quad 0 < z < c\,, && (8)\\
&= \infty\,, \qquad \text{elsewhere}\,. && (9)
\end{aligned}
$$

What is an expression for the energy of the (n_x, n_y, n_z) state?

d) Suppose the rectangular box has edges of length a, $b = 2a$ and $c = 3a$. What are the energies of the 5 lowest-energy states? Are any of these levels degenerate? In general, breaking symmetry tends to remove degeneracies in quantum mechanics; comment on how this statement applies to this problem.