$f_p(x) = A\, e^{ipx/\hbar}$ $\qquad$ ($p = \hbar k$, <u>any</u> real k (+) or (−))

Adjust A so that $\langle f_{p'} | f_p \rangle = \delta(p - p')$

$$\langle f_{p'} | f_p \rangle = |A|^2 \underbrace{\int dx\, e^{i(p-p')x/\hbar}}_{2\pi\,\delta\left(\frac{p-p'}{\hbar}\right) = 2\pi\hbar\,\delta(p-p')} = \underbrace{|A|^2\, 2\pi\hbar}_{\text{want this to be } 1}\, \delta(p - p')$$

using $\delta(c \cdot x) = \frac{1}{|c|}\delta(x)$

$\Rightarrow A = \frac{1}{\sqrt{2\pi\hbar}}$

$$\Rightarrow \boxed{f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{+ip\cdot x/\hbar}}$$

<u>Any</u> real p, (+) or (−)

Are the f_p's a complete set?

Fourier Analysis (Plancherel's Thm) says that <u>any</u> $f(x)$ can be written

$$f(x) = \frac{1}{\sqrt{2\pi}} \int dk\, F(k)\, e^{+ikx}, \text{ where}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int dx\, f(x)\, e^{-ikx}$$

Now, <u>any</u> $\Psi(x,t)$ is a fn of x (at arbitrary t)

$$\Rightarrow \Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk\, \Phi_k(k,t)\, e^{+ikx} \text{ where}$$

$$\Phi_k(k,t) = \frac{1}{\sqrt{2\pi}} \int dx\, \Psi(x,t)\, e^{-ikx}$$

Change of variable: $k \to p = \hbar k$, $dk = dp/\hbar$

Define $\Phi(p,t) = \dfrac{\Phi_k(k,t)}{\sqrt{\hbar}}$ (where $p = \hbar k$)

No subscript

$$\Rightarrow \Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \Phi(p,t)\, e^{ipx/\hbar}$$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \Psi(x,t)\, e^{-ipx/\hbar}$$

$$\Psi(x,t) = \int dp\, \Phi(p,t)\, f_p(x) \quad \Rightarrow f_p\text{'s are complete}$$

Note! Previously we wrote similar relations when $\Psi(x,t)$ was a free particle state ($V=0$). But <u>any</u> fn $\Psi(x,t)$ can be Fourier analyzed. In the special case of free particle, then

$$\Phi_k(k,t) = \phi(k)\, e^{-i\omega t}, \text{ where } \omega = \frac{\hbar k^2}{2m}$$

But this particular time-dependence in $\Phi(k,t)$ is true <u>only</u> for the special case of a free particle

$\Phi(p,t)$ is called the momentum-space wavefunction. It is the Fourier transform of $\Psi(x,t)$ and tells "how much $p = \hbar k = h/\lambda$" is in Ψ. $\Phi(p,t)$ contain all the same info as $\Psi(x,t)$.

Ready now to re-state Postulate 3

Previously, Postulate 3 was stated as

$$\text{Prob}(\text{find position in } x \to x+dx) = |\Psi|^2 dx$$

Our re-statement will look very different, but will be same.

Postulate 3 If a system is in state $\Psi(x,t)$, and a measurement of observable Q is made on the system, where the corresponding operator $\hat{Q}$ has eigenfunctions $f_n(x)$ and eigenvalues q_n:

$$\hat{Q}\, f_n(x) = q_n\, f_n(x), \qquad \text{then}$$

the strongest predictive statement that can be made about the result of that measurement is:

$$\text{Prob (measure } q_n) = |\langle f_n | \Psi \rangle|^2 \qquad \text{(discrete spectrum)}$$

If spectrum is continuous, $\hat{Q}\, f_q(x) = q\, f_q(x)$, any real value of q, then

$$\text{Prob (measure } q \to q + dq) = |\langle f_q | \Psi \rangle|^2\, dq$$

Example of P3:

Suppose system has discrete energy eigenvalues:

$$\hat{H}\, \psi_n(x) = E_n\, \psi_n(x) \qquad (n = 1, 2, 3, \ldots)$$

and system is in state that is linear combo

$$\Psi(x,t) = \sum_n c_n(t)\, \psi_n(x) = \sum_n c_n \cdot e^{-iE_n t/\hbar}\, \psi_n(x)$$

$$\left(c_n(t) = c_n \cdot e^{-iE_n t/\hbar} \quad \text{where } c_n = \int dx\, \psi_n^*\, \Psi \right),$$

then a measurement of energy will yeild value E_n with probability =

$$|\langle \psi_n | \Psi \rangle|^2 = |c_n|^2$$

If the system is in particular eigenstate n_0:

$$\Psi(x,t) = \psi_{n_0}(x)\, e^{-iE_{n_0}t/\hbar} \quad \Longrightarrow$$

$$\Psi(x,t) = \sum_n c_n e^{-iE_nt/\hbar} \psi_n(x)$$ where $c_{n_0} = 1$ and <u>all other</u> c_n's $= 0$

then measurement of energy will yeild E_{n_0} w/ probability $|c_{n_0}|^2 = 1$.

$\Rightarrow$ Eigenstate of energy ~~will yield a~~ is state of definite energy.

Same <u>argument</u> applies to any observable: eigenstate of $\hat{Q}$ is state of definite Q (value of Q = eigenvalue of eigenstate)

*

Previously, we <u>asserted</u> (Notes SE-5) that expectation value of Q =

$$\langle \hat{Q} \rangle = \int dx\, \Psi^* \hat{Q}\, \Psi = \langle \Psi | \hat{Q}\, \Psi \rangle$$

Can now show this follows from P3:

$$\langle \Psi | \hat{Q}\, \Psi \rangle \overset{\text{hermiticity!}}{=} \langle \hat{Q}\, \Psi | \Psi \rangle \overset{\text{completeness!}}{=} \langle \hat{Q} \sum_n c_n f_n | \Psi \rangle$$

$$\underset{(\hat{Q} f_n = q_n f_n)}{=} \langle \sum_n c_n q_n f_n | \Psi \rangle \underset{(q_n^* = q_n)}{=} \sum_n c_n^* q_n \underbrace{\langle f_n | \Psi \rangle}_{c_n}$$

$$= \sum_n q_n |c_n|^2 = \sum_n q_n \cdot \mathrm{Prob}(q_n)$$

= weighted average of q_n's ✓