University of Colorado, Department of Physics
PHYS3220, Spring 11, HW#5
due Wed, Feb 9, in class
- Please note: Late Homeworks will not be accepted -

1. Superposition of stationary states
   The normalized state of a system is given by a superposition of stationary states as:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \chi_n(x) \exp(-iE_n t) \tag{1}$$

   a) Show that from the normalization of $\Psi(x,t)$ follows that

$$\sum_{n=1}^{\infty} |c_n|^2 = 1 \tag{2}$$

   Give a physical interpretation of Eq. (2).

   b) Solve Eq. (1) for c_5.

2. Energy measurement I
   A system is initially in the state

$$\Psi(x,t=0) = \frac{1}{\sqrt{7}}\left(\sqrt{2}\chi_1(x) + \sqrt{3}\chi_2(x) - i\chi_3(x) + \chi_4(x)\right)$$

   where $\chi_n(x)$ are eigenstates of the system's Hamiltonian such that $\hat{H}\chi_n(x) = n^2 E_0 \chi_n(x)$

   a) Is $\Psi(x,t)$ normalized for $t > 0$?

   b) If the energy of state is measured, what values will be obtained and with what probabilities?

   c) Calculate the expectation value of the total energy of the system.

   d) Suppose that a measurement yields $4E_0$, write down the wave function immediately after the measurement.

3. Energy measurement II
   Consider a particle in an (1D) infinite square well of width a. The normalized energy eigenstates of this system are given by

$$\Psi_n(x,t=0) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

   and the corresponding energy eigenvalues are given by $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

- Problem continues on the back -

Suppose that the wave function of the particle at time $t = 0$ is given by

$$\Psi(x, t = 0) = \begin{cases} A\frac{x}{a}, & \text{if} \quad 0 \leq x \leq \frac{a}{2} \\ A\left(1 - \frac{x}{a}\right), & \text{if} \quad \frac{a}{2} < x \leq a \\ 0, & \text{else} \end{cases}$$

where $A = \sqrt{\frac{12}{a}}$ such that $\int\limits_{-\infty}^{\infty} |\Psi(x, t = 0)|^2 dx = 1$. Calculate the probability that the measurement of the energy yields the eigenvalue E_3.
(Hint: Problem 1b) and its solution should be useful here.)

4. Parseval's theorem for Fourier transforms
For a function $f(x)$ its Fourier transform $F(k)$ is defined by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$

while

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

is called the inverse Fourier transform of $F(k)$. Prove Parseval's theorem

$$\int\limits_{-\infty}^{\infty} |f(x)|^2 dx = \int\limits_{-\infty}^{\infty} |F(k)|^2 dk$$

(This theorem tells you that if a function $f(x)$ is normalized, then its Fourier transform $F(k)$ is normalized too and vice versa.)

5. Some physics with plane waves
Consider the plane wave characterized by a positive constant k_0

$$\Psi(x, t) = A \exp(ik_0 x - i\hbar k_0^2 t/2m) \tag{3}$$

(Note: You do not have to determine A for solving the following problems.)

   a) Find the Fourier transform $\Phi(k, t)$ of this wavefunction. Describe in words: What is the result telling you about plane waves?
   b) Find the probability current

$$J(x, t) = \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right)$$

   for the wavefunction given in Eq. (3). Try to simplify your result as much as possible and give a brief interpretation of the expression you get.
   c) If you flip the sign of k_0, describe what has changed physically and mathematically about the state. How is this reflected in the results from part b)?