University of Colorado, Department of Physics
PHYS3220, Fall 09, HW#6
due Wed, Sep 30, 2PM at start of class

1. Momentum space (Total: 15 pts)
   In class we have defined the momentum space wave function $\Phi(p,t)$ as

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \Psi(x,t) \exp(-ipx/\hbar) dx$$

   where

$$\Psi(x,t) = \frac{1}{\hbar\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \Phi(p,t) \exp(ipx/\hbar) dp$$

   is a solution of the time-dependent Schrödinger equation in configuration (position) space, that is

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

   Show that the expectation values of x and p can be written in terms of $\Phi(p,t)$ as:

$$< x > \;=\; \frac{1}{\hbar} \int\limits_{-\infty}^{\infty} \Phi^*(p,t) \left(i\hbar\frac{\partial}{\partial p}\right) \Phi(p,t) dp \tag{1}$$

$$< p > \;=\; \frac{1}{\hbar} \int\limits_{-\infty}^{\infty} \Phi^*(p,t) p \Phi(p,t) dp \tag{2}$$

2. Uncertainty Principle $\Delta x \Delta p \geq \hbar/2$ (Total: 10 pts)
   We never notice the Uncertainty Principle for macroscopic objects. Let's see how big an effect the Uncertainty Principle produces for an object that is very small but still large as compared to atoms. Consider a 1μm ($= 1 \times 10^{-6}$ m) diameter droplet of oil suspended in a vacuum chamber. With care, it is possible to determine the droplet's position to within an uncertainty of about 10% of its diameter. Make an order-of-magnitude estimate of the corresponding uncertainty in the speed of the particle. Would this uncertainty in the speed be easy or hard to measure? (You will need to estimate the density of oil to get the mass of the particle.)

3. Stationary states (Total: 20 pts)

   a) We have shown in class that the solution of the time-dependent Schrödinger equation is necessarily complex. Show that this statement does *not* hold for solutions of the time-independent Schrödinger equation.

   b) Show that two stationary states for *unequal* energies E_1 and E_2 are orthogonal. (Two states $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are called orthogonal, if $\int\limits_{-\infty}^{\infty} \Psi_1^*(x,t)\Psi_2(x,t)dx = 0$.)

   c) Consider a one-dimensional particle which is confined by a potential $V(x)$ within the region $0 \leq x \leq a$ and whose wave function is $\Psi(x,t) = \sin(\pi x/a)\exp(-i\omega t)$. Find the potential $V(x)$.

4. Superposition of stationary states (Total: 15 pts)
   Let $\chi_1(x)$ and $\chi_2(x)$ be normalized energy eigenfunctions of an one-dimensional system for unequal energies E_1 and E_2. Let $\Psi(x,t)$ be the wave function of the system, and suppose that at $t = 0$ it is given by

$$\Psi(x, t = 0) = A\chi_1(x) + \frac{1}{\sqrt{3}}\chi_2(x)$$

   a) Determine A such that $\Psi(x, t = 0)$ is normalized. Is A real or complex? (Hint: The statement proved in problem 3b) will be useful here.)

   b) Write down the wave function $\Psi(x,t)$ at time t. Is $\Psi(x,t)$ an energy eigenstate? Explain.

   c) Does the probability density $|\Psi(x,t)|^2$ vary with time?