Normalizing: Schrödinger Eq'n, if \( V = V(x) \) (no \( t \) !)

\[
\frac{\hbar}{i} \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x) \Psi(x,t)
\]

We argued \( \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1 \) if \( |\Psi|^2 \) is to be a "probability density". Need to check that if it's true at one \( t \), it'll still be true later!

\[
\frac{\partial}{\partial t} |\Psi(x,t)|^2 = \frac{\partial}{\partial t} \left[ \Psi^*(x,t) \, \Psi(x,t) \right]
\]

\[
= \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t} \Psi
\]

Now \( \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{1}{i \hbar} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi \cdot \frac{1}{\hbar}
\]

So \( \left( \frac{\partial \Psi}{\partial t} \right)^* = -\frac{\hbar^2}{2m} \cdot \frac{1}{-i \hbar} \frac{\partial^2 \Psi^*}{\partial x^2} + V^* \Psi^* \cdot \frac{1}{-\hbar} \)

\[= V, \text{ for real} \quad \text{P.S.} \]
\[ \frac{\partial \psi^*}{\partial t} = \frac{\hbar^2}{2m} \cdot \frac{1}{i \hbar} \frac{\partial^2 \psi^*}{\partial x^2} + V \psi^* \]

Thus
\[ \frac{\partial |\psi|^2}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{1}{i \hbar} \left[ \psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2} \right] + V |\psi|^2 \left( \frac{1}{i \hbar} \frac{1}{i \hbar} \right) \]

**Trick:** This is \( \frac{\partial}{\partial x} \left[ \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right] \)

Check for yourself, cross terms cancel!

So \( \frac{\partial |\psi|^2}{\partial t} \) is not zero (!) but \( \int_{-\infty}^{\infty} \frac{\partial |\psi|^2}{\partial t} \, dx \) is!

\[ \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |\psi|^2 \, dx = \frac{i \hbar}{2m} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left[ \text{stuff} \right] \, dx \]

\[ = \frac{i \hbar}{2m} \left[ \text{stuff} \bigg|_{-\infty}^{\infty} \right] = 0, \text{ because } (\text{if!}) \]

\( \psi(\infty) \text{ vanishes.} \)

That means \( \frac{d}{dt} \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 0, \text{ "Prob. is conserved"} \) over time.
Momentum operator:

We've argued \( \langle x \rangle = \int x |\psi(x,t)|^2 \, dx \)

What about \( \langle p \rangle = m \langle v \rangle \)?

I cannot just try \( m \int v |\psi(x,t)|^2 \, dx \), because I have no quantum formula for \( V(x) \) (it turns out you can't, due to Heisenberg uncertainty).

Griffiths says \( \langle v \rangle = \frac{d\langle x \rangle}{dt} \) (seems reasonable!)

This deriv is a lot like what we just did (prev. 2 pages), but there's an extra "\( x \)" inside the integral.

But \( \frac{dx}{dt} \) is not velocity, it's \( \frac{d\psi}{dt} \) [\( (x,t) \) are our two variables!]

So \( \langle p \rangle = m \frac{d\langle x \rangle}{dt} = m \int x \frac{d}{dt} |\psi(x,t)|^2 \, dx \)

using result from prev pages,
\[ \langle p \rangle = m \int x \cdot \frac{\hat{c} \hbar}{2m} \frac{\partial}{\partial x} \left[ \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right] dx \]

Integrate by parts, toss "surface term" (assuming \( \frac{\partial \psi}{\partial x} \) vanishes fast enough at \( \alpha \) to kill the "x" that comes along too, now!)

\[ \langle p \rangle = \frac{\hat{c} \hbar}{2} (-i) \int \left[ \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right] dx \]

Now do it again, \( \int -\psi \frac{\partial \psi^*}{\partial x} \) dx = \[ \left. \psi^* \psi \right|_0^a + \int \psi^* \frac{\partial \psi}{\partial x} dx \]

\[ = -uv + \int uv \, du \]

so two terms are both same, and

\[ \langle p \rangle = -\hat{c} \hbar \int \psi^* \frac{\partial \psi}{\partial x} dx \]

This looks like \( \int \psi^* \hat{p} \psi \) dx with \( \hat{p} = -\hat{c} \hbar \frac{\partial}{\partial x} \), the "momentum operator"
OK, check it out:

\[ \langle p \rangle = \int p(x) \rho(x) \, dx \]

Classically, \[ \langle p \rangle = \int p(x) \rho(x) \, dx \]

In QM, you'd expect \[ p \to \Psi^*(x,t) \Psi(x,t) \]

And that's close, but \( p \) is no longer a function of \( x \), it's the operator \(-i\hbar \frac{\partial}{\partial x}\), and you "sandwich" it:

\[ \langle p \rangle = \int \Psi^*(x,t) \hat{p} \Psi(x,t) \, dx \]

\[ \Rightarrow \text{with } \hat{p} = -i\hbar \frac{\partial}{\partial x}. \]

Does this make sense? You know, de Broglie says

\[ \Psi(\text{free particle}) = Ae^{i(kx - \omega t)} \]

has momentum \( p = \hbar k \).

But notice \(-i\hbar \frac{\partial \Psi_{\text{free}}}{\partial x} = -i\hbar (i\hbar) \Psi_{\text{free}} = \hbar k \Psi_{\text{free}}\)

So \( \hat{p} \Psi_{\text{free}} = (\hbar k) \Psi_{\text{free}} \). Nice! \( \hat{p} \) is novel...

operator a number, the momentum according to de Broglie
If $\hat{\Theta} \Phi = c \Phi$ (for some special functions $\Phi$) we say $\Phi$ is an "eigenfunction of $\hat{\Theta}$" with "eigenvalue $c$".

Apparently $\psi_{\text{free}}$ is an eigenfunction of $\hat{p} = -i \hbar \frac{\partial}{\partial x}$ with eigenvalue $\hbar k$ (which DeBroglie says is the number $p$).

In classical mechanics all dynamical quantities "$Q$" depend on $x$ and/or $p$:

- E.g. position = $x$
- Kin Energy = $\frac{p^2}{2m}$, Total Energy = $\frac{p^2}{2m} + V(x)$
- Ang momentum = $\vec{x} \times \vec{p}$

In quantum, we just replace every "$p$" in these classical eqns with $-i \hbar \frac{\partial}{\partial x}$, i.e. with $\hat{p}$, and get quantum operators.
So $\langle \hat{Q} \rangle = \int \Psi^*(x,t) \hat{Q} \Psi(x,t) \, dx$

where $\hat{Q}$ is just $Q(x,p)$, where you let $p \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial x}$

e.g. $\langle \text{Energy} \rangle = \int \Psi^* \left[ \frac{p^2}{2m} + V(x) \right] \Psi \, dx$

but $p$ is really $\frac{\hbar}{i} \frac{\partial}{\partial x}$, so $p^2 = -\frac{\hbar^2}{i^2} \frac{\partial^2}{\partial x^2}$

right !!

So $\langle \text{Energy} \rangle = \int \Psi^* \cdot \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi \, dx$

This is called $\hat{H}$, the Hamiltonian.

It is also the right side of Schrodinger's Eq'n,

which is $\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$. See p.1 of Griffiths!

So $\langle \text{Energy} \rangle = \langle \hat{H} \rangle$. 