Formula Sheet for Final Exam (These formulas will be given.)

The classical wave equation: $\dfrac{\partial^2 f(x,t)}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f(x,t)}{\partial t^2}$

The time-dependent Schrödinger Equation:

$$\boxed{i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi}$$

The standard deviation $\sigma = \sqrt{\left\langle (\Delta x)^2 \right\rangle} = \sqrt{\left\langle \left(x - \langle x\rangle\right)^2 \right\rangle} = \sqrt{\left\langle x^2 \right\rangle - \langle x\rangle^2}$

Momentum operator: $\hat{p}_x = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x}$

Fourier Transform formulae (Plancherel's Theorem) :

$$f(x) = \tfrac{1}{\sqrt{2\pi}}\int dk\ F(k)\exp(ikx)$$
$$F(k) = \tfrac{1}{\sqrt{2\pi}}\int dx\ f(x)\exp(-ikx)$$

A useful form of the delta function: $\delta(x) = \tfrac{1}{2\pi}\int \exp(ikx)\,dk$

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^*}{\partial x} - \Psi^*\frac{\partial \Psi}{\partial x}\right) = \frac{\hbar}{m}\,\mathrm{Im}\left(\Psi^*\frac{\partial \Psi}{\partial x}\right) = \frac{1}{m}\,\mathrm{Re}\left(\Psi^*\hat{p}\Psi\right)$$

Momentum space wavefunction :

$$\Psi(x,t) = \tfrac{1}{\sqrt{2\pi\hbar}}\int dp\ \Phi(p,t)\exp(ipx/\hbar)$$
$$\Phi(p,t) = \tfrac{1}{\sqrt{2\pi\hbar}}\int dx\ \Psi(x,t)\exp(-ipx/\hbar)$$

Position-momentum commutator: $[\hat{x}, \hat{p}_x] = i\hbar$

$\dfrac{d\langle Q\rangle}{dt} = \dfrac{i}{\hbar}\left\langle [\hat{H}, \hat{Q}] \right\rangle$ (assuming Q operator is independent of time)

Heisenberg Uncertainty Principle: $\sigma_A\,\sigma_B \geq \left|\dfrac{1}{2i}\left\langle [\hat{A}, \hat{B}] \right\rangle\right| = \dfrac{1}{2}\left|\left\langle [\hat{A}, \hat{B}] \right\rangle\right|$

$[\hat{L}_i, \hat{L}_j] = i\hbar\hat{L}_k$

Radial TISE for H-atom: $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_{eff}u = Eu$, $V_{eff} = V(r) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)$

$Y_\ell^\ell(\theta,\phi) = (\sin\theta)^\ell e^{i\ell\phi}$

$\hat{L}_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1)-m(m\pm1)}\,|\ell,m\pm1\rangle$

Spin ½ matrices: $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

End of Formulas

Exam 1 Review Topics: **Ch 1 and Ch 2 in Griffiths, Homeworks 1 thru 5, and Lecture Notes up thru/including "Dirac Delta function"**

- probability density and the wavefunction
- normalization of the wavefunction
- computation of expectation values and standard deviation, given a wavefunction

Classical wave equation, the form of a traveling wave : f(x,t) = f(x – v t), and superposition of solutions due to linearity of the equation.

- Complex variables: polar form vs. cartesian form; complex conjugation; modulus (amplitude) of a complex variable
- Separation of variables.
- Deriving the TISE from the TDSE, starting with separation of variables
- Relation between solutions to the TISE [$\hat{H}\psi_n(x) = E_n\psi_n(x)$] and solutions to the TDSE [$\Psi(x,t) = \sum_n c_n \exp[-iE_n t/\hbar]\psi_n(x)$.] where expansion coefficients c_n gotten from "Fourier's Trick"
- Solutions of the infinite square well.
- Qualitative solutions to the TISE: Sketching the solutions $\psi_n(x)$, given V(x)
- Energy eigenvalue equation and properties of the energy eigenfunctions (they form a complete, orthonormal set)
- Free particles: plane-waves states $\Psi(x,t) = A\,e^{i(kx-\omega t)}$ and wave packets; phase velocity $v_{phase} = \omega/k$, vs. group velocity $v_{group} = d\omega/dk$; relation between free particle wavefunction $\Psi(x,t)$ and the Fourier transform $\phi(x)$.

