I: Thinking about the wave function

In quantum mechanics, the term wave function usually refers to a solution to the Schrödinger equation,

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t),$$

where $V(x)$ is the potential energy experienced by a particle of mass m and $\Psi(x,t)$ is the wave function in this one-dimensional example.

A. Let's say you have a system where the wave function is of the form:

$$\Psi_1(x,t) = f(x)e^{i\omega t}$$

where $f(x)$ is some real-valued function of x.

1. Is $|\Psi_1(x,t)|^2$ real? Is it positive? Do your answers make sense given the physical meaning (as discussed in class) of $|\Psi_1(x,t)|^2$?

2. Does $\Psi_1(x,t)$ depend on time? Does $|\Psi_1(x,t)|^2$ depend on time?

3. The expectation value of a measurable quantity a for a particle described by a wave function $\Psi(x,t)$ is defined as $\langle a\rangle = \int a|\Psi(x,t)|^2 dx$. Write down the expectation value $\langle x\rangle$ for the system in state $\Psi_1(x,t)$. Does it depend on time? Is it real?

   Describe in words how you interpret $\langle x\rangle$. What information do you get from it?

4. Write down an expression for $\langle g(x)\rangle$ where $g(x)$ is any real-valued function of x. Does it depend on time? Again, how would you physically interpret $\langle g(x)\rangle$?

B. Now let's say your system is a bit more complex (pun intended):

$$\Psi_2(x,t) = f(x)e^{i\omega t} + g(x)e^{2i\omega t}$$

where $f(x)$ and $g(x)$ are real functions of x which are orthogonal to each other.

1. Is $|\Psi_2(x,t)|^2$ real? Is it positive? Do your answers make sense given the physical meaning of $|\Psi_2(x,t)|^2$?

2. Does $|\Psi_2(x,t)|^2$ depend on time?

3. Write down an expression for $\langle x \rangle$. Does it depend on time? Describe the difference(s) between this result and the result for section A.3 above.

   Even though f and g are unknown functions of x, do your best to give a physical description or interpretation of this new result for $\langle x \rangle$ for the state Ψ_2.

✓ Check your results with a tutorial instructor.

C. Now, we will deal with a new wave function at a single moment in time, $\psi_3(x) = \Psi_3(x, t = t_0)$, represented by the graph below (a sine curve from $\pi/4$ to $5\pi/4$ and zero everywhere else).

1. Find a value of A which will normalize $\psi_3(x)$.

2. Using physical arguments (*i.e.*, without doing the integral), what do you think $\langle x \rangle$ is? (If you feel uncertain, you can check by doing the integral)

3. We want to find the standard deviation for x for this system. First, do you think that $\langle x^2 \rangle$ is larger/the same/smaller (circle one) than $\langle x \rangle^2$? Now, actually calculate $\langle x^2 \rangle$.

4. What is σ_x^2? What is the probability that you will find the particle represented by $\psi_3(x)$ in the range $\langle x \rangle \pm \sigma_x$? (Recall that $\sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$).

D. Now we somehow create a system where for an instant, the wave function, $\psi_4(x) = \Psi_4(x, t = t_0)$, looks like the graph below.

1. Find the value of A which will normalize $\psi_4(x)$.

2. Using physical arguments (*i.e.*, without doing the integral), what do you think $\langle x \rangle$ is? (If you feel uncertain, you can check by doing the integral)

3. Estimate $\langle x^2 \rangle$ and σ_x. Indicate on the graph above the range which you think represents $\langle x \rangle \pm \sigma_x$.
   Bonus (*i.e.*, come back to this if you have time after finishing the rest of the tutorial), calculate $\langle x^2 \rangle$ and σ_x^2.

4. How do you physically interpret σ_x?

5. What are the possible values of a measurement of x on any of these identical systems? Do you "expect" to measure x equal to the expectation value of x?

✓ Check your results with a tutorial instructor.