- Procedure for solving the TISE for specific potentials (such a finite square well or scattering from a step): 1) Write general solutions with unknown constants. 2) Apply boundary conditions to determine the constants.
- Tunneling depth
- Reflection and transmission coefficients and relation to the probability current.

Exam 2 Review Topics: **Ch 1, 2, and 3 in Griffiths, Homeworks 1 thru 10, and Lecture Notes up thru/including Angular Momentum (upto p.H-9)**

- Review Exam 1 Material! Make sure you understand the things you missed on exam 1.
- Hermitean operators and their properties (real eigenvalues, eigenstates that are complete, orthonormal)
- Vectors spaces and Hilbert Space
- The Postulates:

  P1: Physical states represented by square-integrable vectors in Hilbert Space

  P2: Observables are represented by Hermitean operators. The result of a measurement is always one of the eigenvalues. Operators for momentum p, position x, and any function of x, p

  P3: $\text{Prob(find } q_n) = \left|\langle f_n \,|\, \Psi \rangle\right|^2 \qquad \text{Prob(find } q \to q + dq) = \left|\langle f_q \,|\, \Psi \rangle\right|^2 dq$

  P4: Wavefunction collapse

  P5: TDSE
- Particle in a 3D box. Degeneracy.
- Commutation relations, operator algebra
- If operators commute, they can have simultaneous eigenfunctions.
- Time-dependence of expectation values
- Heisenberg Uncertainty Principle and time-energy uncertainty
- Angular Momentum operator $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$
- Operator algebra: [A+B, C] = [A, C] + [B, C] ; [AB, C] = A[B, C] + [A, C]B
- Angular momentum raising and lowering operators and proof of angular momentum eigenvalue equations: $\hat{L}^2\, Y = \hbar^2 \ell(\ell+1) Y$, $L_z Y = \hbar\, m\, Y$, possible values of $\boldsymbol{\ell}$, m.

Post-Exam 2 Review Topics:

- General features of the TISE solution of H-atom

  Separation of variables: Energy eigenstates have radial and angular parts:

  $\psi = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$, possible values of n, ℓ, and m.

  Energy depends only on principle quantum number n: $E_n = \frac{E_1}{n^2}$

  Only s-states (ℓ = 0) "touch nucleus" , have $R(r = 0) \neq 0$

  Normalization in 3D

  Radial probability density $P(r)\,dr$ = Probability(find particle in range $r \to r+dr$)

  Selection Rule for emission/absorption of photons: $\Delta\ell = \pm 1$

- Bra-ket Notation , Projection operators
- Matrix formulation of QM:

  states (kets) are column vectors; operators are matrices; bra's are row vectors;
  operator $\hat{A}$ hemitian $\Leftrightarrow$ $A_{mn} = A^*_{nm}$

- Spin ½

  Spin and magnetic moment; Stern-Gerlach apparatus

  Construction of $\hat{S}_x$, $\hat{S}_y$ from raising and lowering operators

  Eigenstates and eigenvalues of $\hat{S}_z$, $\hat{S}_x$, $\hat{S}_y$

THREE IMPORTANT POINTs:

- Most states are NOT energy eigenstates (or eigenstates of *any* operator). Energy eigenstates are *special* states; they are solutions of $\hat{H}\,u_n = E_n\,u_n$ [ket notation . $\hat{H}|n\rangle = E_n|n\rangle$] A *general* state is a *linear combination* of eigenstates: $|\psi\rangle = \sum_n c_n|n\rangle = \sum_n |n\rangle\langle n|\psi\rangle$

- The time dependence of a state is related to the energy eigenstates and eigenvalues

$$\hat{H}\,u_n = E_n\,u_n\,, \qquad \Psi(x,t) = \sum_n c_n(t)\,u_n(x) = \sum_n c_{n0}\,e^{-iE_nt/\hbar}u_n(x)$$

In ket notation: $|\psi(t)\rangle = \sum_n c_n(t)|n\rangle = \sum_n c_{n0}\,e^{-iE_nt/\hbar}|n\rangle$

- Measurement: If system is in state $|\psi\rangle$ and a measurement of the observable A is made [A has operator $\hat{A}$: $\hat{A}|u_n\rangle = \lambda_n|u_n\rangle$], then the possible results of the measurement are one of the eigenvalues λ_n, and the probability of a given result is $\text{Prob(find }\lambda_n) = |\langle u_n|\psi\rangle|^2$.

  Notice that this probability is NOT the expectation value of A: $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$.

  A probability is dimensionless; in contrast, an expectation value has the dimensions of the observable A.